

# the mathematics of burger flipping

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# cooking by flipping



Consider a nice, tasty **food object** ( $\sim$ meat) on a hot grill.



We will discuss strategies for perfect cooking, with an emphasis on **flipping**.

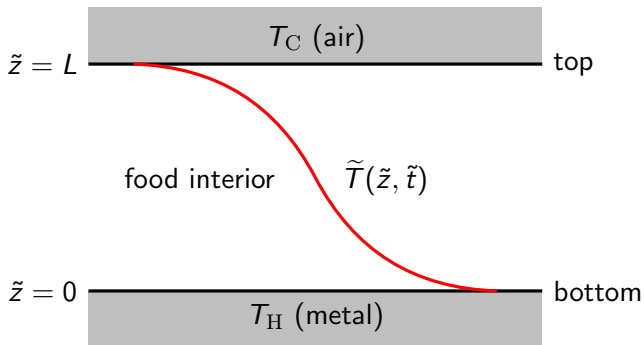


The internet provides a near-infinite number of opinions about the best way to grill meat:

- There seems to be a belief that you should flip *at most* once, to make a better charred surface.
- The Food Lab has a strong contrarian opinion on this:

*Flipping steak repeatedly during cooking can result in a cooking time about **30% faster** than flipping only once. The idea is that with repeated flips, each surface of the meat is exposed to heat relatively evenly, with very little time for it to cool down as it faces upwards. **The faster you flip, the closer your setup comes to approximating a cooking device that would sear the meat from both sides simultaneously.***

[J. Kenji López-Alt, The Food Lab: Flip Your Steaks Multiple Times for Better Results, July 2013]



Plot temperature profiles with  $z$  on the vertical and  $T$  on the horizontal.

We solve the standard heat equation

$$\tilde{T}_{\tilde{t}} = \kappa \tilde{T}_{\tilde{z}\tilde{z}}, \quad 0 < \tilde{z} < L, \quad \tilde{t} > 0,$$

with initial condition

$$\tilde{T}(\tilde{z}, 0) = \tilde{T}_0(\tilde{z}).$$

Newton's law of cooling gives the boundary conditions:

$$-k \tilde{T}_{\tilde{z}}(0, \tilde{t}) = \tilde{h}_0 (T_H - \tilde{T}(0, \tilde{t})), \quad k \tilde{T}_{\tilde{z}}(L, \tilde{t}) = \tilde{h}_1 (T_C - \tilde{T}(L, \tilde{t})),$$

where  $\tilde{h}_0 > 0$  and  $\tilde{h}_1 \geq 0$  are **heat transfer coefficients**.

For  $\tilde{h}_i \rightarrow \infty$  we recover a **perfect conductor**.

For  $\tilde{h}_1 = 0$  we have a **perfect insulator**.

notation	value	d'less	description
$L$	0.01 m	1	food thickness (length scale)
$\kappa$	$1.205 \times 10^{-7} \text{ m}^2/\text{s}$	1	thermal diffusivity of food
$L^2/\kappa$	830 s	1	diffusive time (time scale)
$\Delta T$	175 °C	1	temperature diff. (temp. scale)
$T_H$	200 °C	1	bottom temperature
$T_C$	25 °C	0	top temperature (also initial)
$T_{\text{cook}}$	70 °C	0.257	cooked temperature
$h_0$	900 W/m <sup>2</sup> °C	21.6	bottom heat transfer coeff.
$h_1$	60 W/m <sup>2</sup> °C	1.44	top heat transfer coeff.

<sup>1</sup>Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65



$$\begin{aligned}T_t &= T_{zz}, & 0 < z < 1, & \quad t > 0, \\T(z, 0) &= T_0(z), & 0 < z < 1, & \\T_z(0, t) &= -h_0 (1 - T(0, t)), & t > 0, & \\T_z(1, t) &= -h_1 T(1, t), & t > 0. & \end{aligned}$$

Our model of cooking is very simplified: we assume a point  $z$  is **cooked** if

$$T(z, t) \geq T_{\text{cook}} \quad \text{for any } t.$$

The food is declared cooked **if every  $z$  is cooked**. We ignore burning, drying out, phase changes, . . . .

