Stirring, topology, and the simplest maps

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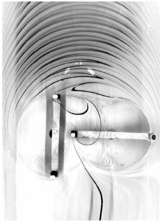
Channel flow: Injection into mixing region

Injection against flow



Injection with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 1] [movie 2] [movie 3]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

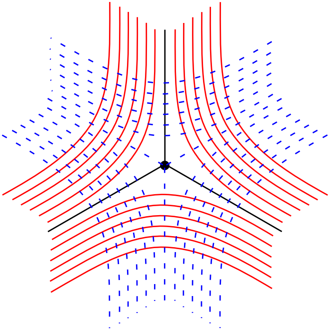
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

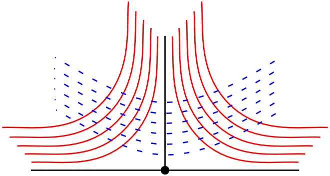
Number 3 is the one we want for good mixing

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

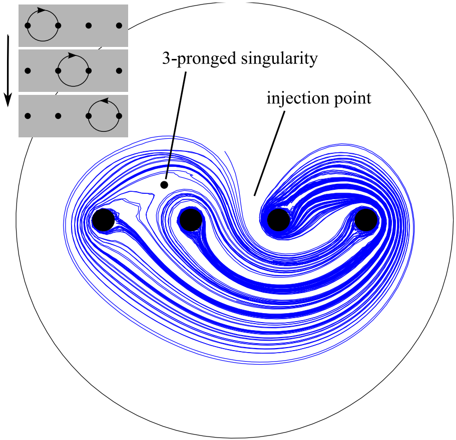


3-pronged singularity



Boundary singularity

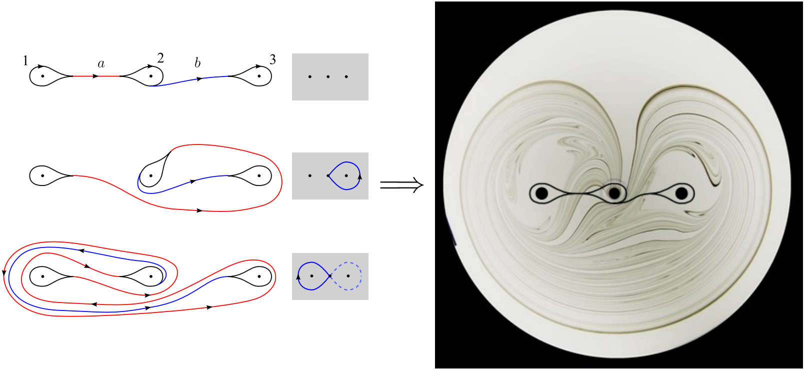
Visualising a singular foliation



- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

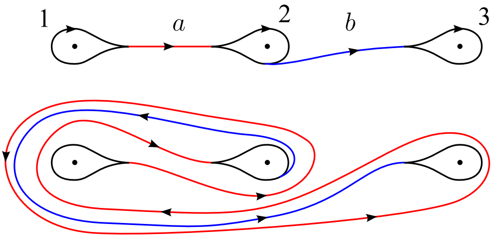
[Boyland et al. (2000); Thiffeault et al. (2008)]

Train tracks



Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the ‘cusps’ that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

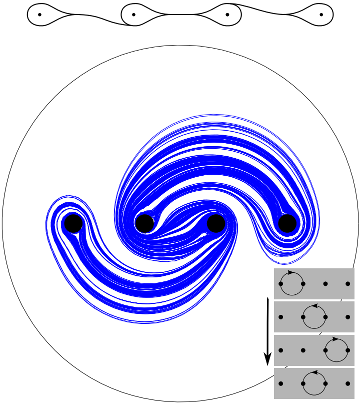
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of a and b, and write as matrix:

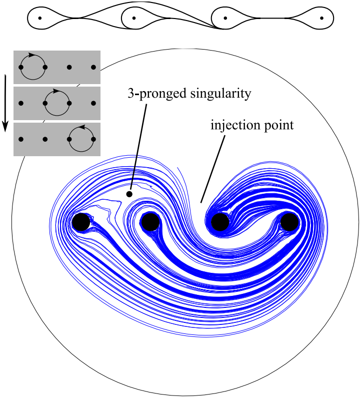
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



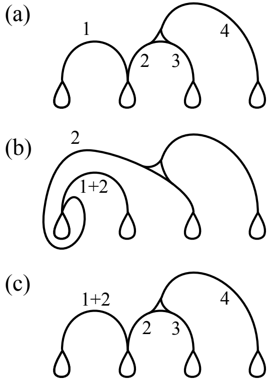
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

Pseudo-Anosovs involve ‘folding’ the foliation



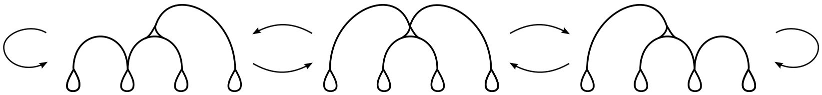
Build pA’s ‘in reverse,’ by regarding them as a sequence of gluings or foldings of pieces of foliation.

Make a transition matrix showing how edges 1–4 are folded:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

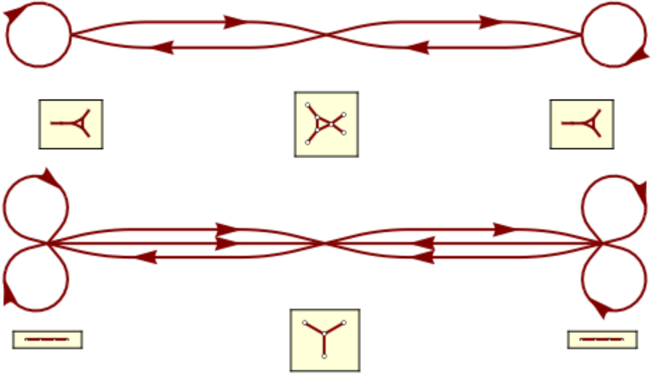
A train track folding automaton

The result is a folding automaton (a graph of train tracks):

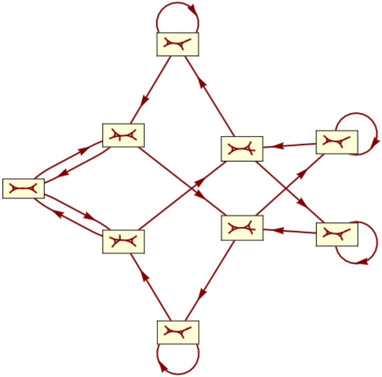


- Each arrow represents a folding of an edge onto another.
- A transition matrix is associated with each arrow.
- pA's are closed paths in this automaton, since they should leave the foliation invariant.
- All pA's are contained therein (up to conjugacy).

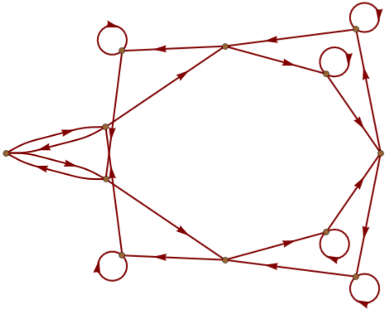
Automata can be simple. . .



Or elegant...

