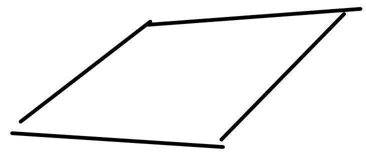
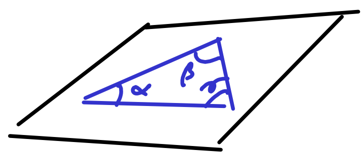
Hyperbolic geometry (Jean-Luc Thiffeault)

A plane is flat:



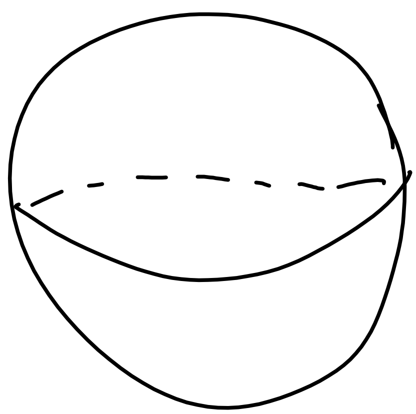
A triangle on a plane has angles that sum to π (or 180°).



$$\alpha+\beta+\gamma=\pi$$

This is a property of Euclidean geometry.

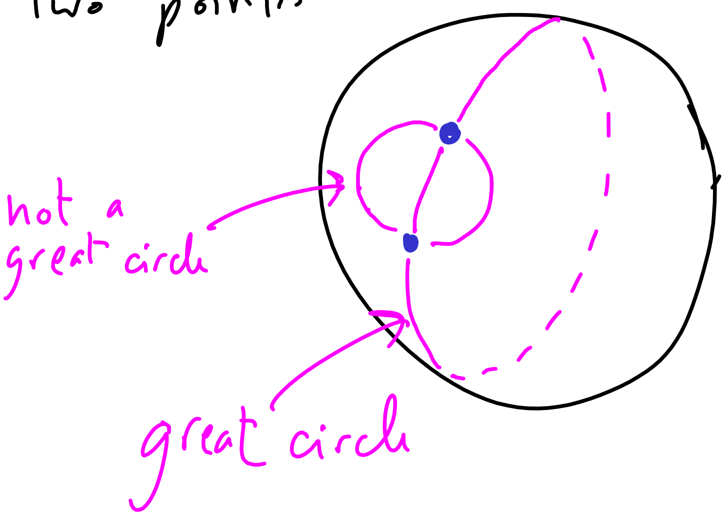
Now let's consider the surface of a sphere:



First, how do we draw a "straight line"?

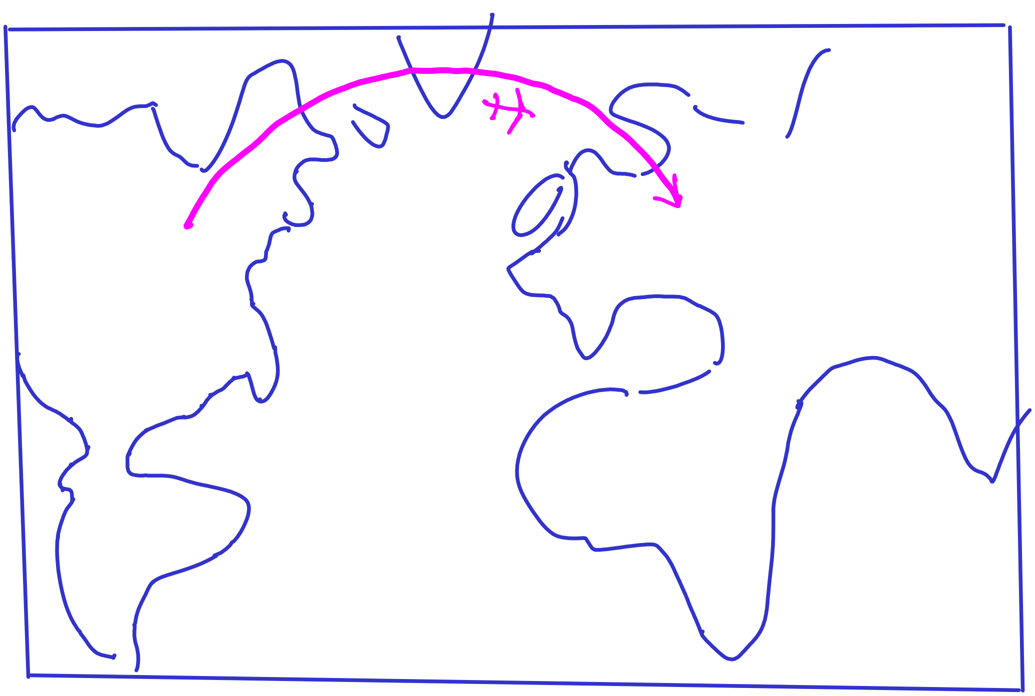
We define it as the shortest distance between points.