[For much more complex models, see for example

Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65

Ou, D. & Mittal, G. S. (2007). *J. Food Eng.* **80**, 33–45

Chapwanya, M. & Misra, N. N. (2015). *Appl. Math. Model.* **39** (14), 4033–4043]



Let's solve the **steady problem** first ( $T_t = 0$ ), which has a linear profile:

$$S(z) = \frac{h_0(1 + h_1 - h_1 z)}{h_0 + h_1 + h_0 h_1} = S(0) - \Delta S z, \quad \Delta S := S(0) - S(1),$$

with

$$S(0) = \left(1 + \frac{h_1/h_0}{1 + h_1}\right)^{-1} \leq 1, \quad S(1) = \left(1 + h_1 + \frac{h_1}{h_0}\right)^{-1} \leq 1.$$

For small  $h_1$  and large  $h_0$ , we have

$$S(0) \simeq 1 - h_1/h_0, \quad S(1) \simeq 1 - h_1,$$

so both temperatures are near 1, that is, the steady profile is nearly uniform.





Reformulate as a homogeneous problem for the temperature deviation

$$\theta(z, t) = T(z, t) - S(z) :$$

$$\begin{aligned}\theta_t &= \theta_{zz}, & 0 < z < 1, & \quad t > 0, \\ \theta(z, 0) &= \theta_0(z) = T_0(z) - S(z), & 0 < z < 1, \\ \theta_z(0, t) &= h_0 \theta(0, t), & & \quad t > 0, \\ \theta_z(1, t) &= -h_1 \theta(1, t), & & \quad t > 0.\end{aligned}$$

$T_0(z) \equiv 0$  when the food starts at room temperature.

Write solution as

$$\theta(z, t) = \sum_{m=1}^{\infty} \hat{\theta}_{0m} e^{-\mu_m^2 t} \phi_m(z),$$

with  $L^2$ -normalized eigenfunctions

$$\phi_m(z) = C_m^{-1} (\sin \mu_m z + (\mu_m/h_0) \cos \mu_m z).$$

The eigenvalues  $\mu_m^2$  satisfy the **transcendental relation**

$$(h_0 + h_1)\mu = (\mu^2 - h_0 h_1) \tan \mu$$

which must be solved numerically for  $\mu$ :

$$\mu_1 \approx 2.0803, \mu_2 \approx 4.7865, \mu_3 \approx 7.6966 \dots$$

Approximate by keeping only the **first mode**:

$$T(z, t) \approx S(z) - \widehat{S}_1 e^{-\mu_1^2 t} \phi_1(z).$$

The criterion for the food to have cooked through to  $z = 1$  without needing to flip is

$$S(1) - \widehat{S}_1 e^{-\mu_1^2 t} \phi_1(1) = T_{\text{cook}}$$

where  $S(1)$  (top) is the **coldest point** of the steady solution.

The 'cookthrough time' for the food to cook without needing a flip is

$$t_{\text{cookthrough}} \approx \frac{1}{\mu_1^2} \log \left( \frac{\widehat{S}_1 \phi_1(1)}{S(1) - T_{\text{cook}}} \right), \quad e^{-(\mu_2^2 - \mu_1^2) t_{\text{cookthrough}}} \ll 1.$$

We have  $t_{\text{cookthrough}} \approx 0.340$ , or **283 s** in dimensional terms.

(accurate to 0.07%)



Two mathematically equivalent approaches to deal with the 'flipping' of the food on the hot surface:

- 1 Flip the food (the  $[0, 1]$  domain itself);
- 2 Flip the boundary conditions.

Respectively **Eulerian** (fixed frame) and **Lagrangian** (moving frame) pictures.

There's pros and cons to both choices; **we choose to flip the food**.

**Careful:** Make sure the  $z$  coordinate refers to the same point in the food when determining if it's cooked.

'Flipping operator':

$$\mathcal{F}f(z) = f(1 - z).$$

Self-adjoint with respect to the standard inner product on  $[0, 1]$ .

