Long-wave Instability in Anisotropic Double-Diffusion

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Overview

- Want to capture asymptotic dynamics near Takens–Bogdanov bifurcation.
- Problem: typical scaling leads to Hamiltonian (and thus conservative) equation, which obviously does not capture a lot of the dynamics.
- Try using different scaling, but then get unremovable resonant terms.
- Solution: extend parameter space to allow removal of resonant terms. Raises codimension, but asymptotic.

Takens–Bogdanov

- TB bifurcation occurs when two modes become unstable at the same parameter values.

- Equations for the reduced dynamics near this bifurcation point capture more of the diverse behaviour of the system than simple steady or Hopf bifurcation.

- For double-diffusive convection in long-wave theory such a bifurcation is present.

- Problem: the reduced equations contain terms of differing order in the standard asymptotic expansion parameter. The asymptotic theory fails to collect a dissipative nonlinear term; the amplitude equations are Hamiltonian to leading order (Childress and Spiegel 1981).

Possible solutions

- Normal form theory: not available for extended (continuum of excited modes) systems.
- Reconstitution: Not asymptotic, so hard to judge validity. May be flawed in some cases (Clune, Depassier, and Knobloch, 1994).
- Nonlocal averaging: Difficult to solve (Pismen, 1988).
- Alternative route: if more parameters were available, could remove resonant terms at the cost of augmenting the codimension of the bifurcation.

To introduce needed extra parameters, we choose anisotropic double-diffusion as our system. (possible transport model for ocean, astrophysics, tokamak plasmas)

Illustration of Procedure

Normal form for three real marginal modes:

$$\begin{aligned}\dot{F} &= G \\ \dot{G} &= H \\ \dot{H} &= -\eta\, H - \nu\, G - \lambda\, F + a\, F^2 \\ &\quad + b\, G^2 + c\, FG + d\, FH\end{aligned}$$

Assuming strongly damped mode ($|\eta| >> |\nu|, |\lambda|$) we should recover the two-mode normal form. One way to do this (Spiegel *et al*) is to use the scaling

$$t = \bar{t}/\delta, \quad \lambda = \delta^2 \bar{\lambda}, \quad \nu = \delta^2\, \bar{\nu}, \quad F = \delta^2 \bar{F}, \quad G = \delta^3 \bar{G}.$$

This leads to a Hamiltonian equation, not two-mode normal form as one would expect. If instead of rescaling the amplitudes one rescales the nonlinear terms

$$t = \bar{t}/\delta, \quad \lambda = \delta^2 \bar{\lambda}, \quad \nu = \delta\, \bar{\nu}, \quad a = \delta^2\, \bar{a}, \quad c = \delta\, \bar{c},$$

we recover two-mode normal form, at the cost of raising the codimension.

Model Equations

The equations for anisotropic double-diffusion are

$$
\begin{aligned}
\sigma^{-1}\frac{d}{dt}\nabla^2\psi &= R\,\partial_x\Theta - S\,\partial_x\Sigma + (D^2 + \Delta\,\partial_x^2)\nabla^2\psi, \\
\frac{d}{dt}\Theta &= \partial_x\psi + (D^2 + \Lambda\,\partial_x^2)\Theta, \\
\mathrm{Le}\,\frac{d}{dt}\Sigma &= \mathrm{Le}\,\partial_x\psi + (D^2 + \Xi\,\partial_x^2)\Sigma;
\end{aligned}
$$

with no-slip, fixed-flux boundary conditions

$$\psi = D\psi = 0, \quad D\,\Theta = D\,\Sigma = 0, \;\; z = 0 \text{ and } 1$$

Fixed flux favors convection cells that are as large as the system will permit. Use this to define small parameter ϵ.

Scaling:

$$\partial_x = \epsilon\,\partial_X, \quad \partial_t = \epsilon^4\,\partial_T, \quad \psi = \epsilon\,\phi_X$$

Order ϵ^0 and ϵ^2

The fixed flux boundary conditions give

$$\Theta_0 = \Theta_0(X,T), \quad \Sigma_0 = \Sigma_0(X,T)$$

at order ϵ^0.

At order ϵ^2, we get the solvability condition (linear at this order):

$$\begin{pmatrix} \frac{1}{720}\,R_0 - \Lambda_0 & -\frac{1}{720}\,S_0 \\ \frac{1}{720}\,\mathrm{Le}_0\,R_0 & -\frac{1}{720}\,\mathrm{Le}_0\,S_0 - \Xi_0 \end{pmatrix} \begin{pmatrix} \Theta_{0XX} \\ \Sigma_{0XX} \end{pmatrix} = 0.$$

The requirement that the matrix have zero eigenvalues means that its trace and determinant must vanish. This is obtained by letting

$$R_0 = 720\,\frac{\Lambda_0^2}{\Lambda_0 - \Xi_0/\mathrm{Le}_0}, \quad S_0 = \frac{720}{\mathrm{Le}_0^2}\,\frac{\Xi_0^2}{\Lambda_0 - \Xi_0/\mathrm{Le}_0},$$

The eigenvector for the matrix is parametrized by $\Sigma_0 = (\mathrm{Le}_0\,\Lambda_0/\Xi_0)\Theta_0$ (it only has one).