On a sphere, the shortest distance is given by the unique* great circle through two points.



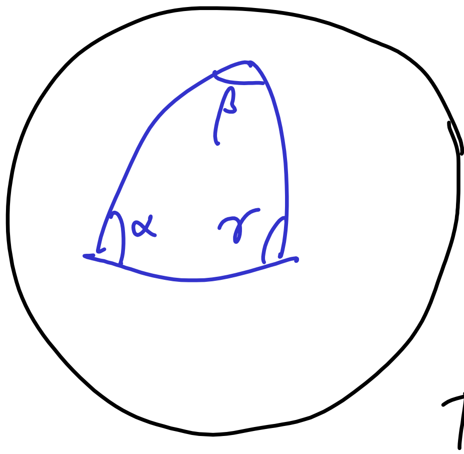
* except for antipodal points

The technical name for such a straight line is a geodesic.



This is why flight routes seem funny when looking at a map: we tend to think of east-west as the "shortest distance", but it's not!

We can draw a triangle made up of geodesics:



For a sphere of radius 1, the triangle will have area.

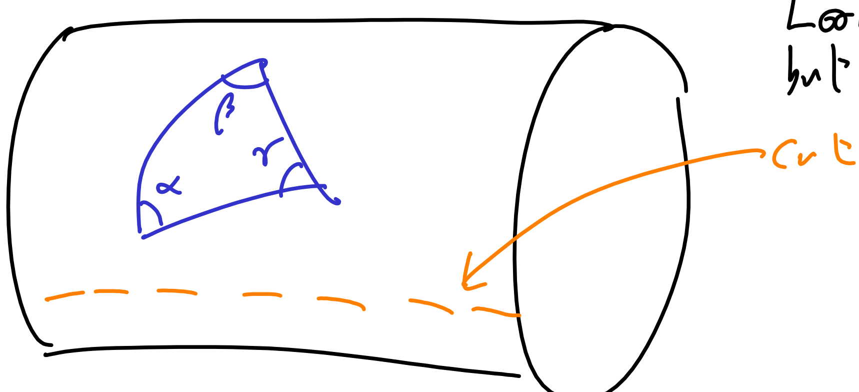
$$A = (\alpha + \beta + \gamma) - \pi$$

$$> 0!$$

The sum of the interior angles is $> \pi$!

This is one feature of non-Euclidean geometry.

What about a cylinder?

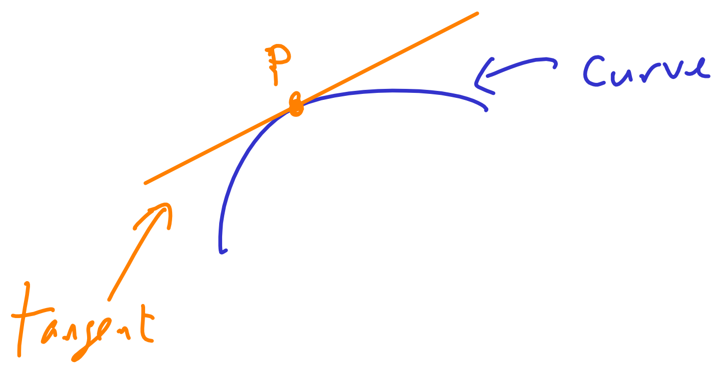


Looks "curved", but if we cut, we get a flat piece of paper, so angles sum to π!

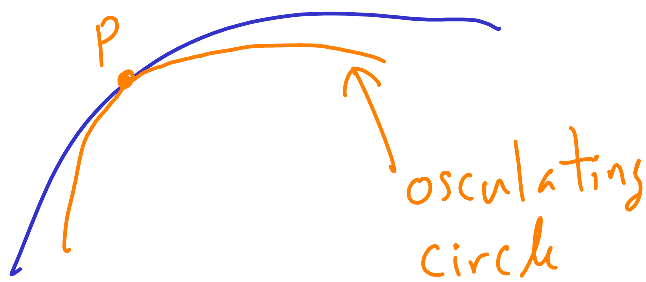
Somehow the cylinder is "not curved"

How to measure curvature?

You might have seen a tangent line to a curve:



A tangent line touches a line at one point: the slope of the line = derivative at that point.



