PDE solution of a simple cooking problem

"Masterclass" — University of Edinburgh

Jean-Luc Thiffeault

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Abstract

I will discuss how to tackle the solution of the "cooking by flipping" problem discussed in yesterday's Maths/Higgs Colloquium. Though at its heart this is a simple linear PDE problem for the heat equation, the time dependence is complex enough to cause some serious difficulties. We will see the advantages and drawbacks of the Eulerian (fixed) and Lagrangian (material) viewpoints. We will then reformulate the problem in terms of a series of periodic delta-function impulses, and see if we can find periodic solutions for the temperature distribution. Hopefully this will serve as a nice illustration of the interplay between modeling and approximations involved in typical applied maths problems.

1 The model equation and the steady profile

The dimensionless system from yesterday is

$$T_t = T_{zz}, \qquad 0 < z < 1, \quad t > 0, \tag{1.1a}$$
$$T(z,0) = T_0(z), \qquad 0 < z < 1, \tag{1.1b}$$
$$T_z(0,t) = -h_0\,(1 - T(0,t)), \qquad t > 0, \tag{1.1c}$$
$$T_z(1,t) = -h_1\,T(1,t), \qquad t > 0, \tag{1.1d}$$

We will typically take $T_0(z) \equiv 0$: the food starts at room temperature. In this model there is no flipping of the food just yet: this is just a traditional heat conduction problem.

For long times the solution to Eq. (1.1) will converge to a steady (time-independent) temperature distribution $T(z,t) = S(z)$, which has a linear profile $S(z) = a + bz$. After

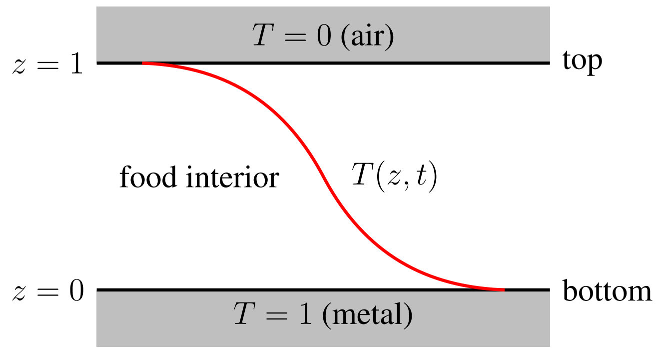

Figure 1: Slab geometry for the food. In general $T(0,t) < 1$ and $T(1,t) > 0$, with equality only for perfect conductors.

applying the boundary conditions Eqs. (1.1c) and (1.1d), we find

$$S(z) = \frac{h_0(1 + h_1 - h_1 z)}{h_0 + h_1 + h_0 h_1} = S(0) - \Delta S\, z, \qquad \Delta S \coloneqq S(0) - S(1), \tag{1.2}$$

with

$$S(0) = \left(1 + \frac{h_1/h_0}{1 + h_1}\right)^{-1} \le 1, \qquad S(1) = \left(1 + h_1 + \frac{h_1}{h_0}\right)^{-1} \le 1. \tag{1.3}$$

For small h_1 and large h_0, we have

$$S(0) \simeq 1 - h_1/h_0, \qquad S(1) \simeq 1 - h_1, \tag{1.4}$$

so both temperatures are near 1, that is, the steady profile is nearly uniform. For $h_0 = \infty$ and $h_1 = 0$, we have $S(z) = 1$.

2 Flipping and Duhamel's principle

Recall from yesterday the *flipping operator*

$$\mathcal{F} f(z) = f(1 - z). \tag{2.1}$$

To incorporate a 'flip' into Eq. (1.1) at a time $t = \Delta t$, we simply restart the problem at $t = \Delta t$, but with an initial condition given by the final $\mathcal{F}T$:

$$T(z, t + \Delta t + \varepsilon) = \mathcal{F}T(z, t + \Delta t - \varepsilon), \qquad \varepsilon \downarrow 0. \tag{2.2}$$

Then, to model multiple flips of equal duration Δt, we repeat this process:

$$T(z, t + k\Delta t + \varepsilon) = \mathcal{F}T(z, t + k\Delta t - \varepsilon), \qquad \varepsilon \downarrow 0, \quad k \in \mathbb{Z}. \tag{2.3}$$

In this manner we can solve the flipping problem in a piecewise manner, using an eigenfunction expansion in each interval, as we saw yesterday. The advantage of using the fixed (material) frame here is that the boundary conditions do not change for each interval. Otherwise we need to also flip the eigenfunctions at each interval, which is not too difficult.

