The Energy-Casimir Method and MHD Stability

Jean-Luc Thiffeault and P. J. Morrison

Institute for Fusion Studies and Department of Physics

The University of Texas at Austin

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e-mail: jeanluc@physics.utexas.edu

This poster and relevant preprints available at

http://w3fusion.ph.utexas.edu/~jeanluc/

Overview

- Many equations of plasma physics have a Hamiltonian formulation in terms of Lie–Poisson brackets.
- We investigate the structure of these Lie–Poisson brackets. The simplest case is the semidirect sum structure.
- Some systems, such as a model of 2-D compressible reduced MHD, have a more complicated structure involving cocycles.
- We look at the role of cocycles in formal stability. The principle is similar that of δW energy methods, but we determine stability criteria using the concept of dynamical accessibility, which uses the bracket directly. This is closely related to the energy-Casimir method.

Hamiltonian Formulation

A system of equations has a Hamiltonian formulation if it can be written in the form

$$\boxed{\dot{\xi}^\lambda(\mathbf{x},t) = \{\xi^\lambda\,, H\}}$$

where H is a Hamiltonian functional, and $\xi(\mathbf{x})$ represents a vector of field variables (vorticity, temperature, ...).

The Poisson bracket $\{\,,\}$ is antisymmetric and satisfies the Jacobi identity,

$$\{F\,,\,\{G\,,H\}\} + \{G\,,\,\{H\,,F\}\} + \{H\,,\,\{F\,,G\}\} = 0.$$

This tells us that there exist local canonical coordinates.

Lie–Poisson Brackets

A particular type of bracket is the Lie–Poisson bracket,

$$\boxed{\{F\,,G\} = \int_\Omega W_\lambda{}^{\mu\nu}\,\xi^\lambda(\mathbf{x},t)\left[\frac{\delta F}{\delta\xi^\mu(\mathbf{x},t)}\,,\frac{\delta G}{\delta\xi^\nu(\mathbf{x},t)}\right]\mathrm{d}^2x}$$

where repeated indices are summed, and $\mathbf{x} = (x, y)$. The 3-tensor W is constant, and determines the structure of the bracket. The inner bracket is the 2-D Jacobian,

$$[\,a\,,b\,] = \frac{\partial a}{\partial x}\,\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\,\frac{\partial a}{\partial y}.$$

The 2-D fluid domain is denoted by Ω.

Example: Compressible Reduced MHD

The four-field model derived by Hazeltine et al. [7] for 2-D compressible reduced MHD (CRMHD) has a Lie–Poisson structure. The model includes compressibility and finite ion Larmor radius effects. The field variables are

ω	vorticity
v	parallel velocity
p	pressure
ψ	magnetic flux

and are functions of (x, y, t). There is also a constant parameter β_{i} that measures compressibility.

The equations of motion for CRMHD are

$$\begin{aligned}
\dot{\omega} &= [\,\omega\,,\phi\,] + [\,\psi\,,J\,] + 2\,[\,p\,,x\,] \\
\dot{v} &= [\,v\,,\phi\,] + [\,\psi\,,p\,] + 2\beta_{\mathrm{i}}\,[\,x\,,\psi\,] \\
\dot{p} &= [\,p\,,\phi\,] + \beta_{\mathrm{i}}\,[\,\psi\,,v\,] \\
\dot{\psi} &= [\,\psi\,,\phi\,]\,,
\end{aligned} \tag{1}$$

where $\omega = \nabla^2\phi$, ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current.

The Hamiltonian is just the energy,

$$H[\omega, v, p, \psi] = \frac{1}{2}\int_\Omega \left(|\nabla\phi|^2 + v^2 + \frac{(p - 2\beta_{\mathrm{i}}\,x)^2}{\beta_{\mathrm{i}}} + |\nabla\psi|^2\right)\,\mathrm{d}^2x.$$

