Final Defense

Classification, Casimir Invariants, and Stability of Lie–Poisson Systems

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Overview

- Many physical systems have a Hamiltonian formulation in terms of Lie–Poisson brackets obtained from Lie algebra extensions.
- We classify low-order brackets, thus showing that there are only a small number of independent normal forms. We make use of Lie algebra cohomology to achieve this.
- We also develop methods for finding the Casimir invariants of Lie–Poisson brackets. We introduce the concept of coextension.
- We look at the stability of equilibria of Lie–Poisson systems, using the method of dynamical accessibility, which uses the bracket directly. This is closely related to the energy-Casimir method.

Hamiltonian Formulation

A system of equations has a Hamiltonian formulation if it can be written in the form

$$\dot{\xi}^{\lambda}(\mathbf{x},t) = \{\xi^{\lambda}\,, H\}$$

where H is a Hamiltonian functional, and $\xi(\mathbf{x})$ represents a vector of field variables (vorticity, temperature, ...).

The Poisson bracket $\{\,,\}$ is antisymmetric and satisfies the Jacobi identity,

$$\{F\,,\{G\,,H\}\} + \{G\,,\{H\,,F\}\} + \{H\,,\{F\,,G\}\} = 0.$$

This tells us that there exist local canonical coordinates.

The Lie–Poisson Bracket

We define the Lie–Poisson bracket for one field variable as

$$\{F\,,G\} := \int_\Omega \omega(\mathbf{x}',t)\left[\,\frac{\delta F}{\delta\omega(\mathbf{x}',t)}\,,\frac{\delta G}{\delta\omega(\mathbf{x}',t)}\,\right]\,\mathrm{d}^2x'$$

The spatial coordinates are $\mathbf{x} = (x, y)$, and the inner bracket is the 2-D Jacobian,

$$[\,a\,,b\,] = \frac{\partial a}{\partial x}\,\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\,\frac{\partial a}{\partial y}.$$

The 2-D fluid domain is denoted by Ω.

The 2-D Euler Equation

Consider the Hamiltonian

$$H[\omega] = \tfrac{1}{2}\int_\Omega |\nabla\phi(\mathbf{x},t)|^2 \, \mathrm{d}^2x, \qquad \frac{\delta H}{\delta\omega} = -\phi,$$

where ϕ is the streamfunction and $\omega = \nabla^2\phi$ is the vorticity.
Inserting this into the Lie–Poisson bracket, we have

$$\begin{aligned}
\dot\omega(\mathbf{x},t) = \{\omega\,,H\} &= \int_\Omega \omega(\mathbf{x}',t)\left[\frac{\delta\omega(\mathbf{x},t)}{\delta\omega(\mathbf{x}',t)}\,,\frac{\delta H}{\delta\omega(\mathbf{x}',t)}\right]\mathrm{d}^2x' \\
&= \int_\Omega \omega(\mathbf{x}',t)\,[\,\delta(\mathbf{x}-\mathbf{x}')\,,-\phi(\mathbf{x}',t)\,]\;\mathrm{d}^2x' \\
&= \int_\Omega \delta(\mathbf{x}-\mathbf{x}')\,[\,\omega(\mathbf{x}',t)\,,\phi(\mathbf{x}',t)\,]\;\mathrm{d}^2x' \boxed{= [\,\omega(\mathbf{x},t)\,,\phi(\mathbf{x},t)\,]}\,,
\end{aligned}$$

which is Euler's equation for the 2-D ideal fluid.

Lie–Poisson Bracket Extensions

Now, say we wish to describe a physical system consisting of several field variables. The most general linear combination of one-field brackets is

$$\{F\,,G\} = \int_\Omega W_\lambda{}^{\mu\nu}\,\xi^\lambda(\mathbf{x}',t)\left[\frac{\delta F}{\delta\xi^\mu(\mathbf{x}',t)}\,,\frac{\delta G}{\delta\xi^\nu(\mathbf{x}',t)}\right]\mathrm{d}^2x'$$

where repeated indices are summed from 0 to n. The 3-tensor W is constant, and determines the structure of the bracket.

We call this type of bracket an extension of the one-field bracket.