Define the **flip-heat map**, where we take an initial heat profile  $T(z, t)$ , flip it, then allow it to evolve for a time  $\Delta t$ .<sup>2</sup>

The temperature profile at time  $t + \Delta t$  is

$$T(z, t + \Delta t) = S(z) + \mathcal{H}_{\Delta t}(\mathcal{F}T(z, t) - S(z)).$$

where the heat operator  $\mathcal{H}_{\Delta t}$  evolves a homogeneous profile by a time  $\Delta t$ .

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<sup>2</sup>Since our initial condition  $T(z, 0) \equiv 0$  is  $\mathcal{F}$ -invariant, it is immaterial whether we flip at the beginning or the end of the interval  $[t, t + \Delta t)$ .

We can solve this recurrence relation to obtain at time  $t_k = k\Delta t$

$$\begin{aligned} T(z, t_k) &= \mathcal{K}_{\Delta t}^k T_0(z) + \sum_{j=0}^{k-1} \mathcal{K}_{\Delta t}^j (1 - \mathcal{H}_{\Delta t}) S(z) \\ &= \mathcal{K}_{\Delta t}^k T_0(z) + (1 - \mathcal{K}_{\Delta t})^{-1} (1 - \mathcal{K}_{\Delta t}^k) (1 - \mathcal{H}_{\Delta t}) S(z) \end{aligned}$$

where the **flip-heat operator** is

$$\mathcal{K}_{\Delta t} := \mathcal{H}_{\Delta t} \mathcal{F}.$$

Since  $\|\mathcal{K}_{\Delta t}\| < 1$ , this converges as  $k \rightarrow \infty$  to

$$U_{\Delta t}(z) := (1 - \mathcal{K}_{\Delta t})^{-1} (1 - \mathcal{H}_{\Delta t}) S(z).$$

This is the fixed point for the flip-heat map. The rate of convergence is determined by the modulus of the eigenvalues of  $\mathcal{K}_{\Delta t}$ .



Even this simple description highlights the **two mathematical factors** that will determine cooking time under repeated flipping:

- The **spectrum of the flip-heat operator**  $\mathcal{K}_{\Delta t} = \mathcal{H}_{\Delta t}\mathcal{F}$ .  
Smaller lead eigenvalue  $\implies$  faster approach to equilibrium.
- The **structure of the equilibrium profile**  $U_{\Delta t}(z)$ .  
If the profile itself is warmer, then cooking will be faster.

The eigenvalues of  $\mathcal{H}_{\Delta t}$  are  $e^{-\mu_m^2 \Delta t}$ .

Denote the eigenvalues of  $\mathcal{K}_{\Delta t}$  by  $\sigma_m$ , and define

$$\nu_m^2 = -\frac{1}{\Delta t} \log |\sigma_m|$$

so we can compare the inverse timescale  $\mu_1^2$  to  $\nu_1^2$  (larger is faster).

We find numerically a residual value

$$\lim_{\Delta t \rightarrow 0} \nu_1 \approx 2.685$$

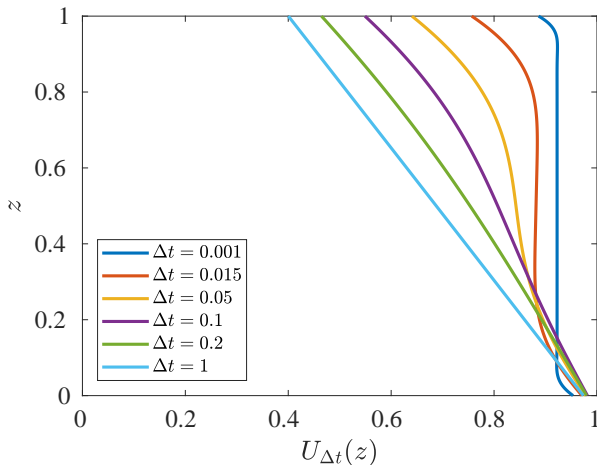
so that

$$\frac{\nu_1^2}{\mu_1^2} \approx \frac{(2.685)^2}{(2.0803)^2} \approx 1.67, \quad \Delta t \rightarrow 0.$$

