A Lagrangian approach to the study of the kinematic dynamo

Jean-Luc Thiffeault

Department of Applied Physics and Applied Mathematics

Columbia University

http://plasma.ap.columbia.edu/~jeanluc/

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with Allen Boozer

Magnetic Field Evolution

The evolution of a magnetic field in resistive MHD is governed by the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

where the Eulerian velocity field $\mathbf{v}(\boldsymbol{x}, t)$ is some prescribed time-dependent flow. $\mathbf{B}$ is the magnetic field, η is the resistivity, and μ_0 is the permeability of free space.

In a chaotic flow, fluid elements are stretched exponentially. The magnetic field grows due to the stretching, and the diffusion is also increased by this process. This enhancement is known as chaotic mixing.

The Kinematic Dynamo

The kinematic dynamo problem consists of studying the induction equation on the assumption that the magnetic field does not react back on the flow $\mathbf{v}$ (the Lorentz force is neglected). This is justified when the field is small.

The fast kinematic dynamo can be formulated as follows:

Starting from a small seed magnetic field, what properties of $\mathbf{v}$ are needed to obtain exponential growth of a large-scale $\mathbf{B}$ such that the growth rate remains nonzero as $\eta \to 0$?

Relevant to astrophysical plasmas, where η is so small that a growth rate going to 0 as $\eta \to 0$ would be too long to account for observed magnetic fields.

Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $\boldsymbol{x}$ satisfies

$$\frac{d\boldsymbol{x}}{dt}(\boldsymbol{\xi}, t) = \mathbf{v}(\boldsymbol{x}(\boldsymbol{\xi}, t), t),$$

where $\boldsymbol{\xi}$ are Lagrangian coordinates that label fluid elements. The usual choice is to take as initial condition $\boldsymbol{x}(\boldsymbol{\xi}, t = 0) = \boldsymbol{\xi}$, which says that fluid elements are labeled by their initial position.

$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t)$ is thus the transformation from Lagrangian ($\boldsymbol{\xi}$) to Eulerian ($\boldsymbol{x}$) coordinates.

For a chaotic flow, this transformation gets horrendously complicated as time evolves.

The Metric Tensor

The Jacobian matrix of the transformation $\boldsymbol{x}(\boldsymbol{\xi}, t)$ is

$$M^i{}_q := \frac{\partial x^i}{\partial \xi^q}$$

For simplicity, we restrict ourselves to divergence-free flows, $\nabla \cdot \mathbf{v} = 0$, so that $\det M = 1$. The Jacobian matrix is a precise record of how a fluid element is rotated and stretched by $\mathbf{v}$. We are interested in the stretching, not the rotation, so we construct the metric tensor

$$\boxed{g_{pq} := \sum_{i=1}^{3} M^i{}_p \, M^i{}_q}$$

which contains only the information on the stretching of fluid elements.

Rates and Directions of Stretching

The metric is a symmetric, positive-definite matrix, so it can be diagonalized with orthogonal eigenvectors $\{\hat{\mathbf{u}}, \hat{\mathbf{m}}, \hat{\mathbf{s}}\}$ and corresponding real, positive eigenvalues $\{\Lambda_{\rm u}, \Lambda_{\rm m}, \Lambda_{\rm s}\}$,

$$g_{pq} = \Lambda_{\rm u}\, \hat{\mathbf{u}}_p\, \hat{\mathbf{u}}_q + \Lambda_{\rm m}\, \hat{\mathbf{m}}_p\, \hat{\mathbf{m}}_q + \Lambda_{\rm s}\, \hat{\mathbf{s}}_p\, \hat{\mathbf{s}}_q$$

The label u indicates an unstable direction: after some time $\Lambda_{\rm u} \gg 1$, growing exponentially for long times. The label s indicates a stable direction: after some time $\Lambda_{\rm s} \ll 1$, shrinking exponentially for long times. The intermediate direction, denoted by m, does not grow or shrink exponentially.

The incompressibility of $\mathbf{v}$ implies that $\Lambda_{\rm u}\, \Lambda_{\rm m}\, \Lambda_{\rm s} = 1$.

Induction Equation in Lagrangian Coordinates

With the help of the chain rule and the metric tensor, we can transform the magnetic induction equation to from Eulerian coordinates $\boldsymbol{x}$ to Lagrangian coordinates $\boldsymbol{\xi}$:

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{\xi}} b^s(\boldsymbol{\xi},t) = \sum_{p,q=1}^{3} \frac{\eta}{\mu_0} \frac{\partial}{\partial \xi^p} \left[g^{pq}(\boldsymbol{\xi},t) \frac{\partial}{\partial \xi^q} b^s(\boldsymbol{\xi},t) \right]$$

where $b^s := \sum_i \left(M^{-1}\right)^s{}_i B^i$ is the magnetic field in the new frame, and $g^{pq} := (g^{-1})^{pq}$. The above equation is a simply a diffusion equation with anisotropic diffusivity $\eta\, g^{pq}$. By construction, the velocity $\mathbf{v}$ has dropped out of the equation entirely.