Properties of W

In order for the extension to be a good Poisson bracket, it must satisfy

1. Antisymmetry: Since the inner bracket [,] is already antisymmetric, W must be symmetric in its upper indices:

$$\boxed{W_\lambda{}^{\mu\nu} = W_\lambda{}^{\nu\mu}} \;.$$

2. Jacobi identity: assuming the inner bracket [,] satisfies Jacobi, it is easy to show that W must satisfy

$$\boxed{W_\lambda{}^{\sigma\mu}\, W_\sigma{}^{\tau\nu} = W_\lambda{}^{\sigma\nu}\, W_\sigma{}^{\tau\mu}} \;.$$

If we look at W as a collection of matrices $W^{(\mu)}$, then this means that those matrices commute.

Example: Compressible Reduced MHD

The four-field model derived by Hazeltine et al. (1987) for 2-D compressible reduced MHD (CRMHD) has a Lie–Poisson structure. The model includes compressibility and finite ion Larmor radius effects. The field variables are

ω	vorticity
v	parallel velocity
p	pressure
ψ	magnetic flux

and are functions of (x, y, t).

There is also a constant parameter β_{e} that measures compressibility.

The equations of motion for CRMHD are

$$
\begin{aligned}
\dot{\omega} &= [\,\omega\,,\phi\,] + [\,\psi\,,J\,] + 2\,[\,p\,,x\,] \\
\dot{v} &= [\,v\,,\phi\,] + [\,\psi\,,p\,] + 2\beta_{\mathrm{e}}\,[\,x\,,\psi\,] \\
\dot{p} &= [\,p\,,\phi\,] + \beta_{\mathrm{e}}\,[\,\psi\,,v\,] \\
\dot{\psi} &= [\,\psi\,,\phi\,]\,,
\end{aligned}
$$

where $\omega = \nabla^2\phi$, ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current.

The Hamiltonian functional is just the total energy,

$$
H[\omega, v, p, \psi] = \frac{1}{2}\int_\Omega \left(|\nabla\phi|^2 + v^2 + \frac{(p - 2\beta_{\mathrm{e}}\,x)^2}{\beta_{\mathrm{e}}} + |\nabla\psi|^2 \right)\,\mathrm{d}^2x.
$$

The equations for CRMHD can be obtained by inserting this Hamiltonian into the Lie–Poisson bracket

$$\{F\,,G\} = \int_\Omega \Bigg(\omega \left[\frac{\delta F}{\delta\omega}\,,\frac{\delta G}{\delta\omega}\right] + v\left(\left[\frac{\delta F}{\delta\omega}\,,\frac{\delta G}{\delta v}\right] + \left[\frac{\delta F}{\delta v}\,,\frac{\delta G}{\delta\omega}\right]\right)$$
$$+\,p\left(\left[\frac{\delta F}{\delta\omega}\,,\frac{\delta G}{\delta p}\right] + \left[\frac{\delta F}{\delta p}\,,\frac{\delta G}{\delta\omega}\right]\right) + \psi\left(\left[\frac{\delta F}{\delta\omega}\,,\frac{\delta G}{\delta\psi}\right] + \left[\frac{\delta F}{\delta\psi}\,,\frac{\delta G}{\delta\omega}\right]\right)$$
$$-\,\beta_{\mathrm{e}}\,\psi\left(\left[\frac{\delta F}{\delta p}\,,\frac{\delta G}{\delta v}\right] + \left[\frac{\delta F}{\delta v}\,,\frac{\delta G}{\delta p}\right]\right)\Bigg)\;\mathrm{d}^2x.$$

Comparing this to our definition of the Lie–Poisson bracket, with the identification $\boxed{(\xi^0,\xi^1,\xi^2,\xi^3) = (\omega,v,p,\psi)}$, we can read off the tensor W...