The equations for CRMHD are obtained by inserting the above Hamiltonian into the Lie–Poisson bracket

$$\{A\,,B\} = \int_\Omega \Bigg(\omega\left[\frac{\delta A}{\delta\omega}\,,\frac{\delta B}{\delta\omega}\right] + v\left(\left[\frac{\delta A}{\delta\omega}\,,\frac{\delta B}{\delta v}\right] + \left[\frac{\delta A}{\delta v}\,,\frac{\delta B}{\delta\omega}\right]\right)$$

$$+\,p\left(\left[\frac{\delta A}{\delta\omega}\,,\frac{\delta B}{\delta p}\right] + \left[\frac{\delta A}{\delta p}\,,\frac{\delta B}{\delta\omega}\right]\right) + \psi\left(\left[\frac{\delta A}{\delta\omega}\,,\frac{\delta B}{\delta\psi}\right] + \left[\frac{\delta A}{\delta\psi}\,,\frac{\delta B}{\delta\omega}\right]\right)$$

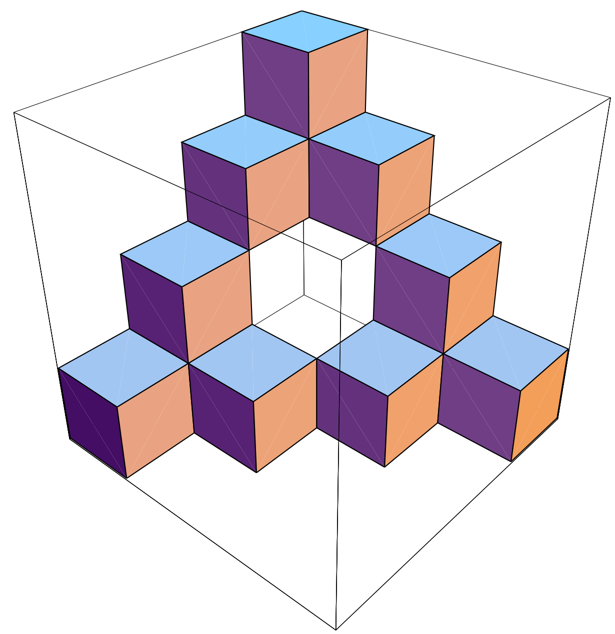
$$-\,\beta_{\rm i}\,\psi\left(\left[\frac{\delta A}{\delta p}\,,\frac{\delta B}{\delta v}\right] + \left[\frac{\delta A}{\delta v}\,,\frac{\delta B}{\delta p}\right]\right)\Bigg)\;{\rm d}^2x.$$

This can be shown to satisfy the Jacobi identity. Comparing this to our definition of the Lie–Poisson bracket with the definition $(\xi^0, \xi^1, \xi^2, \xi^3) = (\omega, v, p, \psi)$, we can read off the tensor W.

The W tensor for CRMHD

Since W is a 3-tensor, we can represent it as a cube:

The small blocks denote nonzero entries. The "shape" of W is constrained by the Jacobi identity. The vertical axis is the lower index of ${W_\lambda}^{\mu\nu}$, with the origin at the top rear. The two horizontal axes are the symmetric upper indices.



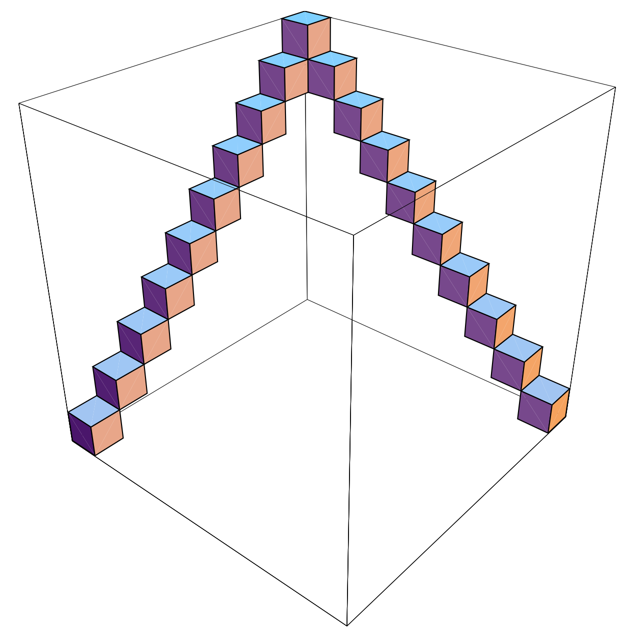
Semidirect Sums

A common form for the bracket is the semidirect sum (SDS), for which W looks like the picture below.