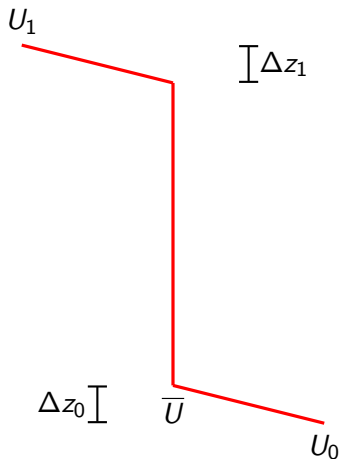
That is, the 'infinitely-fast flipping' limit is **67% faster**. We should regard this as an upper bound (but no proof yet — working on it!).



$$U_{\Delta t}(z) = (1 - \mathcal{K}_{\Delta t})^{-1} (1 - \mathcal{H}_{\Delta t}) S(z)$$



Rapid flipping gives constant interior temperature with boundary layers.



Goal: find  $\bar{U}$ , the limiting interior temperature.



The boundary conditions require

$$\frac{\bar{U} - U_0}{\Delta z_0} \approx -h_0(1 - U_0), \quad \frac{U_1 - \bar{U}}{\Delta z_1} \approx -h_1 U_1.$$

If we want the limit  $\Delta z_i \rightarrow 0$  to exist, we demand

$$\bar{U} - U_0 = O(\Delta z_0) \quad U_1 - \bar{U} = O(\Delta z_1), \quad \Delta z_i \rightarrow 0.$$

Subtracting these, we have

$$\bar{U} = \frac{1}{2}(U_0 + U_1) + O(\Delta z_i).$$



Insert this back into the BCs:

$$-\frac{\Delta U}{\Delta z_0} \approx -h_0(1 - U_0), \quad -\frac{\Delta U}{\Delta z_1} \approx -h_1 U_1.$$

The fluxes  $\Delta U/\Delta z_i$  **must balance**, since otherwise the equilibrium profile would have a net gain or loss of heat during an interval  $\Delta t$ .

This means that  $\Delta z_0 = \Delta z_1$  (the boundary layers have the same thickness), and we find

$$\bar{U} \approx \frac{h_0}{h_0 + h_1}, \quad \Delta t \rightarrow 0.$$

For our parameters this gives  $\bar{U} \approx 0.9375$ , in good agreement with the numerics.

The fact that this is close to 1 (the heating temperature) reflects the possibility of **much faster cooking**.



Things simplify dramatically when the material properties are the same at the top and bottom (e.g., metal press). In that case

$$\mathcal{F}\mathcal{H}_t = \mathcal{H}_t\mathcal{F},$$

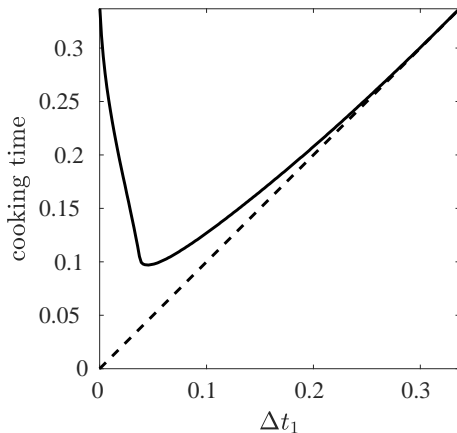
so that  $\mathcal{F}$  and  $\mathcal{H}_t$  share eigenfunctions, and  $\nu_m = \mu_m$ .

This case is fairly inefficient (a good conductor is **bad** on top), but some analytical progress can be made.