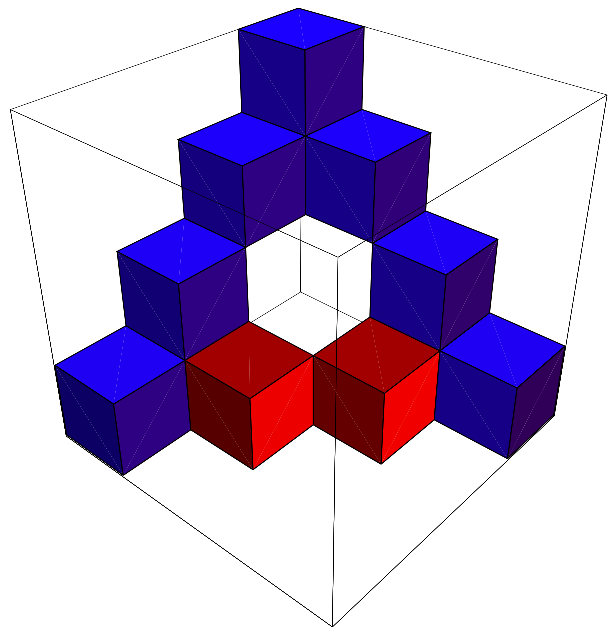
The W tensor for CRMHD

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{\mathrm{e}} & 0 \end{pmatrix},$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\beta_{\mathrm{e}} & 0 & 0 \end{pmatrix}, \qquad W^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that these commute, so that the Jacobi identity holds. (Note the lower-triangular structure.)

Since W is a 3-tensor, we can represent it as a cube:



The vertical axis is the lower index of $W_\lambda{}^{\mu\nu}$, with the origin at the top rear. The two horizontal axes are the symmetric upper indices.

Classification of Brackets

How many independent extensions are there?

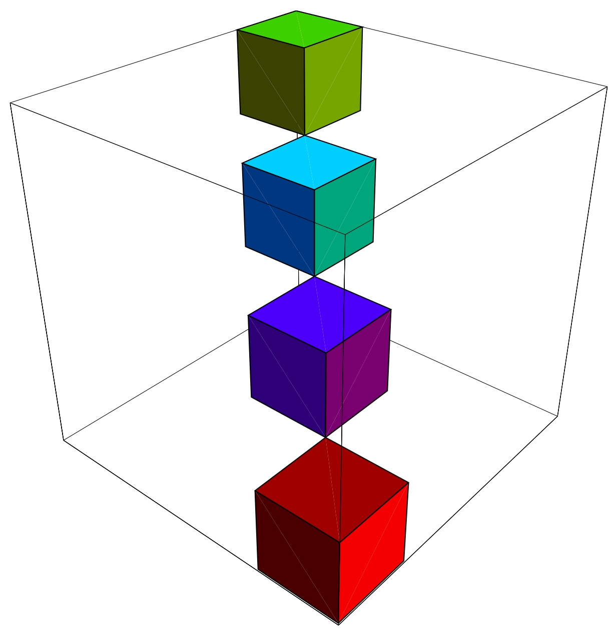
The answer amounts to finding normal forms for W, independent under coordinate transformations.

Threefold process:

1. Decomposition into a direct sum.
2. Transforming the matrices $W^{(\mu)}$ to lower-triangular form.
3. Finally, the hard part is to use Lie algebra cohomology to (almost) achieve the classification.

Classification 1: Direct Sum Structure

A set of commuting matrices, by a coordinate transformation, can always be put in block-diagonal form. Then, the symmetry of the upper indices of W implies the following structure:



Each block corresponds to a degenerate eigenvalue of the $W^{(\mu)}$. We can focus on each block independently.