Instead of solving the periodically-flipped system in this piecewise manner, we can instead appeal to *Duhamel's principle*, which states that initial conditions for a PDE can be written as a source on the right-hand side of the PDE, with a δ function in time to mirror the fact the the initial condition only happens impulsively at the start. Thus, a version of Eq. (1.1) that incorporates flipping as

$$T_t - T_{zz} = \sum_{k=1}^{\infty} \delta(t - k\Delta t)(\mathcal{F}T(z,t) - T(z,t)), \qquad 0 < z < 1, \quad t > 0, \tag{2.4a}$$

$$T(z, 0) = T_0(z), \qquad 0 < z < 1, \tag{2.4b}$$

$$T_z(0, t) = -h_0\,(1 - T(0,t)), \qquad t > 0, \tag{2.4c}$$

$$T_z(1, t) = -h_1\,T(1,t), \qquad t > 0, \tag{2.4d}$$

Some intuition behind the forcing term $\mathcal{F}T(z,t) - T(z,t)$ is that if the temperature distribution T is $\mathcal{F}$-invariant, then there is effectively no forcing due to the flipping. This is not a 'typical' system in that the right-hand side source term in Eq. (2.4a) depends on T itself.

It will be useful to define the operator

$$\mathcal{P} = \tfrac{1}{2}(I - \mathcal{F}) \tag{2.5}$$

which projects onto functions odd about $z = \frac{1}{2}$: we have $\mathcal{P}^2 = \mathcal{P}$.

3 Periodic solutions

One approach now for fully solving the initial-value problem Eq. (2.4) is to Laplace-transform the equation. As usual, the main difficulty is in inverting the transform, and we will instead try a more modest approach. Our main goal for the rest of this lecture will be to find period-Δt solutions of Eq. (2.4). We Fourier transform $T(z,t)$ in time over a period:

$$\widetilde{T}_m(z) = \frac{1}{\Delta t}\int_{(k-1/2)\Delta t}^{(k+1/2)\Delta t} T(z,t)\,\mathrm{e}^{-\mathrm{i}\omega_m t}\,\mathrm{d}t, \qquad \omega_m = 2\pi m/\Delta t = m\omega, \tag{3.1}$$

where we chose the periodic interval so it straddles the kth δ-function in Eq. (2.4). We Fourier transform Eq. (2.4) to obtain

$$\mathrm{i}\omega_m \widetilde{T}_m - \widetilde{T}_{mzz} = \mathcal{E}(z), \qquad 0 < z < 1, \tag{3.2a}$$

$$\widetilde{T}_{mz}(0) = -h_0\,(\delta_{m0} - \widetilde{T}_m(0)), \tag{3.2b}$$

$$\widetilde{T}_{mz}(1) = -h_1\,\widetilde{T}_m(1). \tag{3.2c}$$

with

$$\mathcal{E}(z) := \frac{1}{\Delta t}(\mathcal{F}T(z,0) - T(z,0)) = -\frac{2}{\Delta t}\,\mathcal{P}T(z,0). \tag{3.3}$$

For now we will treat $\mathcal{E}(z)$ as a specified source, and ignore the fact that it depends on T.

