Nonlinear MHD Stability and Dynamical Accessibility

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Overview

- We discuss a unified description of variational methods for establishing stability of plasma equilibria.
- The first method is based upon a Lagrangian approach (in the sense of fluid elements). A Lagrangian equilibrium is static.
- Eulerian (stationary) equilibria can have flow. Their stability can be studied with "Eulerianized" Lagrangian displacements (ELD).
- Another method involves Dynamically Accessible Variations (DAV), which are constrained to satisfy the invariants of the flow. Closely related to the Energy–Casimir method.
- We show the equivalence of the the ELD and DAV methods for the case of MHD equilibria.

History: Some key papers

- Fjørtoft [1950] Geophysical context.
- Lundquist [1951], Bernstein et al. [1958], Woltjer [1958] Static equilibria.
- Kruskal and Oberman [1958], Gardner [1963] Kinetic theory.
- Arnold [1965, 1966, 1969] Nonlinear stability criterion.
- Frieman and Rotenberg [1960], Newcomb [1962], Hameiri [1982, 1998], Hameiri and Holties [1994] Stationary equilibria, Nonlinear eigenvalue problem.
- Holm et al. [1985], Morrison and Eliezer [1986], Finn and Sun [1987] Energy–Casimir method).
- Morrison and Pfirsch [1990], Morrison [1998] Dynamical accessibility.

Equations of Motion

Inviscid, ideally conducting fluid:

$$\rho\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = -\nabla p + \boldsymbol{j}\times\boldsymbol{B},$$

$$\partial_t\rho + \nabla\cdot(\rho\,\boldsymbol{v}) = 0,$$

$$\partial_t s + \boldsymbol{v}\cdot\nabla s = 0,$$

$$\partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = 0.$$

Conserved energy (Hamiltonian):

$$H = \int \mathrm{d}^3x\,\left(\tfrac{1}{2}\,\rho\,|\boldsymbol{v}|^2 + \tfrac{1}{2}\,|\boldsymbol{B}|^2 + \rho\,U(\rho,s)\right)$$

Can have other invariants, such as the helicity and cross-helicity, depending on initial configuration (Padhye and Morrison [1996], Hameiri [1998]).

Static (Lagrangian) Equilibria

Equilibrium quantities are denoted by a subscript, "e".
Setting ∂_t and $\boldsymbol{v}_\mathrm{e}$ to zero, the only condition is

$$\nabla p_\mathrm{e} = (\nabla \times \boldsymbol{B}_\mathrm{e}) \times \boldsymbol{B}_\mathrm{e}, \qquad \nabla \cdot \boldsymbol{B}_\mathrm{e} = 0.$$

To determine a sufficient condition for stability, we consider perturbations about a static equilibrium

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\xi}(\boldsymbol{x}_0, t),$$

where $\boldsymbol{x}$ is the position of a fluid element at time t and $\boldsymbol{\xi}(\boldsymbol{x}_0, t)$ is the Lagrangian displacement, with $\boldsymbol{\xi}(\boldsymbol{x}_0, 0) = 0$.

After computing the variations of the various physical quantities and linearizing the equations of motion with respect to $\boldsymbol{\xi}$ (Bernstein et al. [1958]), we obtain

$$\rho_0 \, \ddot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi}).$$

(Formal) linear stability is then guaranteed if

$$\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) := -\tfrac{1}{2} \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) \, \mathrm{d}^3 x \geq 0.$$

This is Lagrange's principle: the potential energy needs to be positive-definite for stability.

Stationary (Eulerian) Equilibria

The relabeling symmetry allows passage from the Lagrangian to the Eulerian picture via the process of reduction (Marsden and Weinstein [1974], Morrison [1998]). The equilibria then represent stationary flows. Three approaches:

- "Eulerianized" Lagrangian displacements (Frieman and Rotenberg [1960], Newcomb [1962]), by which the displacements are re-expressed in terms of Eulerian variables only.
- Energy–Casimir Method (Holm et al. [1985], Morrison and Eliezer [1986]).
- Dynamically accessible variations (Morrison and Pfirsch [1990], Morrison [1998]), a method for generating variations which preserve the Casimir invariants of the system

“Eulerianized” Lagrangian Displacement (ELD)

Express the Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{x}_0, t)$ in terms of the Eulerian coordinates $\boldsymbol{x}$:

$$\boldsymbol{\eta}(\boldsymbol{x}, t) = \boldsymbol{\xi}(\boldsymbol{x}_0, t)$$

(back)

The variations are (Newcomb [1962])

$$\begin{aligned}
\delta \boldsymbol{v} &= \dot{\boldsymbol{\eta}} + \boldsymbol{v} \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{v}, \\
\delta \rho &= -\nabla \cdot (\rho \, \boldsymbol{\eta}), \\
\delta s &= -\boldsymbol{\eta} \cdot \nabla s, \\
\delta \boldsymbol{B} &= \nabla \times (\boldsymbol{\eta} \times \boldsymbol{B}).
\end{aligned}$$

Energy can be varied with respect to these perturbations: a sufficient stability criterion is obtained. $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ are independent.