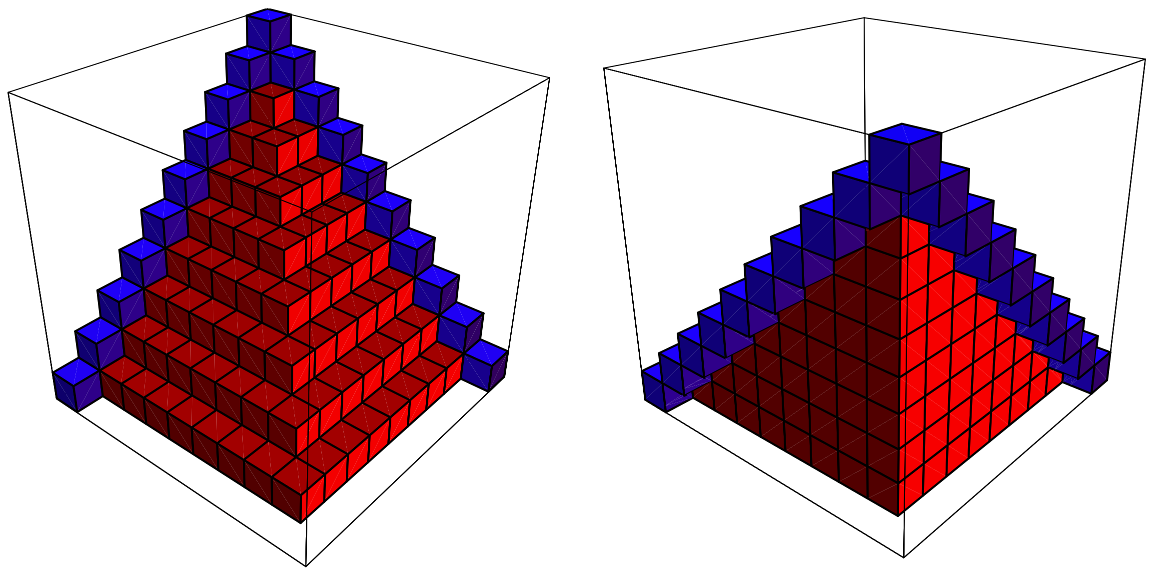
Classification 2: Lower-triangular Form

We focus on a single block, and thus assume that the $W^{(\mu)}$ have $(n+1)$-fold degenerate eigenvalues.

A set of commuting matrices can always be put into lower-triangular form by a coordinate transformation.

Once we do this, by the symmetry of the upper indices of W it is easy to show that only the eigenvalue of $W^{(0)}$ can be nonzero. Furthermore, if it is nonzero it can be rescaled to unity. We assume this is the case.

The most general form of W for an extension is thus



The red cubes form a solvable subalgebra, and are constrained by the commutation requirement. The blue cubes represent unit elements.

Classification 3: Cohomology

The problem of classifying extensions is reduced to classifying the solvable (red) part of the extension. This is achieved by the techniques of Lie algebra cohomology.

Cohomology gives us a class of linear transformations that preserve the lower-triangular structure of the extensions.

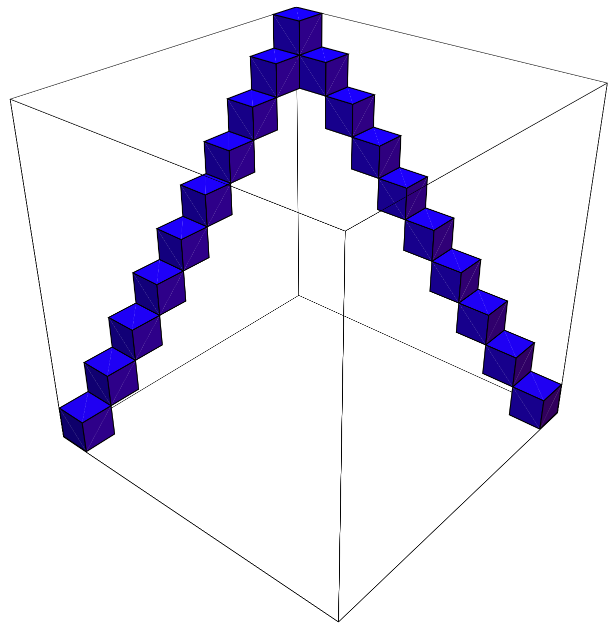
The parts of the extension that can be removed (i.e., made to vanish) by such transformations are called coboundaries.

What is left are nontrivial cocycles.

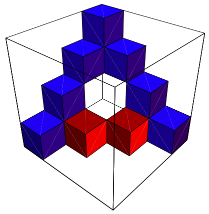
(Cohomology does not quite get it all...)

Pure Semidirect Sum

A common form for the bracket is the semidirect sum (SDS), for which the solvable part of W vanishes:

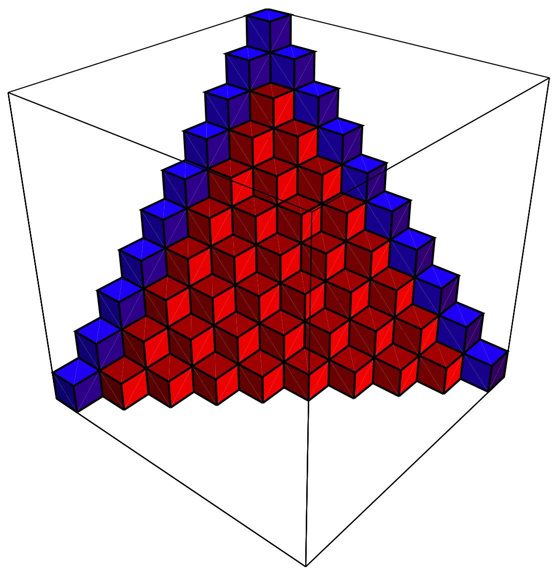


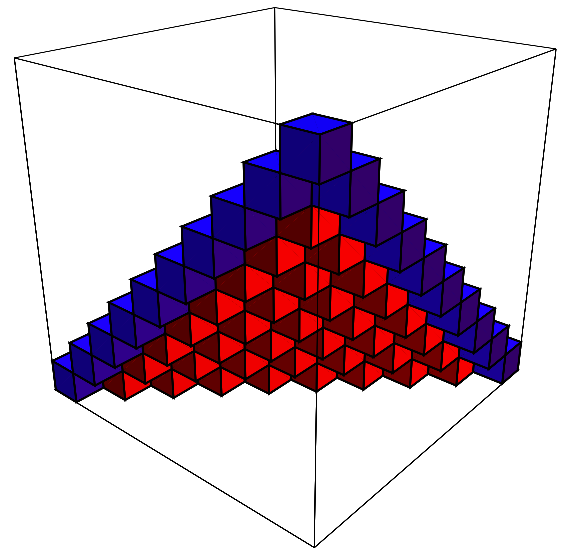
Note that CRMHD does not have a semidirect sum structure because of its extra nonzero blocks, proportional to β_e (a cocycle).



Leibniz Extension

The opposite extreme to the pure semidirect sum is the case for which none of the $W^{(\mu)}$ vanish. Then W must have the structure





This is called the Leibniz extension. All the cubes, red and blue, are equal to unity.

Alternate name: Q*Bert extension. . .

In between these two extreme cases, there are other possible extensions, including the CRMHD bracket.

Order	Number of extensions
1	1
2	1
3	2
4	4
5	9

None of these normal forms contains any free parameter!

(Do not expect this to be true at order 6 and beyond.)

Casimir Invariants

Noncanonical brackets can have Casimir invariants, which are functionals C that commute with every other functional:

$$\{F\,,\,C\} \equiv 0, \quad \text{for all } F.$$

Casimirs are conserved quantities for any Hamiltonian.

For Lie–Poisson brackets, in terms of W, the Casimir condition is

$$W_{\lambda}{}^{\mu\nu}\left[\frac{\delta C}{\delta\xi^{\mu}}\,,\xi^{\lambda}\right] = 0, \quad \nu = 0,\ldots,n.$$

We assume the form

$$C[\xi] = \int_{\Omega} \mathcal{C}(\xi(\mathbf{x}))\,\mathrm{d}^2x. \qquad \left(\frac{\delta C}{\delta\xi} \longrightarrow \frac{\partial\mathcal{C}}{\partial\xi}\right)$$

Assuming the derivation property for $[\ ,\ ]$ and independence of the brackets $\left[\,\xi^{\sigma}\,,\xi^{\lambda}\,\right]$, we can rewrite the Casimir condition as

$$\boxed{W_{\lambda}{}^{\mu\nu}\mathcal{C}_{,\mu\sigma} = W_{,\sigma}{}^{\mu\nu}\mathcal{C}_{,\mu\lambda}}\,, \qquad \lambda, \sigma, \nu = 0, \ldots, n,$$

where $\mathcal{C}_{,\mu} := \partial\mathcal{C}/\partial\xi^{\mu}$.

The key to solving this equation is to take advantage of the lower-triangular structure of the $W^{(\mu)}$, and write

$$g^{\nu\mu}\,\mathcal{C}_{,\mu\sigma} = \widetilde{W}_{\sigma}{}^{\nu\mu}\mathcal{C}_{,\mu n} + \delta^{\nu}{}_{\sigma}\,\mathcal{C}_{,0n}\,,$$

where now the greek indices run from 1 to $n-1$, and

$$g^{\mu\nu} := W_{n}{}^{\mu\nu}$$

is an $n-1$ by $n-1$ symmetric matrix.

If g is nonsingular, with inverse $\bar{g}$, the solution is

$$\mathcal{C}_{,\tau\sigma} = A^{\mu}_{\tau\sigma}\,\mathcal{C}_{,\mu n} + \bar{g}_{\tau\sigma}\,\mathcal{C}_{,0n}\,, \qquad (*)$$

with

$$\boxed{A^{\mu}_{\tau\sigma} := \bar{g}_{\tau\nu}\,\widetilde{W}_{\sigma}{}^{\nu\mu}}\,,$$

where A is the coextension. It satisfies the same properties as W, but with opposite indices:

$$A^{\mu}_{\tau\sigma} = A^{\mu}_{\sigma\tau}\,, \qquad A_{(\tau)}\,A_{(\sigma)} = A_{(\sigma)}\,A_{(\tau)}\,.$$

These conditions are necessary to be able to integrate (*).