The $m = 0$ mode satisfies

$$-\widetilde{T}_{0zz} = \mathcal{E}(z), \tag{3.4a}$$

$$\widetilde{T}_{0z}(0) = -h_0\,(1 - \widetilde{T}_0(0)), \quad \widetilde{T}_{0z}(1) = -h_1\,\widetilde{T}_0(1). \tag{3.4b}$$

The solution is then

$$\widetilde{T}_0(z) = S(z) + \int_0^1 G_0(z, z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0 \tag{3.5}$$

where $S(z)$ is given in Eq. (1.2), and the Green's function is defined as

$$G_0(z, z_0) = \frac{(1 + h_0 z_<)\,(1 + h_1(1 - z_>))}{h_0 + h_1 + h_0 h_1} \tag{3.6}$$

with the customary notation

$$z_< = \min(z, z_0), \qquad z_> = \max(z, z_0). \tag{3.7}$$

For $m > 0$, we write the solution of Eq. (3.2) as

$$\widetilde{T}_m(z) = \int_0^1 G_{\gamma_m}(z, z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0, \qquad \gamma_m = \sqrt{\omega_m/2} \tag{3.8}$$

where the Green's function is

$$G_\gamma(z, z_0) = -\frac{1}{2\gamma D(\gamma)}\,(1 + \mathrm{i})\,F_\gamma^{(0)}(z_<)\,F_\gamma^{(1)}(1 - z_>) \tag{3.9}$$

with

$$F_\gamma^{(k)}(z) = h_k \sinh(1 + \mathrm{i})\gamma z + (1 + \mathrm{i})\gamma \cosh(1 + \mathrm{i})\gamma z, \tag{3.10}$$

$$D(\gamma) = (2\gamma^2 - \mathrm{i}h_0 h_1) \sinh(1 + \mathrm{i})\gamma + (1 - \mathrm{i})(h_0 + h_1)\gamma\,\cosh(1 + \mathrm{i})\gamma. \tag{3.11}$$

The inverse of the transform (3.1) is the Fourier sum

$$T(z,t) = \sum_{m=-\infty}^{\infty} \widetilde{T}_m(z)\,\mathrm{e}^{\mathrm{i}\omega_m t}\,. \qquad (3.12)$$

From Eq. (3.3), we can then write

$$\mathcal{E}(z)\,\Delta t = -2\mathcal{P}T(z,0) = -2\sum_{m=-\infty}^{\infty} \mathcal{P}\widetilde{T}_m(z). \qquad (3.13)$$

Now use Eqs. (3.5) and (3.8) in Eq. (3.13):

$$\mathcal{E}(z)\,\Delta t = -2\mathcal{P}_z S(z) - 2\sum_{m=-\infty}^{\infty} \int_0^1 \mathcal{P}_z\, G_{\gamma_m}(z,z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0\,. \qquad (3.14)$$

We added the subscript z to $\mathcal{P}$ to indicate a projection in term of z rather than z_0. We also used the definition $G_{\gamma_m}(z,z_0) = \overline{G}_{\gamma_{-m}}(z,z_0)$ for $m < 0$, with the overbar denoting complex conjugation.

Define the kernel

$$K(z,z_0) = \sum_{m=-\infty}^{\infty} G_{\gamma_m}(z,z_0) = G_0(z,z_0) + 2\Re\sum_{m=1}^{\infty} G_{\gamma_m}(z,z_0). \qquad (3.15)$$

Using Eq. (3.15) in Eq. (3.14), we finally obtain an inhomogeneous Fredholm integral equation of the second kind for $\mathcal{E}(z)$:

$$-\tfrac{1}{2}\mathcal{E}(z)\,\Delta t = \mathcal{P}_z S(z) + \int_0^1 \mathcal{P}_z K(z,z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0. \qquad (3.16)$$

Since $\mathcal{E}(z_0)$ is odd about $z_0 = 1/2$, we can rewrite the integrand as

$$-\tfrac{1}{2}\mathcal{E}(z)\,\Delta t = \mathcal{P}_z S(z) + \int_0^1 \left[\mathcal{P}_z\mathcal{P}_{z_0} K(z,z_0)\right]\mathcal{E}(z_0)\,\mathrm{d}z_0 \qquad (3.17)$$

where the kernel $\mathcal{P}_z\mathcal{P}_{z_0}K(z,z_0)$ is now invariant under interchange of z and z_0:

$$\mathcal{P}_z\mathcal{P}_{z_0}K(z,z_0) = \tfrac{1}{4}\left(K(z,z_0) - K(1-z,z_0) - K(z,1-z_0) + K(1-z,1-z_0)\right).$$