Some questions / observations that may eventually become theorems:

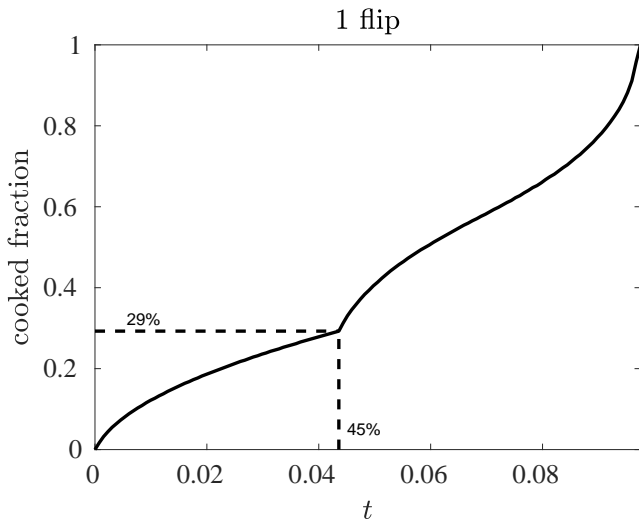
- Is  $z = 1/2$  always the last point to be cooked? **No!**
- If it is, then the total cooking time is **independent of  $\Delta t$  and of how many flips.** **Strange!**

A single flip at  $\Delta t_1$ :



Notice the asymmetry of the minimum: it is better to err on the right.

Minimum  $t_{\text{cook}}$  is **0.0970 (80.5 s)** with a flip at **0.0436 (36 s)**.

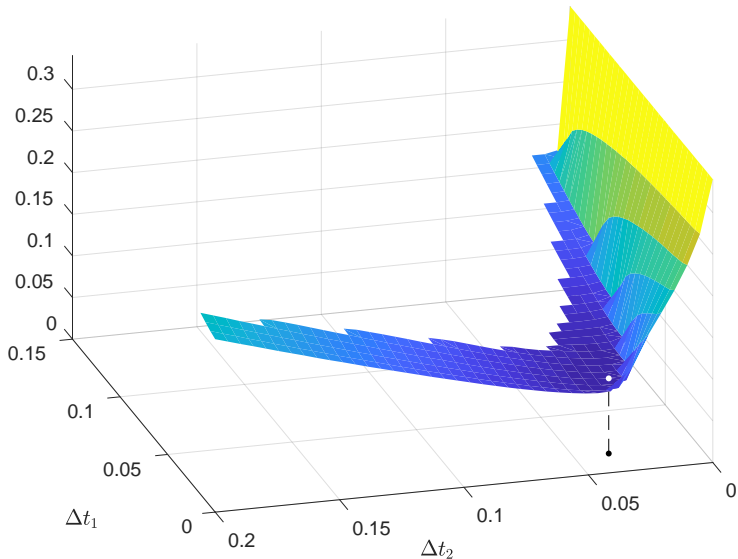


Optimal solutions end in a 'tangency.' Note the longer final phase.

# cooking time for two flips

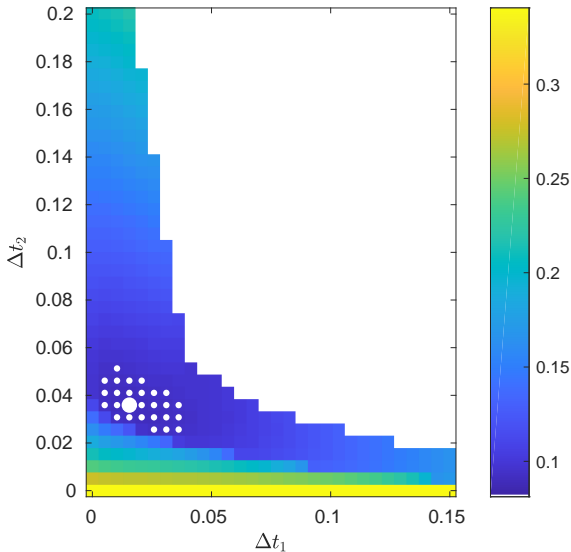


Two flips of duration  $\Delta t_1$  and  $\Delta t_2$ : 8% improvement over 1 flip.