(Singular g quite a bit trickier...)

The $n+1$ independent solutions to the differential equation are

$$\mathcal{C}^{\nu}(\xi^0,\xi^1,\ldots,\xi^n) = \sum_{i\geq 0} D^{(i)\nu}_{\tau_1\tau_2\ldots\tau_{(i+1)}} \frac{\xi^{\tau_1}\xi^{\tau_2}\ldots\xi^{\tau_{(i+1)}}}{(i+1)!} f_i^{\nu}(\xi^n),$$

where f is arbitrary, f_i is the ith derivative of f, and

$$\begin{aligned} D^{(0)\nu}_{\tau} &:= \delta_{\tau}{}^{\nu} \ , \\ D^{(1)\nu}_{\tau_1\tau_2} &:= A^{\nu}_{\tau_1\tau_2} \ , \\ D^{(2)\nu}_{\tau_1\tau_2\tau_3} &:= A^{\mu_1}_{\tau_1\tau_2}\, A^{\nu}_{\mu_1\tau_3} \ , \\ &\vdots \\ D^{(i)\nu}_{\tau_1\tau_2\ldots\tau_{(i+1)}} &:= A^{\mu_1}_{\tau_1\tau_2}\, A^{\mu_2}_{\mu_1\tau_3} \cdots A^{\mu_{(i-1)}}_{\mu_{(i-2)}\tau_i}\, A^{\nu}_{\mu_{(i-1)}\tau_{(i+1)}} \ . \end{aligned}$$

The properties of the coextension imply that the $D^{(i)}$ are symmetric in all their lower indices.

CRMHD Casimirs

For CRMHD ($n = 3$), the Casimirs are

$$C^0 = \int_\Omega \left(\omega\, f^0(\psi) - \frac{1}{\beta_e}\, p\, v\, f^{0\prime}(\psi) \right) \mathrm{d}^2x, \qquad C^2 = \int_\Omega p\, f^2(\psi)\, \mathrm{d}^2x,$$

$$C^1 = \int_\Omega v\, f^1(\psi)\, \mathrm{d}^2x, \qquad C^3 = \int_\Omega f^3(\psi)\, \mathrm{d}^2x.$$

4 arbitrary functions f^0–f^3 of $\xi^3 = \psi$.

C^3 forces the magnetic flux ψ to be tied to the fluid elements, but not so for v and p. (This would be the case for a pure semidirect sum.)

Note that would be hard to derive C^0 from the equations of motion, without the bracket.

Stability

Now that we have developed all this theory for Lie–Poisson brackets, let's put it to use. We determine sufficient conditions for the stability of general systems.

Two methods:

- Energy-Casimir: emphasizes invariants.
- Dynamical Accessibility: uses the bracket directly. Slightly more general. This is our preferred method.

For simplicity, we will contrast CRMHD with the pure semidirect sum.

The Energy-Casimir Method

Requiring that a solution $\xi_{\rm e}$ be a constrained minimum of the Hamiltonian,

$$\delta(H + C)[\xi_{\rm e}] =: \delta F[\xi_{\rm e}] = 0,$$

gives an equilibrium solution. The solutions $\xi_{\rm e}$ is then said to be formally stable if $\delta^2 F[\xi_{\rm e}]$ is definite. This is related to δW energy principles, which extremize the potential energy.

Does not capture equilibria where the bracket is singular.

Dynamical Accessibility

A slightly more general method for establishing formal stability uses dynamically accessible variations (DAV), defined as

$$\delta\xi_{\rm da} := \{\mathcal{G}\,,\xi\} + \tfrac{1}{2}\,\{\mathcal{G}\,,\,\{\mathcal{G}\,,\xi\}\}\,,$$

with $\mathcal{G}$ given in terms of the generating functions χ_μ by

$$\mathcal{G} := \int_\Omega \xi^\mu\,\chi_\mu\,\mathrm{d}^2x.$$

DAV are variations that are constrained to remain on the symplectic leaves of the system. They preserve the Casimirs to second order. Stationary solutions of the Hamiltonian,

$$\delta H_{\rm da}[\xi_{\rm e}] = 0,$$

capture all possible equilibria of the equations of motion.