Prelude: Hamiltonian Formulation

Ideal MHD has a Hamiltonian formulation in terms of a noncanonical bracket (Morrison and Greene [1980])

$$\{F\,,G\} = -\left(\int \mathrm{d}^3x\, F_\rho\, \nabla\cdot\, G_{\boldsymbol{v}} + F_{\boldsymbol{v}}\cdot\left(\frac{(\nabla\times\boldsymbol{v})}{2\rho}\times G_{\boldsymbol{v}}\right)\right.$$

$$\left. +\rho^{-1}\,\nabla s\cdot(F_s\; G_{\boldsymbol{v}}) + \rho^{-1} F_{\boldsymbol{v}}\cdot(\boldsymbol{B}\times(\nabla\times G_{\boldsymbol{B}}))\right) + \left(F \longleftrightarrow G\right).$$

F and G are functionals of the dynamical variables $(\boldsymbol{v},\rho,s,\boldsymbol{B})$, and subscripts denote functional derivatives. The bracket $\{\ ,\ \}$ is antisymmetric and satisfies the Jacobi identity. The equations of motion can be written

$$\partial_t(\boldsymbol{v},\rho,s,\boldsymbol{B}) = \{(\boldsymbol{v},\rho,s,\boldsymbol{B})\,,H\,\}\,.$$

Dynamical Accessibility

Another method establishing formal stability uses dynamically accessible variations (DAV), defined for the variable ζ as

$$\delta\zeta_{\mathrm{da}} := \{\mathcal{G}\,,\zeta\}\,, \qquad \delta^2\zeta_{\mathrm{da}} := \tfrac{1}{2}\,\{\mathcal{G}\,,\,\{\mathcal{G}\,,\zeta\}\}\,,$$

with $\mathcal{G}$ given in terms of the generating functions χ_μ by

$$\mathcal{G} := \sum_\mu \int \zeta^\mu\,\chi_\mu\,\mathrm{d}^3x.$$

DAV are variations that are constrained to remain on the symplectic leaves of the system. They preserve the Casimir invariants to second order (but there is no need to explicitly know the invariants).

Energy Associated with DAVs

Stationary solutions ζ_e of the Hamiltonian,

$$\delta H_{\rm da}[\zeta_e] = 0,$$

capture all possible equilibria of the equations of motion.
The energy of the perturbations is

$$\delta^2 H_{\rm da}[\zeta_e] = \frac{1}{2}\int \left(\delta\zeta^\sigma_{\rm da}\,\frac{\delta^2 H}{\delta\zeta^\sigma\,\delta\zeta^\tau}\,\delta\zeta^\tau_{\rm da} + \delta^2\zeta^\nu_{\rm da}\,\frac{\delta H}{\delta\zeta^\nu}\right)\,{\rm d}^3x,$$

with $\zeta = (\boldsymbol{v}, \rho, s, \boldsymbol{B})$ and repeated indices are summed.
Positive-definiteness of $\delta^2 H_{\rm da}[\zeta_e]$ implies formal stability, which implies linear stability, but not nonlinear stability. (Requires convexity, Holm et al. [1985].)

DAV for MHD

The form of the dynamically accessible variations is

$$\begin{aligned}
\rho\,\delta\boldsymbol{v}_{\mathrm{da}} &= (\nabla\times\boldsymbol{v})\times\boldsymbol{\chi}_0 + \rho\,\nabla\chi_1 - \chi_2\,\nabla s + \boldsymbol{B}\times(\nabla\times\boldsymbol{\chi}_3),\\
\delta\rho_{\mathrm{da}} &= \nabla\cdot\boldsymbol{\chi}_0,\\
\delta s_{\mathrm{da}} &= \rho^{-1}\,\boldsymbol{\chi}_0\cdot\nabla s,\\
\delta\boldsymbol{B}_{\mathrm{da}} &= \nabla\times\left(\rho^{-1}\,\boldsymbol{B}\times\boldsymbol{\chi}_0\right).
\end{aligned}$$

$\boldsymbol{\chi}_0$, χ_1, χ_2, and $\boldsymbol{\chi}_3$ are the arbitrary generating functions of the variations. The variations for ρ, s, and $\boldsymbol{B}$ are the same as for the ELD, with $\boldsymbol{\chi}_0 = -\rho\,\boldsymbol{\eta}$.

The combination of arbitrary functions in the definition of $\delta\boldsymbol{v}_{\mathrm{da}}$ makes that perturbation arbitrary, in the same manner as the ELD perturbation $\delta\boldsymbol{v}$, as we now show.