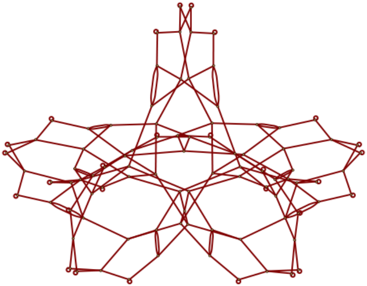


$n = 5, 2 \times$ 3-prong

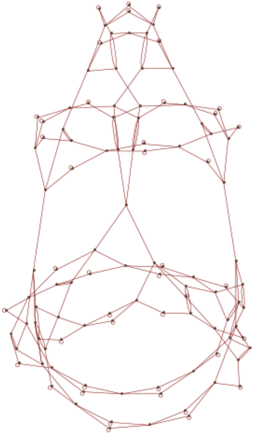


$n = 7, 2 \times$ 4-prong

Or pretty. . .

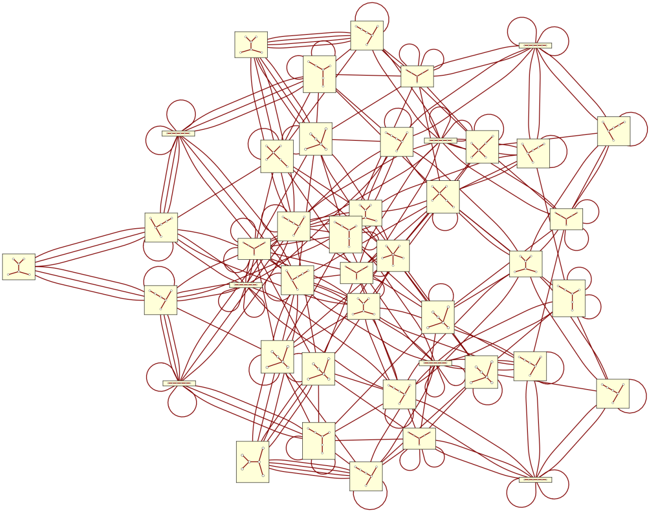


$n = 7, 4 \times 3$-prong
"The maple leaf"



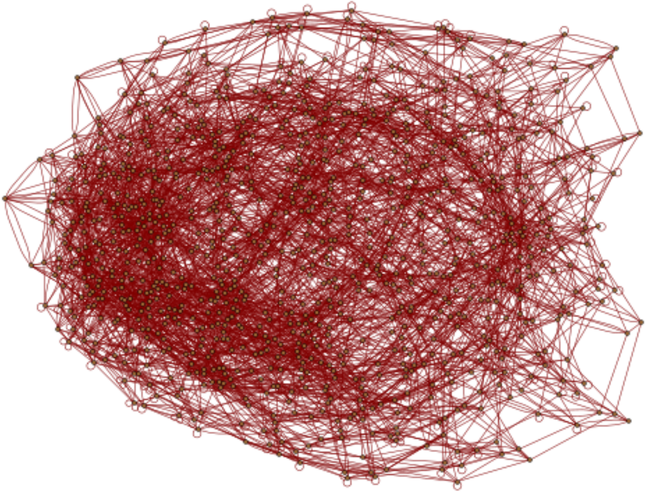
$n = 7, 2\times 3$-prongs, 1×4-prong
"The scarab"

Or rather large. . .



$n = 6$

Or just ridiculous. . .



$n = 7, 2 \times 3$-prongs (977 train tracks!)

The Minimiser problem

- On a given surface, which pA has the least λ?
- Known for $n = 3, 4, 5, 7$ [Song et al. (2002); Ham & Song (2007); Lanneau & Thiffeault (2009a,b)]
- Method: look at all closed paths until column or row norm exceeded.
- Combinatorics explode: on a computer,
  - $n = 3$: trivial;
  - $n = 4$: milliseconds;
  - $n = 5$: seconds;
  - $n = 7$: about 9 months (just finished!);
  - $n = 6$: decades??
- Minimiser is simple for n odd! New ideas are needed...
- Maximiser (completely diffferent question...)

Orientable minimiser

- No punctures: surface of genus g;
- If the foliation is orientable (vector field), then things are much simpler;
- Action of the pA on first homology captures dilatation λ;
- Polynomials of degree $2g$;
- Procedure:
  - We have a guess for the minimiser;
  - Find all integer-coefficient, reciprocal polynomials that could have smaller largest root;
  - Show that they can't correspond to pAs;
  - For the smallest one that can, construct pA.