Note that CRMHD does not have a semidirect sum structure because of its extra nonzero blocks. These extra blocks, proportional to $\beta_{\rm i}$, are called cocycles.

Reduced MHD (two fields: ω and ψ) has a semidirect sum structure.

This structure arises in systems with a vorticity-like field variable that *advects* the other quantities of the model.



Casimir Invariants

Noncanonical brackets can have Casimir invariants, which are functionals C which commute with every other functional:

$$\{F\,,\,C\} \equiv 0, \quad \text{for all } F.$$

Casimirs are conserved quantities for any Hamiltonian.

For CRMHD, they are

$$C^0 = \int_\Omega \left(\omega\, f(\psi) - \frac{1}{\beta_{\rm i}}\, p\, v\, f'(\psi)\right) \mathrm{d}^2x, \qquad C^2 = \int_\Omega p\, h(\psi)\, \mathrm{d}^2x,$$

$$C^1 = \int_\Omega v\, g(\psi)\, \mathrm{d}^2x, \qquad C^3 = \int_\Omega k(\psi)\, \mathrm{d}^2x.$$

The Energy-Casimir Method

Requiring that a solution ξ_e be a constrained minimum of the Hamiltonian,

$$\delta(H + C)[\xi_\mathrm{e}] =: \delta F[\xi_\mathrm{e}] = 0,$$

gives an equilibrium solution. The solutions ξ_e is then said to be formally stable if $\delta^2 F[\xi_\mathrm{e}]$ is definite. This is related to δW energy principles, which extremize the potential energy.

Dynamical Accessibility

A slightly more general method for establishing formal stability uses dynamically accessible variations (DAV), defined as

$$\delta\xi_{\mathrm{da}} := \{\mathcal{G}\,,\xi\} + \tfrac{1}{2}\,\{\mathcal{G}\,,\,\{\mathcal{G}\,,\xi\}\}\,,$$

with $\mathcal{G}$ given in terms of the generating functions χ_μ by

$$\mathcal{G} := \int_\Omega \xi^\mu\,\chi_\mu\,\mathrm{d}^2x.$$

DAV are variations that are constrained to remain on the symplectic leaves of the system. They preserve the Casimirs to second order. Stationary solutions of the Hamiltonian,

$$\delta H_{\mathrm{da}}[\xi_{\mathrm{e}}] = 0,$$

capture *all* possible equilibria of the equations of motion.

Energy Associated with DAVs

The energy of the perturbations is

$$\delta^2 H_{\rm da}[\xi_{\rm e}] = \tfrac{1}{2}\int_\Omega \left(\delta\xi^\sigma_{\rm da}\,\frac{\delta^2 H}{\delta\xi^\sigma\,\delta\xi^\tau}\,\delta\xi^\tau_{\rm da} - {W_\lambda}^{\mu\nu}\delta\xi^\lambda_{\rm da}\left[\chi_\mu\,,\frac{\delta H}{\delta\xi^\nu}\right]\right)\,{\rm d}^2x.$$

In order to determine sufficient conditions for stability, we need to write $\delta^2 H_{\rm da}$ in terms of the $\delta\xi^\lambda_{\rm da}$ only (no explicit χ_μ dependence). In principle, this can always be done.

Equilibrium Solutions of Semidirect Sums

An equilibrium $(\omega_{\rm e}, \{\xi^\mu_{\rm e}\})$ of the equations of motion for an SDS satisfies

$$\dot{\omega}_{\rm e} = \left[\, \delta H/\delta\xi^0 \,, \omega_{\rm e} \,\right] + \sum_{\mu=1}^{n} \left[\, \delta H/\delta\xi^\mu \,, \xi^\mu_{\rm e} \,\right] = 0,$$

$$\dot{\xi}^\mu_{\rm e} = \left[\, \delta H/\delta\xi^0 \,, \xi^\mu_{\rm e} \,\right] = 0, \qquad \mu = 1, \dots, n,$$

where we have labeled the 0th variable by ω. We can satisfy the $\dot{\xi}^\mu_{\rm e} = 0$ equations by letting

$$\frac{\delta H}{\delta\xi^0} = -\Phi(u), \qquad \xi^\mu_{\rm e} = \Xi^\mu(u), \quad \mu = 1, \dots, n,$$

for arbitrary functions $u(\mathbf{x})$, $\Phi(u)$, and $\Xi^\mu(u)$.

