Topological optimization with braids

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The taffy puller

[Photo and movie by M. D. Finn.]

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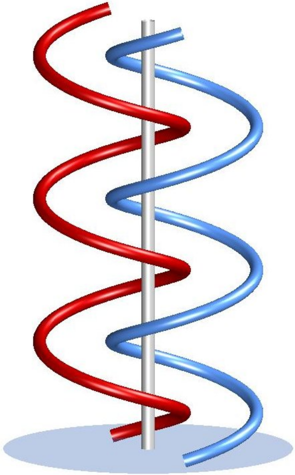
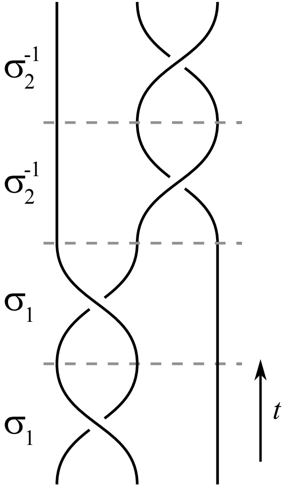
Braid description of taffy puller





The three rods of the taffy puller in a space-time diagram. Defines a braid on $n = 3$ strings, $\sigma_1^2 \sigma_2^{-2}$ (three periods shown on the left).

Topological entropy and pA maps

The effectiveness of a taffy puller is given by how fast it 'stretches' the taffy.

This is where the connection between braids and mapping class groups becomes important.

- The taffy is embedded in an imaginary 'surface,' the disk D_n.
- The rods are punctures, which return to their initial position setwise.
- A mapping class is induced by the rod motion (braid) and stretches the taffy.
- The growth of the taffy is the induced growth on $\pi_1(D_n)$.
- For pseudo-Anosov maps, this is the same as the topological entropy.
- Good taffy pullers should be pseudo-Anosov.

The entropy for 3-braids

For 3-braids, we can use the reduced Burau representation (with $t = -1$) to get the entropy.

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

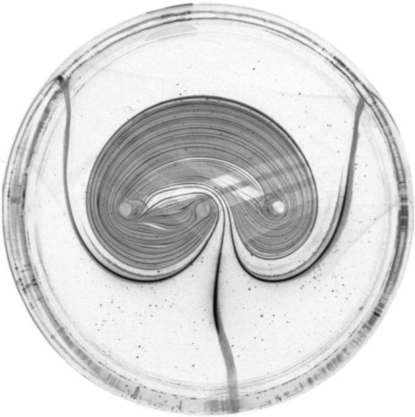
$$[\sigma_1^2 \sigma_2^{-2}] = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

The log of the spectral radius of gives the entropy:

$$h = \log\left(3 + 2\sqrt{2}\,\right) = \log \chi^2 = \log\left(\text{Silver Ratio}\right)^2$$

The Silver Ratio shows up a lot in taffy pullers.

Experiment of Boyland, Aref & Stremler



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[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

The Golden braid

Burau representation:

$$[\sigma_1 \, \sigma_2^{-1}] = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Again the log of the spectral radius of gives the entropy:

$$h = \log\big((3+\sqrt{5}\,)/2\big) = \log \phi^2 = \log\big(\text{Golden Ratio}\big)^2$$

This matrix trick only works for 3-braids, unfortunately.

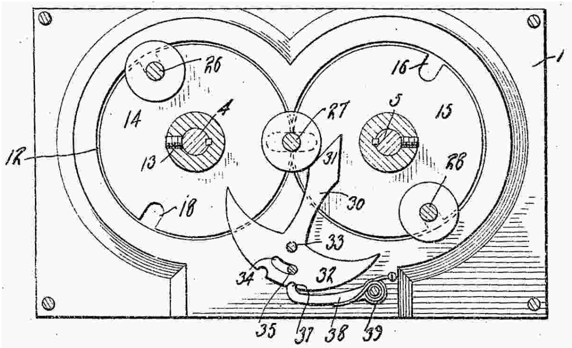
For $n > 3$ the Burau representation gives a lower bound on entropy.