Equivalence of ELD and DAV

The compelling choice is $\boldsymbol{\chi}_0 = \rho\, \boldsymbol{\eta}$, from which the equivalence of the $\boldsymbol{v}$ perturbations requires that

$$\dot{\boldsymbol{\eta}} = \rho\, \nabla \chi_1 - \chi_2\, \nabla s + \boldsymbol{B} \times (\nabla \times \boldsymbol{\chi}_3).$$

The ELD and the DAV will be equivalent if it is possible to choose χ_1, χ_2, and $\boldsymbol{\chi}_3$ to span the same space as $\dot{\boldsymbol{\eta}}$, and vice-versa.

$\dot{\boldsymbol{\eta}}$ can represent any perturbation, up to boundary conditions.

Local Euler–Clebsch representation for magnetic field:

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta \quad [+\nabla\gamma \times \nabla\Psi(\alpha, \beta, \gamma)]$$

[More generally, Boozerize.]

Pick a third, independent function γ. Covariant representation:

$$\boldsymbol{\chi}_3 = a\,\nabla\alpha + b\,\nabla\beta + c\,\nabla\gamma$$

$$\nabla\times\boldsymbol{\chi}_3 = \nabla a\times\nabla\alpha + \nabla b\times\nabla\beta + \nabla c\times\nabla\gamma$$

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{\chi}_3) = J\left(\frac{\partial a}{\partial\gamma} - \frac{\partial c}{\partial\alpha}\right)\nabla\alpha - J\left(\frac{\partial c}{\partial\beta} - \frac{\partial b}{\partial\gamma}\right)\nabla\beta$$

$$J := \nabla\alpha\cdot(\nabla\beta\times\nabla\gamma)$$

$$\dot{\boldsymbol{\eta}} = \left(\rho\,\frac{\partial\chi_1}{\partial\alpha} - \chi_2\,\frac{\partial s}{\partial\alpha} + J\left(\frac{\partial a}{\partial\gamma} - \frac{\partial c}{\partial\alpha}\right)\right)\nabla\alpha$$
$$+\left(\rho\,\frac{\partial\chi_1}{\partial\beta} - \chi_2\,\frac{\partial s}{\partial\beta} - J\left(\frac{\partial c}{\partial\beta} - \frac{\partial b}{\partial\gamma}\right)\right)\nabla\beta + \left(\rho\,\frac{\partial\chi_1}{\partial\gamma} - \chi_2\,\frac{\partial s}{\partial\gamma}\right)\nabla\gamma$$

Covariant representation of $\dot{\boldsymbol{\eta}}$:

$$\dot{\boldsymbol{\eta}} = A\,\nabla\alpha + B\,\nabla\beta + C\,\nabla\gamma$$

Equate coefficients:

$$\rho\,\frac{\partial\chi_1}{\partial\alpha} - \chi_2\,\frac{\partial s}{\partial\alpha} + J\left(\frac{\partial a}{\partial\gamma} - \frac{\partial c}{\partial\alpha}\right) = A \qquad \text{(A)}$$

$$\rho\,\frac{\partial\chi_1}{\partial\beta} - \chi_2\,\frac{\partial s}{\partial\beta} - J\left(\frac{\partial c}{\partial\beta} - \frac{\partial b}{\partial\gamma}\right) = B \qquad \text{(B)}$$

$$\rho\,\frac{\partial\chi_1}{\partial\gamma} - \chi_2\,\frac{\partial s}{\partial\gamma} = C \qquad \text{(C)}$$

The function a only appears in (A), so solve for $\partial a/\partial\gamma$ and integrate; b only appears in (B), so solve for $\partial b/\partial\gamma$ and integrate. Use χ_2 to satisfy (C).

Conclusions

- The three approaches, using Lagrangian perturbations vs energy–Casimir and dynamical accessibility, lead to essentially the same stability criterion.
- The dynamical accessibility method can be used directly at the Hamiltonian level. One needs to know the Poisson bracket and Hamiltonian.
- For energy–Casimir, one also needs the Casimir invariants, but not necessarily the bracket.
- In both approaches other invariants (non-Casimir, e.g., momentum) can be incorporated.
- Dynamical accessibility has also been applied to Vlasov–Maxwell equilibria (Morrison and Pfirsch [1989, 1990]).

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Expression for $\mathbf{F}(\boldsymbol{\xi})$

$$\mathbf{F}(\boldsymbol{\xi}) := \nabla_0 \left[\rho_0 \left(\frac{\partial p_0}{\partial \rho_0} \right)_{s_0} \nabla_0 \cdot \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \nabla_0)\, p_0 \right] + \boldsymbol{j}_0 \times \boldsymbol{Q} - \boldsymbol{B}_0 \times (\nabla_0 \times \boldsymbol{Q})$$

$$\boldsymbol{Q} := \nabla_0 \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)$$

(back)