DAVs for Semidirect Sums

The dynamically accessible variations for an SDS are

$$\delta\omega_{\mathrm{da}} = [\,\omega\,,\chi_0\,] + \sum_{\nu=1}^{n} [\,\xi^\nu\,,\chi_\nu\,],$$
$$\delta\xi^\mu_{\mathrm{da}} = [\,\xi^\mu\,,\chi_0\,], \qquad \mu = 1,\ldots,n.$$

Notice how *all* the $\delta\xi^\mu_{\mathrm{da}}$ depend only on χ_0: the only allowed perturbations are *rearrangements* of the vorticity ω.

CRMHD Equilibria

An equilibrium of Equations (1) satisfies

$$\psi_{\mathrm{e}} = \Psi(u),$$

$$\phi_{\mathrm{e}} = \Phi(u),$$

$$v_{\mathrm{e}} = \bigl(k_1(u) + (k_2(u) + 2x)\,\Phi'(u)\bigr)\,/\,\bigl(1 - |\Phi'(u)|^2/\beta_{\mathrm{i}}\bigr)\,,$$

$$p_{\mathrm{e}} = \bigl(k_1(u)\,\Phi'(u) + \beta_{\mathrm{i}}\,(k_2(u) + 2x)\bigr)\,/\,\bigl(1 - |\Phi'(u)|^2/\beta_{\mathrm{i}}\bigr)\,,$$

$$\omega_{\mathrm{e}}\,\Phi'(u) - J_{\mathrm{e}} = k_3(u) + v_{\mathrm{e}}\,k_1'(u) + p_{\mathrm{e}}\,k_2'(u) + {\beta_{\mathrm{i}}}^{-1}\,p_{\mathrm{e}}\,v_{\mathrm{e}}\,\Phi''(u),$$

with primes defined by $f'(u) = (d\Psi(u)/du)^{-1}\,df(u)/du$, and $u(\mathbf{x})$, $\Psi(u)$, $\Phi(u)$, and the $k_i(u)$ arbitrary functions.

This is very different from the SDS case. In particular, the cocycle allows the equilibrium “advected” quantities v_{e} and p_{e} to depend explicitly on x.

DAVs for CRMHD

The dynamically accessible variations for CRMHD are given by

$$\delta\omega_{\rm da} = [\,\omega\,,\chi_0\,] + [\,v\,,\chi_1\,] + [\,p\,,\chi_2\,] + [\,\psi\,,\chi_3\,],$$
$$\delta v_{\rm da} = [\,v\,,\chi_0\,] - \beta_{\rm i}\,[\,\psi\,,\chi_2\,],$$
$$\delta p_{\rm da} = [\,p\,,\chi_0\,] - \beta_{\rm i}\,[\,\psi\,,\chi_1\,],$$
$$\delta\psi_{\rm da} = [\,\psi\,,\chi_0\,].$$

Note that the DAV for ω is the same as for a semidirect sum. However, the “advected” quantities v, p, and ψ now have *independent* variations, which can be specified by χ_2, χ_1, and χ_0, respectively.

CRMHD Stability

The terms that involve gradients in the perturbation energy are

$$\delta^2 H_{\mathrm{da}} = \int_\Omega \Big(|\nabla \delta\phi - \nabla(\Phi'(u)\, \delta\psi)|^2 \\ + (1 - |\Phi'(u)|^2)|\nabla \delta\psi|^2 + \cdots \Big)\, \mathrm{d}^2 x.$$

These terms must be positive, so we require

$$|\Phi'(u)| < 1, \tag{2}$$

part of the the sufficient condition for stability.