[Fried (1986); Kolev (1989)]

The quest for the Golden Ratio

I used to think that the 'Golden Ratio' device was impractical to build, since each rod moves in a 'Figure-eight.' This is hard to do mechanically.

However, this is before I started searching Google patents. Nitz (1918):



Some examples on D_4

The Burau representation gives the exact entropy for $n = 3$ because

$$D_3 \simeq \text{torus}/\{\text{hyperelliptic involution}\}$$

There is a subclass of mapping classes on D_4 that also descend from the torus.

In fact the most common taffy puller arises in this way.

Four-pronged Silver Ratio taffy puller

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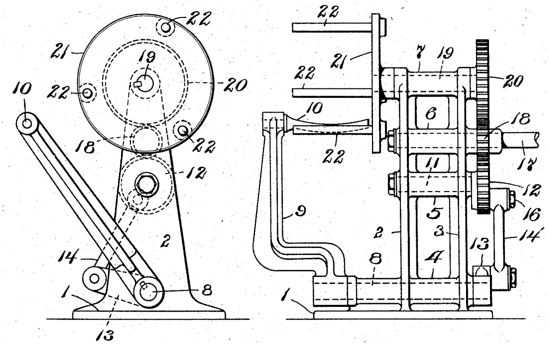
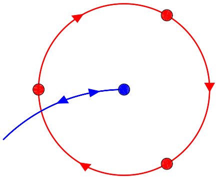
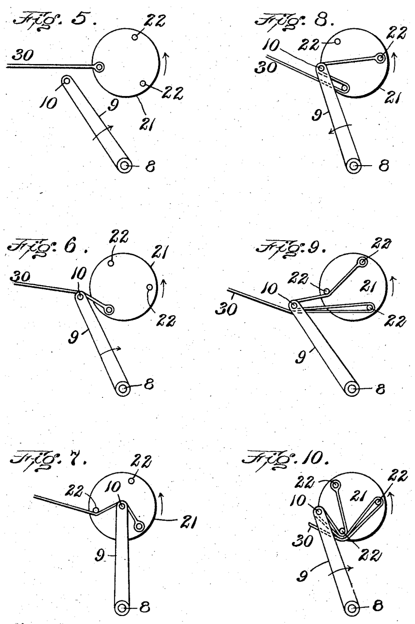
http://www.youtube.com/watch?v=Y7tlHDsquVM

[MacKay (2001); Halbert & Yorke (2014)]

There is actually an earlier 4-rod design by Thibodeau (1904) which has (Golden ratio)^2 growth:



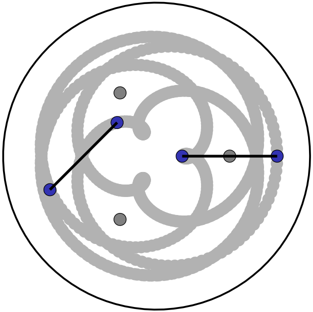




Thibodeau gives very nice diagrams for the action of his taffy puller.

(He has at least 5 patents for taffy pullers.)

Experimental device for kneading bread dough:



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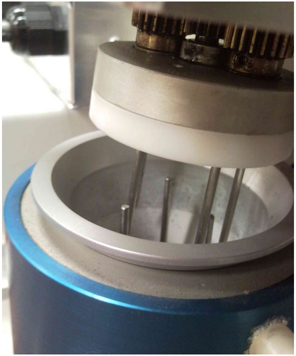


[Department of Food Science, University of Wisconsin. Photos by J-LT.]

The mixograph as a braid

Encode the topological information as a sequence of generators of the Artin braid group B_n.

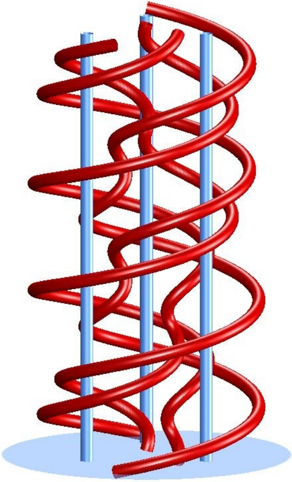
Conjugate to the 7-braid

$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

We feed this braid to the Bestvina–Handel algorithm, which determines the Thurston type of the braid (pseudo-Anosov) and finds the growth as the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$\simeq 4.186$