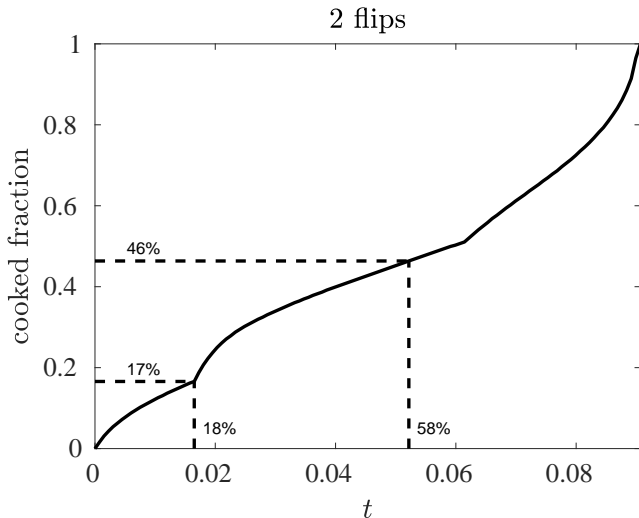


# cooking time for two flips (cont'd)



White dots have cooking times within 5% of min.

# optimal solution for two flips

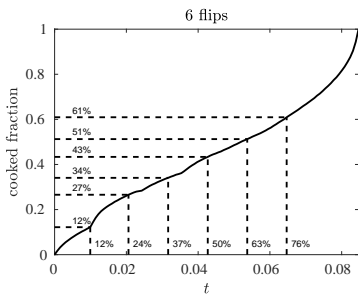
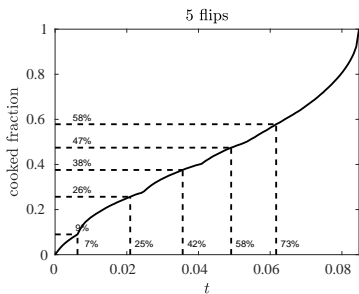
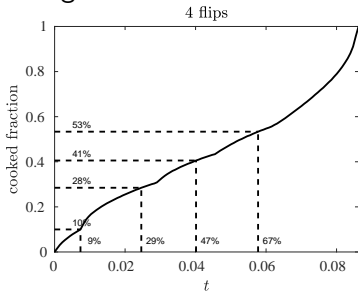
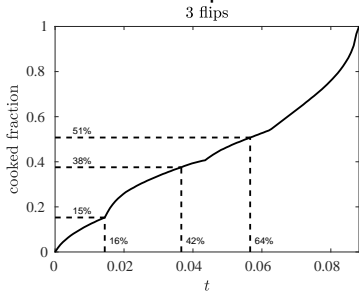


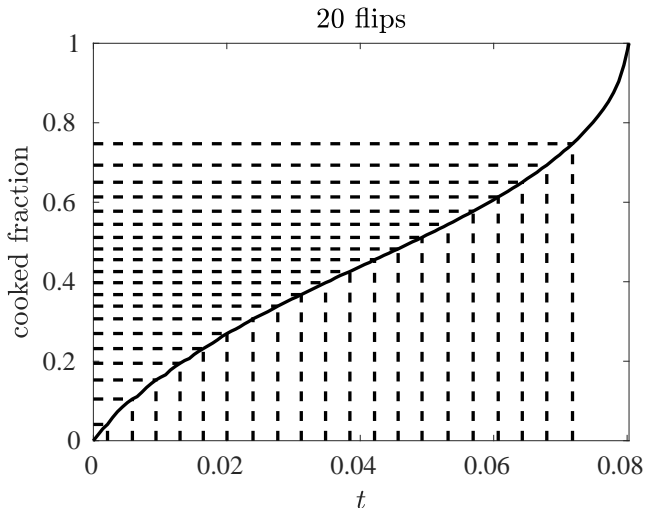
Note the nonsmooth points aren't always at flips!

# optimal solution for more flips



MATLAB's nonlinear optimizer chews through these:

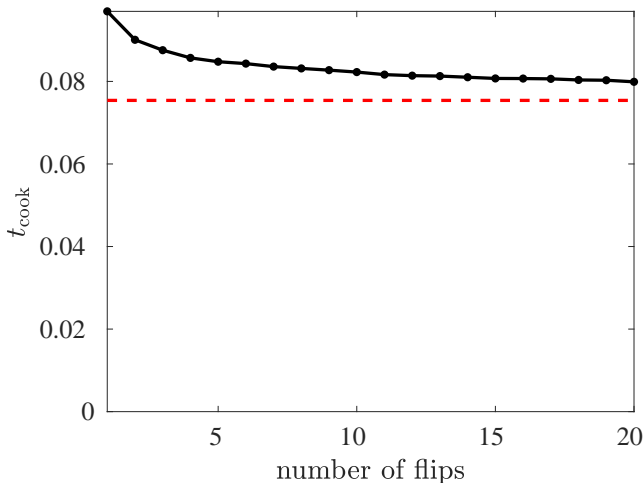




For many flips, the intervals are fairly equal, except the last one.