The remaining terms are a quadratic form in δv_{da}, δp_{da}, and $\delta\psi_{\mathrm{da}}$, which can be written

$$\begin{pmatrix} 1 & -{\beta_{\mathrm{i}}}^{-1}\,\Phi' & -k_1' - {\beta_{\mathrm{i}}}^{-1}\,p_{\mathrm{e}}\,\Phi'' \\ -{\beta_{\mathrm{i}}}^{-1}\,\Phi' & {\beta_{\mathrm{i}}}^{-1} & -k_2' - {\beta_{\mathrm{i}}}^{-1}\,v_{\mathrm{e}}\,\Phi'' \\ -k_1' - {\beta_{\mathrm{i}}}^{-1}\,p_{\mathrm{e}}\,\Phi'' & -k_2' - {\beta_{\mathrm{i}}}^{-1}\,v_{\mathrm{e}}\,\Phi'' & \Theta(x,y) \end{pmatrix}$$

where

$$\Theta(x,y) := -k_3'(u) - v_{\mathrm{e}}\,k_1''(u) - p_{\mathrm{e}}\,k_2''(u) + \omega_{\mathrm{e}}\,\Phi''(u) - {\beta_{\mathrm{i}}}^{-1}\,p_{\mathrm{e}}\,v_{\mathrm{e}}\,\Phi'''(u) + \Phi'(u)\,\nabla^2\Phi'(u).$$

For positive-definiteness of this quadratic form, we require the principal minors of this matrix to be positive.

$$\mu_1 = |1| > 0,$$

$$\mu_2 = \begin{vmatrix} 1 & -{\beta_\mathrm{i}}^{-1}\,\Phi'(u) \\ -{\beta_\mathrm{i}}^{-1}\,\Phi'(u) & {\beta_\mathrm{i}}^{-1} \end{vmatrix} = {\beta_\mathrm{i}}^{-1}\left(1 - \frac{|\Phi'(u)|^2}{\beta_\mathrm{i}}\right) > 0,$$

The positive-definiteness of μ_2, combined with condition (2), implies

$$\boxed{|\Phi'(\psi_\mathrm{e})|^2 < \min(1, \beta_\mathrm{i})}$$

which is part of a sufficient condition for stability. Thus the cocycle modifies the stability directly: it always makes the stability condition *worse*, because $\beta_\mathrm{i} > 0$.

Finally, if we require that the *determinant* of the matrix be positive, we have a sufficient condition for formal stability.

Conclusions

- The Lie–Poisson structure of a system tells us the form of the perturbations allowed by the constraints.
- These perturbations can be used to establish sufficient conditions for formal stability.
- Equilibrium solutions for semidirect sums involve advected quantities that are tied to the fluid elements. When a cocycle is present in the bracket, such as for CRMHD, the equilibria are richer.
- In the case of CRMHD, the cocycle has a *destabilizing* effect on the system, as compared to a semidirect sum structure.

References

[1] V. I. Arnold, "Conditions for nonlinear stability of the stationary plane curvilinear flows of an ideal fluid," Doklady Mat. Nauk. **162**, 773 (1965).

[2] V. I. Arnold, "Variational principle for three dimensional steady-state flows of an ideal fluid," J. Appl. Math. Mech. **29**, 1002 (1965).

[3] V. I. Arnold, "Sur un principe variationnel pour les écoulements stationaires des liquides parfaits et ses applications aux problèmes de stabilité non linéaires," Journal de Mécanique **5**, 29 (1966).

[4] J. M. Finn and G.-Z. Sun, "Nonlinear stability and the energy-Casimir method", Comments Plasma Phys. Controlled Fusion **11**, 7 (1987).

[5] T. K. Fowler, "Liapunov's stability criteria for plasmas," J. Math. Phys. **4**, 559 (1963).

[6] C. S. Gardner, "Bound on the energy available from a plasma," Phys. Fluids **6**, 839 (1963).

[7] R. D. Hazeltine, C. T. Hsu, and P. J. Morrison, "Hamiltonian four-field model for nonlinear tokamak dynamics," Phys. Fluids **30**, 3204 (1987).

[8] D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, "Nonlinear stability of fluid and plasma equilibria," Physics Reports **123**, 1 (1985).

[9] M. B. Isichenko, "Nonlinear hydrodynamic stability," Phys. Rev. Lett. **80**, 972 (1998).

[10] M. D. Kruskal and C. R. Oberman, "On the stability of a plasma in static equilibrium," Phys. Fluids **1**, 275 (1958).

[11] P. J. Morrison, "Hamiltonian description of the ideal fluid," Rev. Modern Phys. **70**, 467 (1998).