The Supreme Court vs Mapping Classes

Early in the 20th century the taffy patent wars raged. A central issue was whether a 2-rod device was the same as a 3-rod device. Shockingly, this went all the way to the US Supreme Court, whose opinion was delivered by Chief Justice William Howard Taft (*Hildreth v. Mastoras*, 1921):

> *The machine shown in the Firchau patent [has two pins that] pass each other twice during each revolution [. . .] and move in concentric circles, but do not have the relative in-and-out motion or Figure 8 movement of the Dickinson machine. With only two hooks there could be no lapping of the candy, because there was no third pin to re-engage the candy while it was held between the other two pins. The movement of the two pins in concentric circles might stretch it somewhat and stir it, but it would not pull it in the sense of the art.*

The Supreme Court opinion displays the fundamental insight that at least three rods are required for positive entropy.

Optimizing over generators

So how do we find the 'best' taffy puller or mixing device?

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1 \sigma_2^{-1}$ has entropy $\log \phi^2$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2} \log \phi^2 = \log \phi$.
- Always assume the mapping class is pA.

Optimal braid in B_3

For the braid group with 3 strings (B_3), things are pretty easy since everything is linear algebra.

We use the Burau representation ($t = -1$):

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

The optimization per generator leads exactly to a Joint Spectral Radius problem:

$$\mathrm{JSR}(\mathcal{M}) = \limsup_{k\to\infty} \max_{A_i\in\mathcal{M}} \mathrm{spr}(A_1 A_2 \cdots A_k)^{1/k}$$

$$\mathcal{M} = \left\{[\sigma_1^{\pm 1}], [\sigma_2^{\pm 1}]\right\}$$

JSR problems can be quite difficult, but luckily this is a relatively easy one: $\mathrm{JSR}(\mathcal{M}) = \phi$!

Optimal braid in B_n, $n \geq 4$

For $n \geq 4$, the Burau representation

$$[\sigma_i] = I_{i-2} \oplus \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

provides only a lower bound on entropy.

So we can't expect to get the entropy just by multiplying matrices.

However, we might as well get the largest lower bound by solving the JSR problem for the set

$$\mathcal{M} = \left\{[\sigma_i^{\pm 1}]\right\}_{1 \leq i \leq n-1}$$

and find again $\mathrm{JSR}(\mathcal{M}) = \phi$.

This doesn't solve the problem since it's only a lower bound on the optimal TEPG.

Upper bound

But consider the map that takes a braid word γ to a non-negative matrix:

$$|\gamma| = \left|\sigma_{\mu_1}^{\pm 1} \cdots \sigma_{\mu_k}^{\pm 1}\right| = \left|\sigma_{\mu_1}^{\pm 1}\right| \cdots \left|\sigma_{\mu_k}^{\pm 1}\right|$$

with

$$\left|\sigma_i^{\pm 1}\right| = I_{i-2} \oplus \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

This gives an upper bound on the TEPG:

$$\text{TEPG} \leq \log \text{JSR}\left(\left\{\left|\sigma_i^{\pm 1}\right|\right\}_{1 \leq i \leq n-1}\right) = \log \phi \,.$$

Now we look for braids with TEPG $= \log \phi$.

Summary: TEPG and optimal braids for all n

- In B_3 and B_4, the optimal TEPG is log[Golden Ratio].
- Realized by $\sigma_1\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$, respectively.
- In B_n, $n > 4$, the optimal TEPG is $<$ log[Golden Ratio].
- But can approach optimal TEPG using very long braids.

Of course, this is completely generator-dependent.

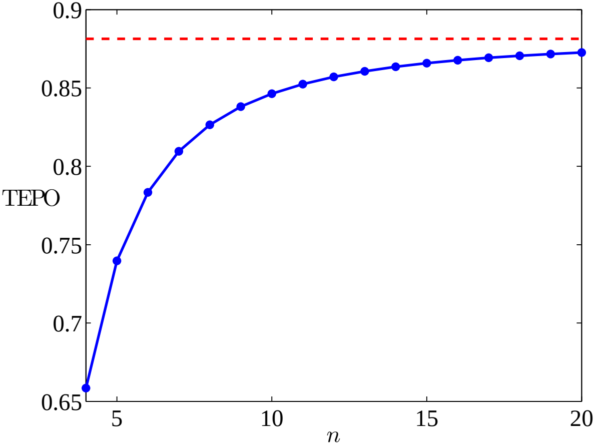
Are there other, somewhat natural, ways of creating a cost function that aligns better with engineering contraints?