When $\eta = 0$, the above is the well-known result that in ideal MHD the magnetic field is frozen into the fluid.

Helicity vs Energy Evolution

The magnetic helicity $h := \mathbf{A} \cdot \mathbf{B}$, where $\mathbf{A}$ is the magnetic potential with $\nabla \times \mathbf{A} = \mathbf{B}$, evolves according to

$$\frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h = -\eta \left(\mathbf{j} \cdot \mathbf{B} + \mathbf{A} \cdot \nabla \times \mathbf{j} \right)$$

where $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ is the current. Some terms were absorbed by an appropriate choice of gauge. Helicity can only be created if $\eta \neq 0$. The magnetic energy $E_B = \mathbf{B}^2/2\mu_0$ satisfies

$$\frac{\partial E_B}{\partial t} + \mathbf{v} \cdot \nabla E_B = -\eta \, \mathbf{j}^2$$

By transforming the evolution equations to Lagrangian coordinates, we want to examine the relative magnitude of helicity creation and energy dissipation. Since it is commonly wisdom in dynamo theory that helicity generation is required to create a large-scale magnetic field, we want to see if the power required is prohibitive.

Comparison

If we compare the generation of helicity and dissipation of energy, the resistivity η drops out. Thus, the ratio is independent of the resistivity for as long as ideal evolution occurs. To leading order, in Lagrangian coordinates, we have

$$\frac{\mathbf{A} \cdot \nabla \times \mathbf{j}}{\mathbf{j}^2} \sim \Lambda_{\mathrm{u}}^{-1} \left(\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}}\right)^{-2}$$

$$\frac{\mathbf{j} \cdot \mathbf{B}}{\mathbf{j}^2} \sim \Lambda_{\mathrm{u}}^{-1} \left(\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}}\right)^{-1}$$

where ∇_0 denotes a gradient with respect to the Lagrangian variables. Both terms seem of the same order, and are growing exponentially.

However . . .

We have applied [Thiffeault and Boozer, submitted to Chaos] the techniques of differential geometry to study the coordinate transformation $\boldsymbol{x}(\boldsymbol{\xi}, t)$. By requiring that the curvature tensor, an invariant under coordinate transformations, satisfy some consistency conditions (constraints), we have found that

$$\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}} \sim \Lambda_{\mathrm{u}}^{-1}\,\Lambda_{\mathrm{m}}$$

This complicates the problem considerably because now terms of lower order must be considered. The asymptotic behavior of Lagrangian derivatives such as

$$\hat{\mathbf{u}} \cdot \nabla_0 \ln \Lambda_i \sim \Lambda_{\mathrm{u}}^{1/2}$$

must also be considered in detail.

Revised Scalings

Using these new scalings, we obtain the new form

$$\frac{\mathbf{A} \cdot \nabla \times \mathbf{j}}{\mathbf{j}^2} \sim \Lambda_{\mathrm{u}}^{-1/2}$$

$$\frac{\mathbf{j} \cdot \mathbf{B}}{\mathbf{j}^2} \sim \Lambda_{\mathrm{u}}^{-1}$$

Now the major contribution to helicity creation comes from the $\mathbf{A} \cdot \nabla \times \mathbf{j}$ term, and both terms shrink less rapidly. Nevertheless, the rate of creation of helicity is still exponentially smaller than the rate of energy dissipation.

Even though these results are for ideal evolution, for very small η the magnetic field will have built up huge gradients by the time the evolution ceases to be ideal. Exponentially large amounts of power thus seem required to create helicity.

Summary

- Using Lagrangian coordinates, we have revised earlier estimates of the growth rates of energy dissipation and helicity generation.
- We assumed ideal evolution and compared the helicity creation to the energy dissipated. Ratio is independent of the resistivity.
- After adjusting the growth rate for new geometrical constraints ($\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}} \sim \Lambda_{\mathrm{u}}^{-1} \Lambda_{\mathrm{m}}$), we find that the helicity creation is exponentially smaller than the energy dissipation. The situation is worse than that: our analysis does not take into account the fact that the helicity created will be fractal in nature, and so probably very little of it will contribute to a large-scale magnetic field.
- Suggest close inspection of energy dissipation in numerical simulations.