Energy of DAVs

The energy associated with the variations is

$$\delta^2 H_{\rm da}[\xi_{\rm e}] = \tfrac{1}{2}\int_\Omega \left(\delta\xi^\sigma_{\rm da}\,\frac{\delta^2 H}{\delta\xi^\sigma\,\delta\xi^\tau}\,\delta\xi^\tau_{\rm da} - W_\lambda{}^{\mu\nu}\delta\xi^\lambda_{\rm da}\left[\chi_\mu\,,\frac{\delta H}{\delta\xi^\nu}\right]\right){\rm d}^2x$$

In order to determine conditions for stability, we need to write $\delta^2 H_{\rm da}$ in terms of the $\delta\xi^\lambda_{\rm da}$ only (no explicit χ_μ dependence). In principle, this can always be done.

Positive-definiteness of $\delta^2 H_{\rm da}[\xi_{\rm e}]$ is a sufficient condition for formal stability.

Equilibrium Solutions of Semidirect Sum

An equilibrium $(\omega_{\rm e}, \{\xi_{\rm e}^\mu\})$ of the equations of motion for an SDS satisfies

$$\dot\omega_{\rm e} = \left[\,\delta H/\delta\xi^0\,, \omega_{\rm e}\,\right] + \sum_{\nu=1}^{n}\left[\,\delta H/\delta\xi^\nu\,, \xi_{\rm e}^\nu\,\right] = 0,$$

$$\dot\xi_{\rm e}^\mu = \left[\,\delta H/\delta\xi^0\,, \xi_{\rm e}^\mu\,\right] = 0, \qquad \mu = 1,\dots,n,$$

where we have labeled the 0th variable by ω. We can satisfy the $\dot\xi_{\rm e}^\mu = 0$ equations by letting

$$\frac{\delta H}{\delta\xi^0} = -\Phi(u), \qquad \xi_{\rm e}^\mu = \Xi^\mu(u), \quad \mu = 1,\dots,n,$$

for (so far) arbitrary functions $u(\mathbf{x})$, $\Phi(u)$, and $\Xi^\mu(u)$. The $\dot\omega_{\rm e} = 0$ condition gives a differential equation for u.

CRMHD Equilibria

An equilibrium of the CRMHD equations satisfies

$$\begin{aligned}
\psi_{\mathrm{e}} &= \Psi(u), \\
\phi_{\mathrm{e}} &= \Phi(u), \\
v_{\mathrm{e}} &= \big(k_1(u) + (k_2(u) + 2x)\,\Phi'(u)\big) \,/\, \big(1 - |\Phi'(u)|^2/\beta_{\mathrm{e}}\big)\,, \\
p_{\mathrm{e}} &= \big(k_1(u)\,\Phi'(u) + \beta_{\mathrm{e}}\,(k_2(u) + 2x)\big) \,/\, \big(1 - |\Phi'(u)|^2/\beta_{\mathrm{e}}\big)\,, \\
\omega_{\mathrm{e}}\,\Phi'(u) - J_{\mathrm{e}} &= k_3(u) + v_{\mathrm{e}}\,k_1'(u) + p_{\mathrm{e}}\,k_2'(u) + {\beta_{\mathrm{e}}}^{-1}\,p_{\mathrm{e}}\,v_{\mathrm{e}}\,\Phi''(u),
\end{aligned}$$

with primes defined by $f'(u) = (d\Psi(u)/du)^{-1}\,df(u)/du$, and $u(\mathbf{x})$, $\Psi(u)$, $\Phi(u)$, and the $k_i(u)$ arbitrary functions.

This is very different from the SDS case. In particular, the cocycle allows the equilibrium "advected" quantities v_{e} and p_{e} to depend explicitly on x.

DAVs for Semidirect Sum

The dynamically accessible variations for an SDS are

$$\delta\omega_{\rm da} = [\,\omega\,, \chi_0\,] + \sum_{\nu=1}^{n} [\,\xi^{\nu}\,, \chi_{\nu}\,],$$
$$\delta\xi^{\mu}_{\rm da} = [\,\xi^{\mu}\,, \chi_0\,], \qquad \mu = 1, \ldots, n.$$

Notice how all the $\delta\xi^{\mu}_{\rm da}$ depend only on χ_0: the allowed variations are tied to the fluid elements.