Topological Entropy per Operation (TEPO)

- The problem with counting generators is that in an engineering context you don't want to leave rods fixed while you move others.
- Define an 'operation' as a block of pairwise-commuting generators, such as $\sigma_1\sigma_3^{-1}\sigma_5$. These are motions that can be done simultaneously.
- So the braid $(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_2^{-1})$ has cost 2, since it contains two operations.
- σ_1^2 also has cost 2.
- The Topological Entropy per Operation (TEPO) of a braid γ is

$$\mathrm{TEPO}(\gamma) = \frac{h(\gamma)}{\text{min. number of operations in } \gamma}$$

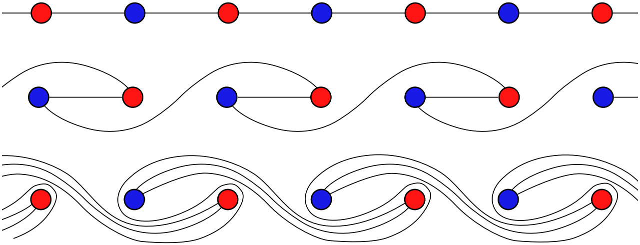
TEPO in B_n



TEPO as a function of n, the number of strings. The asymptote (dashed) is the rigorous upper bound $\log(1+\sqrt{2}) \simeq 0.8814$.

Periodic array of rods

The large-n limit of the TEPO is easy to understand: it is simply an infinite array of punctures undergoing a motion like $\sigma_1\sigma_2^{-1}$:

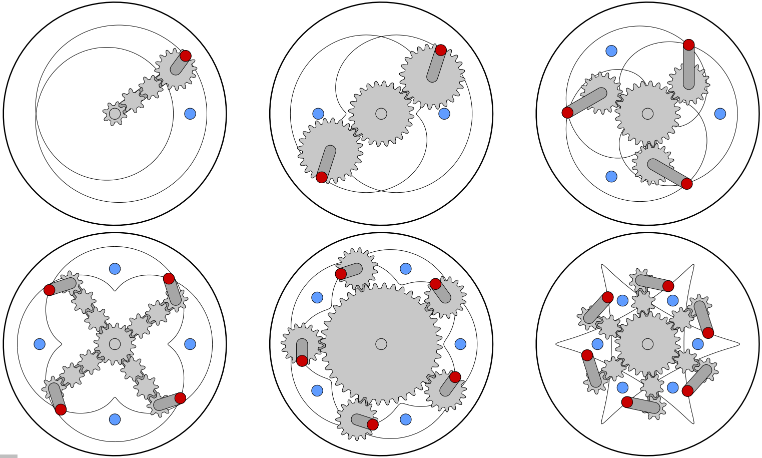


The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!

[Thiffeault & Finn (2006); Finn & Thiffeault (2011)]

Silver mixers

- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.



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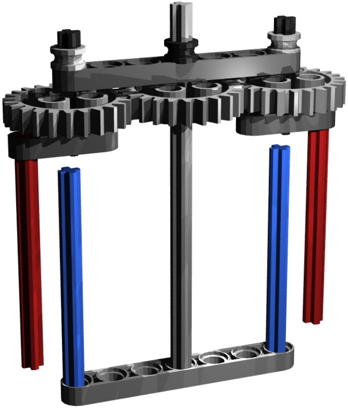
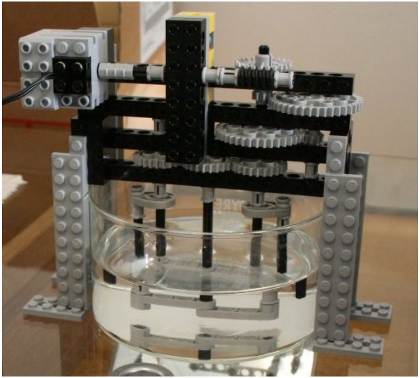
Build it!