Newton's formulas

We need an efficient way to bound the number of polynomials with largest root smaller than λ. Given a reciprocal polynomial

$$P(x) = x^{2g} + a_1\, x^{2g-1} + \ldots + a_2\, x^2 + a_1\, x + 1$$

we have Newton's formulas for the traces,

$$\mathrm{Tr}(\phi_*^k) = -\sum_{m=1}^{k-1} a_m \mathrm{Tr}(\phi_*^{k-m}) - k a_k,$$

where

- ϕ is a (hypothetical) pA associated with $P(x)$;
- ϕ_* is the matrix giving the action of the pA ϕ on first homology;
- $\mathrm{Tr}(\phi_*)$ is its trace.

Bounding the traces

The trace satisfies

$$|\mathrm{Tr}(\phi_*^k)| = \left|\sum_{m=1}^{g}(\lambda_m^k + \lambda_m^{-k})\right| \leq g(r^k + r^{-k})$$

where λ_m are the roots of ϕ_*, and $r = \max_m(|\lambda_m|)$.

- Bound $\mathrm{Tr}(\phi_*^k)$ with $r < \lambda$, $k = 1, \ldots, g$;
- Use these g traces and Newton's formulas to construct candidate $P(x)$;
- Overwhelming majority have fractional coeffs $\rightarrow$ discard!
- Carefully check the remaining polynomials:
  - Is their largest root real?
  - Is it strictly greater than all the other roots?
  - Is it really less than λ?
- Largest tractable case: $g = 8$ (10^{12} polynomials).

Lefschetz's fixed point theorem

This procedure still leaves a fair number of polynomials — though not enormous (10's to 100's, even for $g = 8$.)
The next step involves using Lefschetz's fixed point theorem to eliminate more polynomials:

$$L(\phi) = 2 - \mathrm{Tr}(\phi_*) = \sum_{p \in \mathrm{Fix}(\phi)} \mathrm{Ind}(\phi, p)$$

where

- $L(\phi)$ is the Lefschetz number;
- $\mathrm{Fix}(\phi)$ is set of fixed points of ϕ;
- $\mathrm{Ind}(\phi, p)$ is index of ϕ at p.

We can easily compute $L(\phi^k)$ for every iterate using Newton's formula.

Eliminating polynomials

Outline of procedure: for a surface of genus g,

- Use the Euler–Poincaré formula to list possible singularity data for the foliations;
- For each singularity data, compute possible contributions to the index (depending on how the singularities and their separatrices are permuted);
- Check if index is consistent with Lefschetz's theorem.

With this, we can reduce the number of polynomials to one or two!

Minimisers for orientable foliations

g	polynomial	minimiser
2	$X^4 - X^3 - X^2 - X + 1$	$\simeq 1.72208$ †
3	$X^6 - X^4 - X^3 - X^2 + 1$	$\simeq 1.40127$
4	$X^8 - X^5 - X^4 - X^3 + 1$	$\simeq 1.28064$
5	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	$\simeq 1.17628$ $*$
6	$X^{12} - X^7 - X^6 - X^5 + 1$	$\gtrsim 1.17628$
7	$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$	$\gtrsim 1.11548$
8	$X^{16} - X^9 - X^8 - X^7 + 1$	$\gtrsim 1.12876$

- † Zhirov (1995)'s result; also for nonorientable [Lanneau–T];
- $*$ Lehmer's number; realised by Leininger (2004)'s pA;
- For genus 6 to 8 we have not explicitly constructed the pA;
- Genus 6 is the first nondecreasing case.

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs.
- Proof of minimiser on disc for $n = 7$ and surface up to genus 8 (orientable case)
- Maximiser? (Some results — the silver mixer)

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