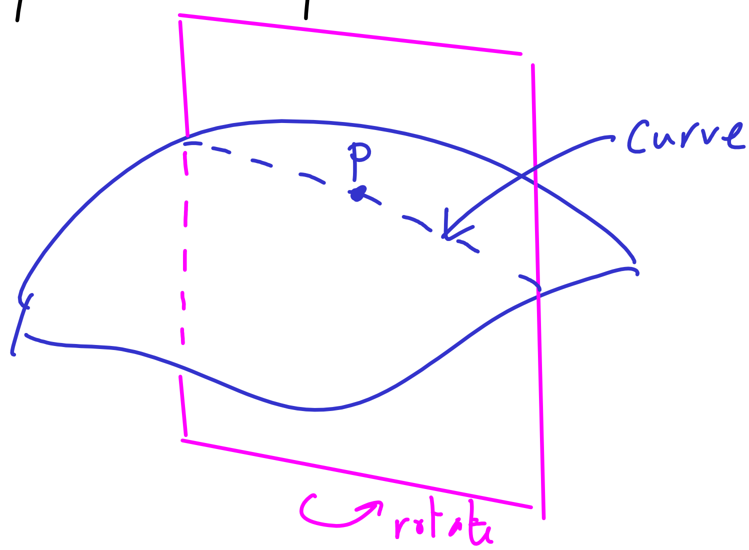
We can also draw a circle which is tangent, but such that its second derivative also agrees with the curve at point P.

This is called the osculating circle at point P.

The circle has a radius R.

$k = \frac{1}{R}$ is called the curvature of the curve at P.

Now for a surface: "cut" along a plane:



We compute the curvature k at P.

As we rotate the plane at P, can show that there will be a maximum and minimum of k:

$$k_1 = \max k \quad , \quad k_2 = \min k$$

Define:

$$K = k_1 k_2$$

Gaussian curvature

One detail: we assign a different sign depending on which side of the surface the osculating circle lies.

$K \neq 0$ means the surface is intrinsically curved

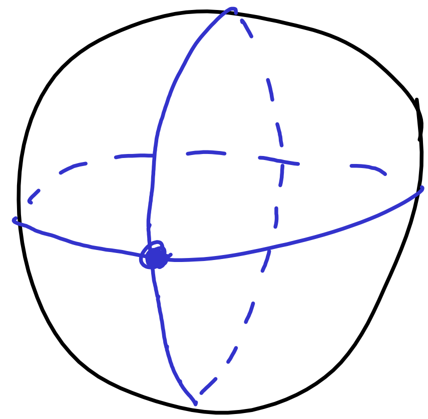
Sphere of radius R has Gaussian curvature

$$K = \frac{1}{R^2}$$

at every point

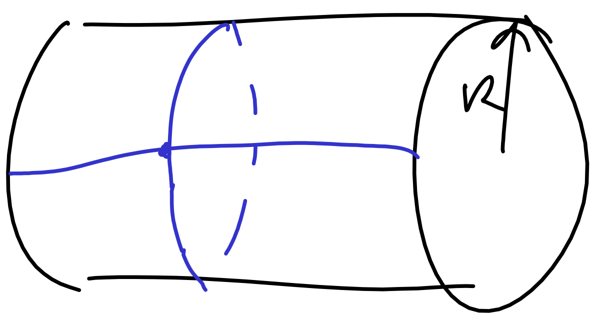
$k_1 = \frac{1}{R}$

$k_2 = \frac{1}{R}$



Cylinder has

$$k_1 = \frac{1}{R}, \quad k_2 = 0$$ ← since $R = \infty$ for a straight line

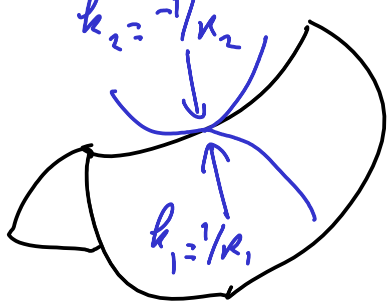


So $K = 0$ everywhere!

Are there surfaces with $K < 0$?

Saddle (Pringle potato chip).