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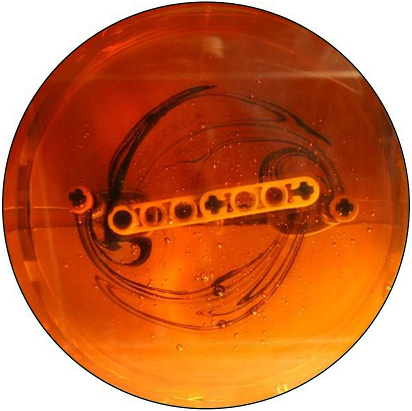
[M. D. Finn and J-LT, *SIAM Review* **53**, 723 (2011)]

Experiment: Silver mixer with four rods



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Conclusions

- Taffy pullers are a great setting for getting intuition about pA maps.
- Inventors 'discovered' a large number of devices in the early 20th century [see 'A Mathematical History of Taffy Pullers']
- Can optimize to find the best rod motions, but depends on choice of 'cost function.'
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!
- See also [Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology,* **11** (4), 2265–2296] for the point-pushing case.
- Are there other relevant optimization problems? Is there something more 'intrinsic'?

References

Bestvina, M. & Handel, M. (1995). *Topology,* **34** (1), 109–140.

Boyland, P. L., Aref, H., & Stremler, M. A. (2000). *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology,* **11** (4), 2265–2296.

D'Alessandro, D., Dahleh, M., & Mezić, I. (1999). *IEEE Transactions on Automatic Control,* **44** (10), 1852–1863.

Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743.

Franks, J. & Rykken, E. (1999). *Proc. Amer. Math. Soc.* **127**, 2183–2192.

Fried, D. (1986). *Topology,* **25** (4), 455–470.

Gouillart, E., Finn, M. D., & Thiffeault, J.-L. (2006). *Phys. Rev. E,* **73**, 036311.

Halbert, J. T. & Yorke, J. A. (2014). *Topology Proceedings,* **44**, 257–284.

Ham, J.-Y. & Song, W. T. (2007). *Experiment. Math.* **16** (2), 167–179.

Kolev, B. (1989). *C. R. Acad. Sci. Sér. I,* **309** (13), 835–838. English translation at `http://arxiv.org/abs/math.DS/0304105`.

MacKay, R. S. (2001). *Phil. Trans. R. Soc. Lond. A,* **359**, 1479–1496.

Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46.

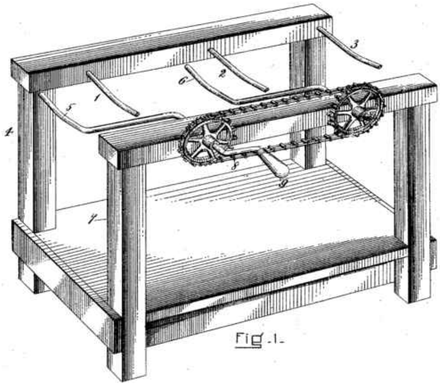
Thiffeault, J.-L. & Finn, M. D. (2006). *Phil. Trans. R. Soc. Lond. A,* **364**, 3251–3266.

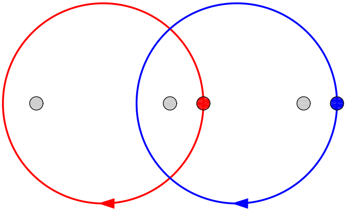
Thiffeault, J.-L., Finn, M. D., Gouillart, E., & Hall, T. (2008). *Chaos,* **18**, 033123.

Thurston, W. P. (1988). *Bull. Am. Math. Soc.* **19**, 417–431.