For the symmetric case $h_0 = h_1$, we have $K(z,z_0) = K(1-z,1-z_0)$ and can rewrite Eq. (3.17) as

$$-\tfrac{1}{2}\mathcal{E}(z)\,\Delta t = \mathcal{P}_z S(z) + \int_0^1 K(z,z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0\,, \qquad h_0 = h_1. \qquad (3.18)$$

4 The limit of rapid flips

As $\Delta t \to 0$, $\gamma_m = \sqrt{m\pi/\Delta t}$ becomes large for $m \neq 0$, and we can approximate G_{γ_m} in Eq. (3.9) by

$$G_\gamma(z, z_0) \sim \frac{1}{4\gamma}(1 - \mathrm{i})\, \mathrm{e}^{-(1+\mathrm{i})\gamma|z-z_0|}, \qquad \gamma \to \infty. \tag{4.1}$$

We can easily check that the real part of this converges to a representation of δ'' as $\gamma \to \infty$:

$$2\Re G_\gamma(z, z_0) = -\frac{1}{2\gamma^4}\, \delta''(z - z_0), \qquad \gamma \to \infty. \tag{4.2}$$

This makes the sum in Eq. (3.15) explicit ($\sum_{m=1}^{\infty} m^{-2} = \pi^2/6$):

$$K(z, z_0) \sim G_0(z, z_0) - \tfrac{1}{12}(\Delta t)^2\, \delta''(z - z_0), \qquad \Delta t \to 0. \tag{4.3}$$

Note that the boundary conditions only enter through the first term, and Δt only through the second. Now insert this back into Eq. (3.17):

$$-\tfrac{1}{2}\mathcal{E}(z)\, \Delta t = \mathcal{P}_z S(z) + \int_0^1 \mathcal{P}_z \mathcal{P}_{z_0} \left[G_0(z, z_0) - \tfrac{1}{12}(\Delta t)^2\, \delta''(z - z_0)\right] \mathcal{E}(z_0)\, \mathrm{d}z_0 \tag{4.4}$$

and carry out the integral over δ'':

$$-\tfrac{1}{2}\mathcal{E}(z)\, \Delta t = \mathcal{P}_z S(z) - \tfrac{1}{12}(\Delta t)^2 \mathcal{E}''(z) + \int_0^1 \left[\mathcal{P}_z \mathcal{P}_{z_0} G_0(z, z_0)\right] \mathcal{E}(z_0)\, \mathrm{d}z_0. \tag{4.5}$$

5 Symmetric boundary conditions

For symmetric boundary conditions, the symmetry $G_0(z, z_0) = G_0(1 - z, 1 - z_0)$ and Eq. (3.5) give us

$$\int_0^1 \left[\mathcal{P}_z \mathcal{P}_{z_0} G_0(z, z_0)\right] \mathcal{E}(z_0)\, \mathrm{d}z_0 = \int_0^1 G_0(z, z_0)\, \mathcal{E}(z_0)\, \mathrm{d}z_0 = \widetilde{T}_0(z) - S(z). \tag{5.1}$$

Hence, after setting $\mathcal{E}(z) = -\widetilde{T}_0''(z)$ from Eq. (3.4a), we obtain the ODE

$$\tfrac{1}{12}(\Delta t)^2 \widetilde{T}_0''''(z) - \tfrac{1}{2}\Delta t\, \widetilde{T}_0''(z) + \widetilde{T}_0(z) = \tfrac{1}{2}. \tag{5.2}$$

The boundary layer as $\Delta t \to 0$ is evident. We write

$$\widetilde{T}_0(z) = \tfrac{1}{2} + B(z) \tag{5.3}$$

where the odd function $B = -\mathcal{P}B$ satisfies

$$\tfrac{1}{12}(\Delta t)^2 B''''(z) - \tfrac{1}{2}\,\Delta t\, B''(z) + B(z) = 0. \tag{5.4}$$

The boundary conditions on B are deduced from those on $\widetilde{T}_0$ in Eq. (3.4b) as well as from $B(z)$ being odd about $z = \frac{1}{2}$:

$$B'(0) = h_0(B(0) - \tfrac{1}{2}), \qquad B(\tfrac{1}{2}) = B''(\tfrac{1}{2}) = 0. \tag{5.5}$$

But oops! We're missing one boundary condition for our fourth order equation! We need one more condition:

$$B'''(0) = h_0(B''(0) - 1/\Delta t), \tag{5.6}$$

which comes from examining the integral equation. We show how to derive this in Appendix A.