DAVs for CRMHD

The dynamically accessible variations for CRMHD are given by

$$\begin{aligned}
\delta\omega_{\rm da} &= [\,\omega\,,\chi_0\,] + [\,v\,,\chi_1\,] + [\,p\,,\chi_2\,] + [\,\psi\,,\chi_3\,],\\
\delta v_{\rm da} &= [\,v\,,\chi_0\,] - \beta_{\rm e}\,[\,\psi\,,\chi_2\,],\\
\delta p_{\rm da} &= [\,p\,,\chi_0\,] - \beta_{\rm e}\,[\,\psi\,,\chi_1\,],\\
\delta\psi_{\rm da} &= [\,\psi\,,\chi_0\,].
\end{aligned}$$

The DAV for ω is the same as for a semidirect sum.

However, the "advected" quantities v, p, and ψ now have independent variations, which can be specified by χ_2, χ_1, and χ_0, respectively.

CRMHD Stability

The terms that involve gradients in the perturbation energy are

$$\delta^2 H_{\rm da} = \int_\Omega \Big(|\nabla \delta\phi_{\rm da} - \nabla(\Phi'(u)\,\delta\psi_{\rm da})|^2 + (1 - |\Phi'(u)|^2)|\nabla\delta\psi_{\rm da}|^2 + \cdots \Big)\, {\rm d}^2 x.$$

These terms must be positive, so we require

$$\boxed{|\Phi'(u)| < 1}\,, \qquad (\,|\nabla\phi_{\rm e}| < |\nabla\psi_{\rm e}|\,),$$

a necessary condition for formal stability.

The remaining terms are a quadratic form in δv_{da}, δp_{da}, and $\delta\psi_{\mathrm{da}}$, which can be written

$$\begin{pmatrix} 1 & -{\beta_{\mathrm{e}}}^{-1}\,\Phi' & -k_1' - {\beta_{\mathrm{e}}}^{-1}\,p_{\mathrm{e}}\,\Phi'' \\ -{\beta_{\mathrm{e}}}^{-1}\,\Phi' & {\beta_{\mathrm{e}}}^{-1} & -k_2' - {\beta_{\mathrm{e}}}^{-1}\,v_{\mathrm{e}}\,\Phi'' \\ -k_1' - {\beta_{\mathrm{e}}}^{-1}\,p_{\mathrm{e}}\,\Phi'' & -k_2' - {\beta_{\mathrm{e}}}^{-1}\,v_{\mathrm{e}}\,\Phi'' & \Theta(x,y) \end{pmatrix}$$

where

$$\Theta(x,y) := -k_3'(u) - v_{\mathrm{e}}\,k_1''(u) - p_{\mathrm{e}}\,k_2''(u) \\ + \omega_{\mathrm{e}}\,\Phi''(u) - {\beta_{\mathrm{e}}}^{-1}\,p_{\mathrm{e}}\,v_{\mathrm{e}}\,\Phi'''(u) + \Phi'(u)\,\nabla^2\Phi'(u).$$

For positive-definiteness of this quadratic form, we require the principal minors of this matrix to be positive.

$$\mu_1 = |1| > 0,$$

$$\mu_2 = \begin{vmatrix} 1 & -{\beta_{\rm e}}^{-1}\,\Phi'(u) \\ -{\beta_{\rm e}}^{-1}\,\Phi'(u) & {\beta_{\rm e}}^{-1} \end{vmatrix} = {\beta_{\rm e}}^{-1}\left(1 - \frac{|\Phi'(u)|^2}{\beta_{\rm e}}\right) > 0,$$

The positive-definiteness of μ_2, combined with condition $|\Phi'(u)| < 1$, implies

$$\boxed{|\Phi'(u)|^2 < \min(1, \beta_{\rm e})}$$

which is part of a sufficient condition for stability. Thus the cocycle modifies the stability directly: it always makes the stability condition worse, because $\beta_{\rm e} > 0$.

Finally, if we require that the determinant of the matrix be positive, we have a sufficient condition for formal stability.

Conclusions

- Classified Lie–Poisson bracket extensions, and found that for low orders there are very few independent brackets, with no free parameters.
- Developed techniques for finding Casimir invariants of Lie–Poisson brackets (coextension).
- Can use brackets or Casimirs to obtain general criteria for stability of Lie–Poisson systems.
- Equilibrium solutions for semidirect sum involve advected quantities that are tied to the fluid elements. Cocycles lead to richer equilibria (Destabilizing for CRMHD).