Taffy puller porn

Jenner (1905)



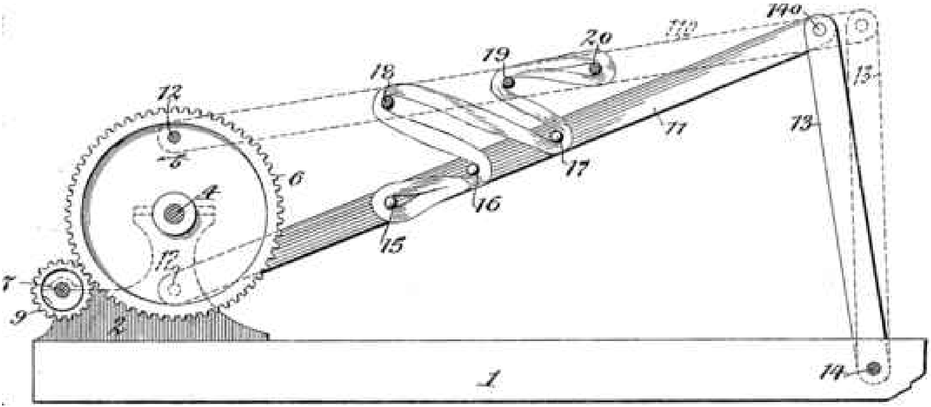


$x^4 - 8x^3 - 2x^2 - 8x + 1$

$\lambda = (\varphi + \sqrt{\varphi})^2$

Taffy puller porn (2)

Shean & Schmeltz (1914)

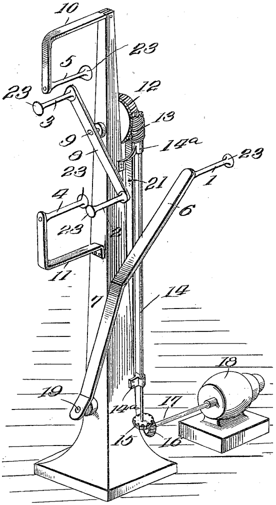


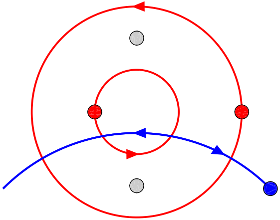
$x^2 - 4x + 1$

$\lambda = 2 + \sqrt{3}$

Taffy puller porn (3)

McCarthy & Wilson (1915)



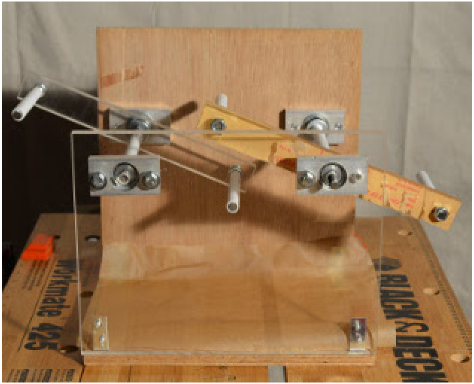


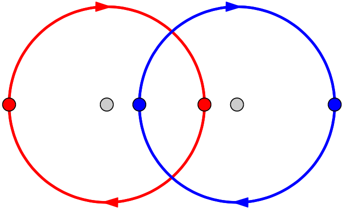
$x^4 - 20x^3 - 26x^2 - 20x + 1$

$\lambda = 21.2667$

Taffy puller porn (4)

Flanagan & J-LT (2015)





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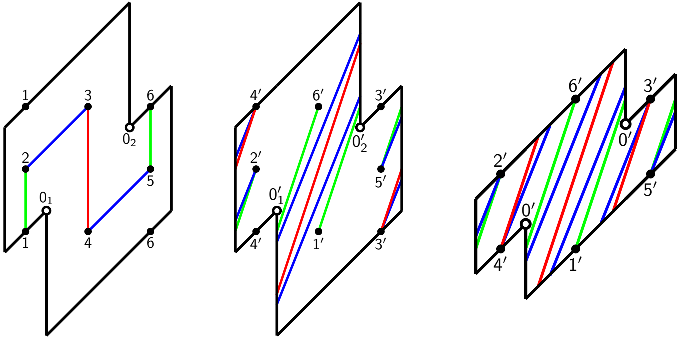
$x^2 - 4x + 1$

$\lambda = 2 + \sqrt{3}$

Nice application of Franks & Rykken (1999)



$$\phi(x) = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix} \cdot x$$