[Smoothness of curve suggests an effective continuum description.]

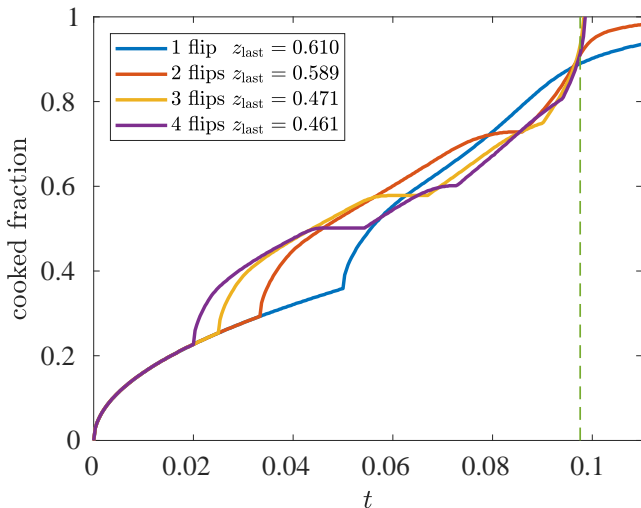
## optimal solution for many flips (cont'd)



The optimal time converges to  $\approx 0.0754$  (63 s), vs 0.0970 (80.5 s) for a single flip.

This is a theoretical **maximum increase of 29%** (same as Food Lab prediction!).

# the strange symmetric case ( $h_0 = h_1 = \infty$ )



The vertical line is the optimal time 0.0976.

None of the curves are optimized (equal  $\Delta t$ ).

For those interested, tomorrow I will discuss a different approach to the whole problem. Instead of regarding the process as a piecewise map, use **Duhamel's principle** to rewrite as a 'kicked' system:

$$T_t - T_{zz} = \sum_{k=1}^{\infty} \delta(t - k\Delta t) (\mathcal{F}T(z, t) - T(z, t)), \quad 0 < z < 1, \quad t > 0.$$

The quantity  $\mathcal{E}(z) \Delta t = \mathcal{F}T(z, t) - T(z, t)$  is the **jump** at each flip. For periodic solutions, we can show that the jump must satisfy

$$-\frac{1}{2} \mathcal{E}(z) \Delta t = \mathcal{P}_z \mathcal{S}(z) + \int_0^1 [\mathcal{P}_z \mathcal{P}_{z_0} K(z, z_0)] \mathcal{E}(z_0) dz_0$$

which is a **Fredholm integral equation** for  $\mathcal{E}(z)$  with a kernel  $K$ . ( $\mathcal{P}_z$  is a projection onto odd modes.)



- Highly-simplified model of cooking allows a fair bit of analytic progress. **Richer than expected.**
- **Many mysteries remain:** for instance, relationship between last cooked point and cooking time for general BCs.
- Also related to work on **heat exchangers.**  
[Marcotte, F., Doering, C. R., Thiffeault, J.-L., & Young, W. R. (2018). *SIAM Journal on Applied Mathematics*, **78** (1), 591–608]
- **Future work:** Add some randomness to the flipping process, to model uncertainties or **ensembles** (the original problem with Diaconis & Holmes).





- Chapwanya, M. & Misra, N. N. (2015). *Appl. Math. Model.* **39** (14), 4033–4043.
- Marcotte, F., Doering, C. R., Thiffeault, J.-L., & Young, W. R. (2018). *SIAM Journal on Applied Mathematics*, **78** (1), 591–608.
- Ou, D. & Mittal, G. S. (2007). *J. Food Eng.* **80**, 33–45.
- Stroshine, R. (1998).
- Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65.