The natural final step is to rescale $Z = z/\sqrt{\Delta t}$ to blow up the boundary layer:

$$\tfrac{1}{12} b''''(Z) - \tfrac{1}{2}\, b''(Z) + b(Z) = 0 \tag{5.7a}$$

with $b(Z) = B(Z\sqrt{\Delta t})$ and boundary conditions

$$b'(0) = h_0\sqrt{\Delta t}\,(b(0) - \tfrac{1}{2}), \tag{5.7b}$$

$$b'''(0) = h_0\sqrt{\Delta t}\,(b''(0) - 1), \tag{5.7c}$$

$$b(\infty) = 0. \tag{5.7d}$$

For $\Delta t \to 0$, what happens next depends on the size of h_0. If $h_0\sqrt{\Delta t} \to 0$, then we find $b(Z) \equiv 0$! We have lost the boundary layer. However, if we have a perfect conductor such that $h_0\sqrt{\Delta t} = \infty$, we take $b(0) = \frac{1}{2}$, $b''(0) = 1$, and find the solution

$$b(Z) = \tfrac{1}{6}\,\mathrm{e}^{-\rho_+ Z}\left(3\cos\rho_- Z + \sqrt{3}\,\sin\rho_- Z\right) \tag{5.8}$$

where

$$\rho_\pm = \sqrt{\sqrt{3} \pm \tfrac{3}{2}}. \tag{5.9}$$

The general (nonsymmetric) case is more complicated, so we shall not treat it here. Finding the last 'missing boundary condition' from the integral equation appears to be quite challenging in the general case (and I haven't quite sorted it out. . .).

A The missing boundary condition

For the symmetric case, we use the integral equation Eq. (3.18) evaluated at $z = 0$,

$$-\tfrac{1}{2}\Delta t\,\mathcal{E}(0) = \mathcal{P}_z S(0) + \int_0^1 K(0, z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0\,, \tag{A.1}$$

and its derivative also evaluated at $z = 0$:

$$-\tfrac{1}{2}\Delta t\,\mathcal{E}'(0) = [\mathcal{P}_z S]'(0) + \int_0^1 \partial_z K(0, z_0)\,\mathcal{E}(z_0)\,\mathrm{d}z_0. \tag{A.2}$$

Combine these two equations,

$$\begin{aligned}-\tfrac{1}{2}\Delta t\,(\mathcal{E}'(0) - h_0\mathcal{E}(0)) &= [\mathcal{P}_z S]'(0) - h_0\mathcal{P}_z S(0) \\ &\quad + \int_0^1 [\partial_z K(0, z_0) - h_0 K(0, z_0)]\,\mathcal{E}(z_0)\,\mathrm{d}z_0\end{aligned} \tag{A.3}$$

Now the integrand vanishes except for the $m = 0$ term from Eq. (3.15), for which it is equal to $-h_0$:

$$-\tfrac{1}{2}\Delta t\,(\mathcal{E}'(0) - h_0\,\mathcal{E}(0)) = [\mathcal{P}_z S]'(0) - h_0\mathcal{P}_z S(0) - h_0\int_0^1 \mathcal{E}(z_0)\,\mathrm{d}z_0 \tag{A.4}$$

but then that integral vanishes since $\mathcal{E}(z)$ is odd. We are left with

$$\mathcal{E}'(0) - h_0\,\mathcal{E}(0) = h_0/\Delta t\,. \tag{A.5}$$

Putting $\mathcal{E} = -B''$ then gives us Eq. (5.6). For $h_0 \to \infty$, this gives $-\mathcal{E}(0) = 1/\Delta t$. For $\Delta t \to 0$ and finite h_0 it's enough to take $\mathcal{E}'(0) = h_0/\Delta t$ to capture the boundary layer solution.