$K < 0$



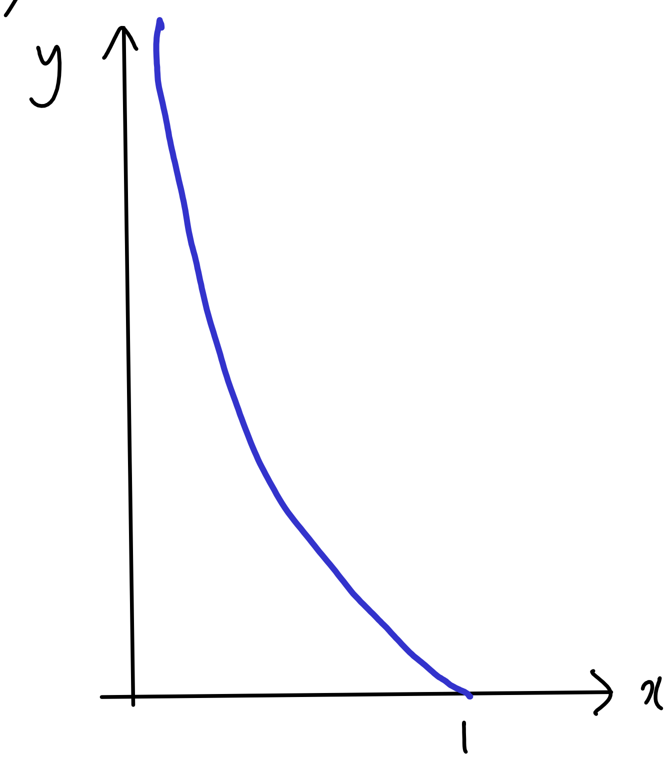
But the curvature is not the same everywhere.

There is a special curve called a tractrix

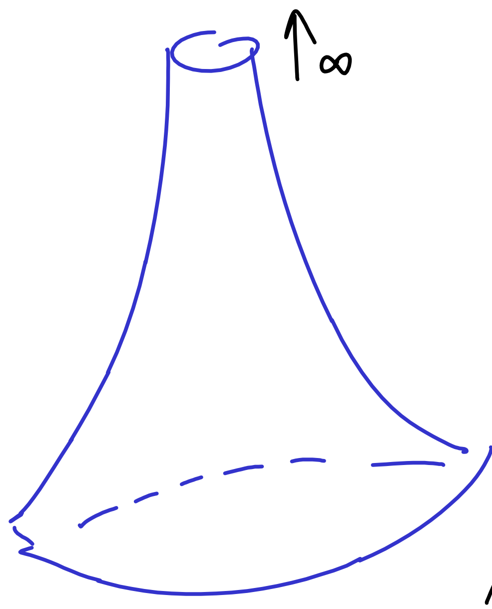
$$y(x) = \operatorname{sech}^{-1} x - \sqrt{1 - x^2}$$

sech = "hyperbolic secant"

(similar to secant, ... but different. Not important for us)



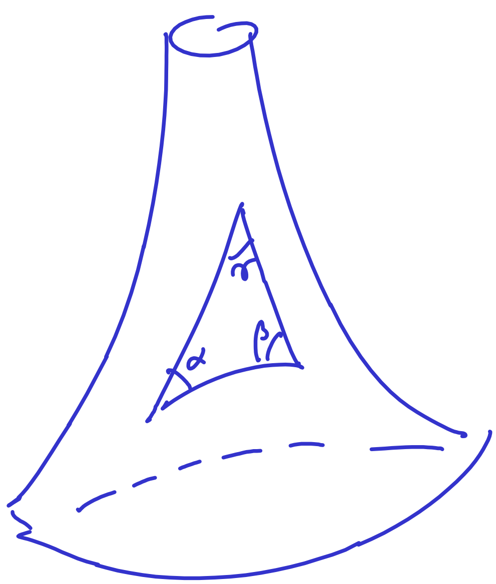
If we spin the tractrix along the y axis we get a surface of revolution.



called a pseudosphere or tratrichoid

This surface has constant negative curvature $K = -1$

A surface with negative curvature everywhere is called hyperbolic

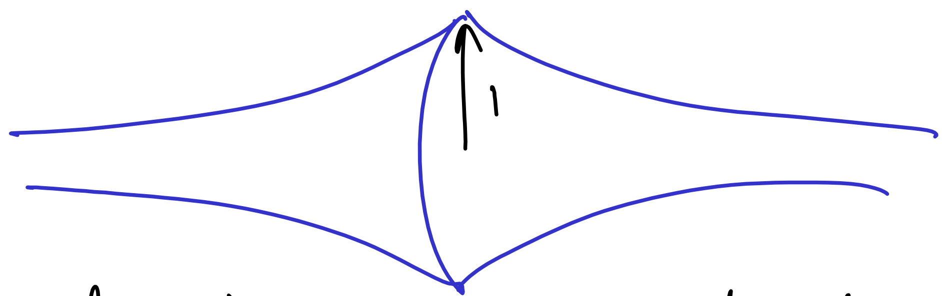


If we draw a triangle with geodesic sides, the angles are less than π:

$$\alpha + \beta + \gamma < \pi$$

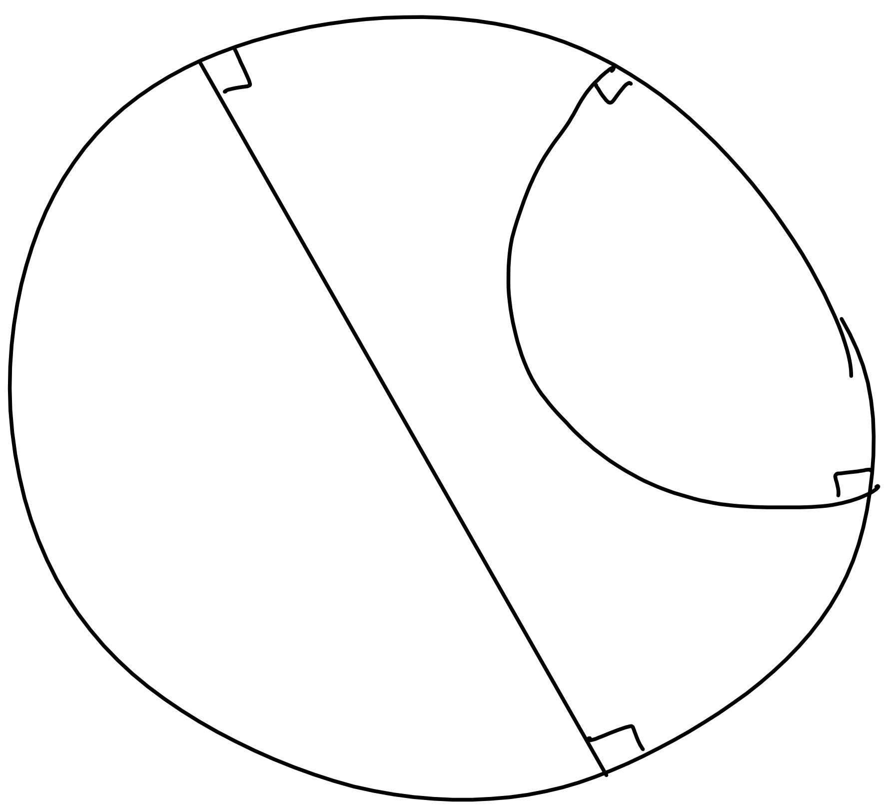
which is a feature of surfaces of negative curvature.

(Why is it called the pseudosphere?

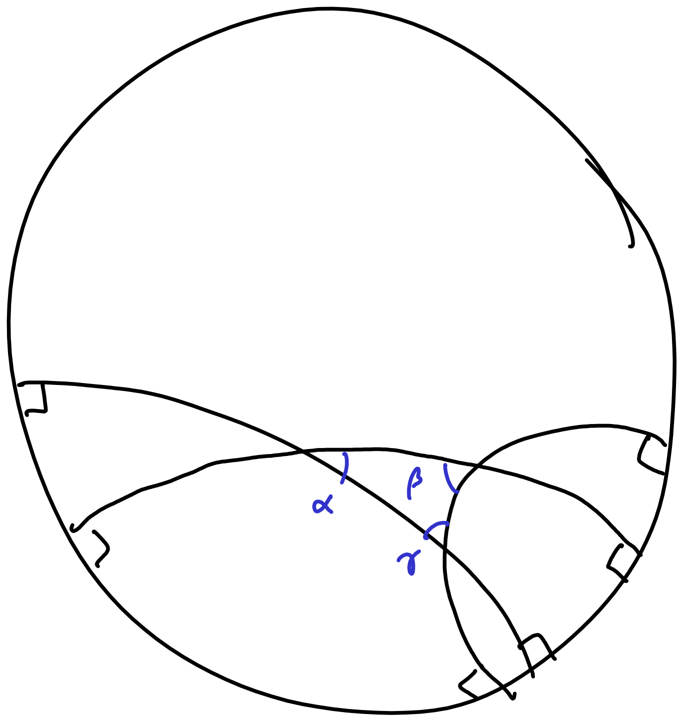


This has the same area and volume as a sphere of radius 1!)

In fact since it is hard to draw triangles on a curved surface, we use the Poincaré disk model:



Straight lines are defined as semicircles perpendicular to boundary!



$\alpha + \beta + \gamma < \pi$

This model turns out to have constant Gaussian curvature $= -1$.

The Poincaré disk has many fascinating properties, such that it can be tiled with any regular polygon (unlike the flat plane)

See famous pictures by M. C. Escher!