Order ϵ^4

Get two solvability conditions again, this time involving T:

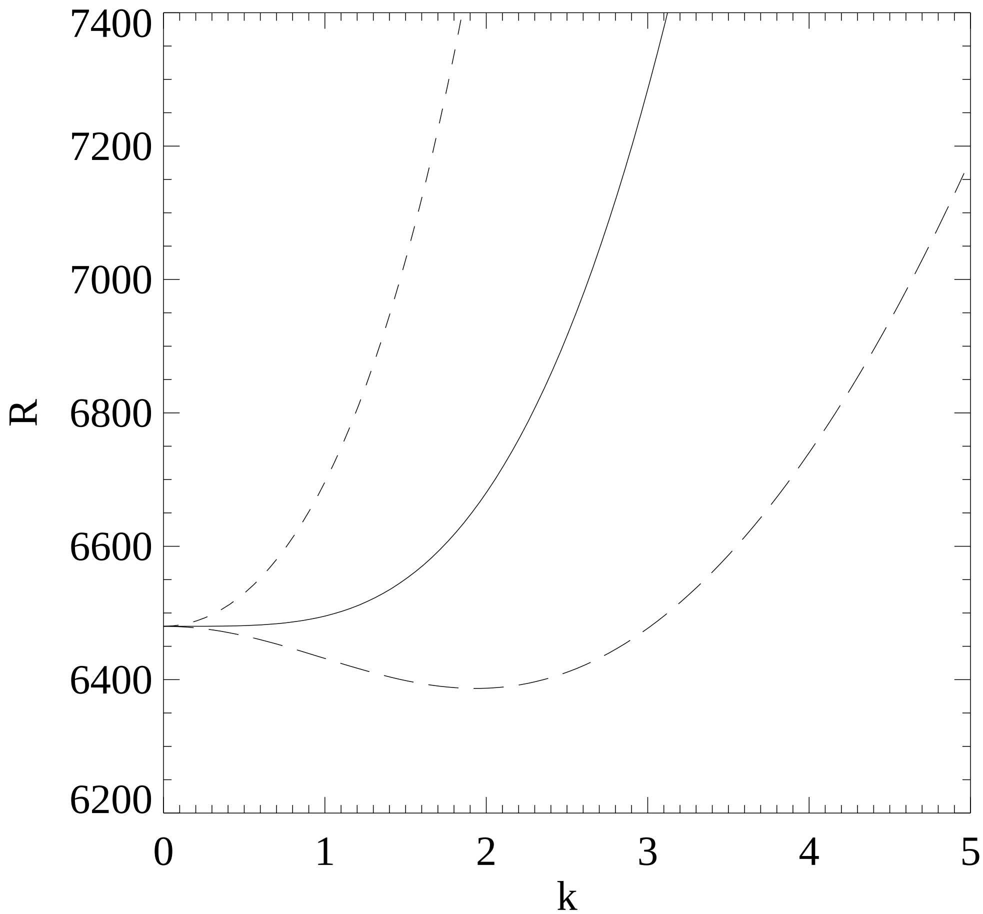
$$\begin{aligned}\Theta_{0T} &= \cdots \\ \Sigma_{0T} &= (\mathrm{Le}_0\,\Lambda_0/\Xi_0)\Theta_{0T} = \cdots\end{aligned}$$

Must be compatible since Θ_{0T} and Σ_{0T} are related. This is not satisfied automatically; this is why we now make use of the extra parameters. By letting

$$\begin{aligned}\mathrm{Le}_0 &= 1 \\ 5(\Lambda_0 + \Xi_0) &= 11(1 + \Delta_0) \\ R_2 - \frac{\Lambda_0}{\Xi_0} S_2 &= \frac{720\Lambda_0(\Lambda_2 - \Xi_2 + \mathrm{Le}_2\,\Xi_0)}{\Lambda_0 - \Xi_0}\end{aligned}$$

the two become compatible. This increases the codimension by three.

Marginal Stability Curves



$\Delta_0 = 0.1$ (long-dashed), 1.2727 (solid), 5 (dashed)

Order ϵ^6

We get a solvability condition involving only the ϵ^2 integration constants

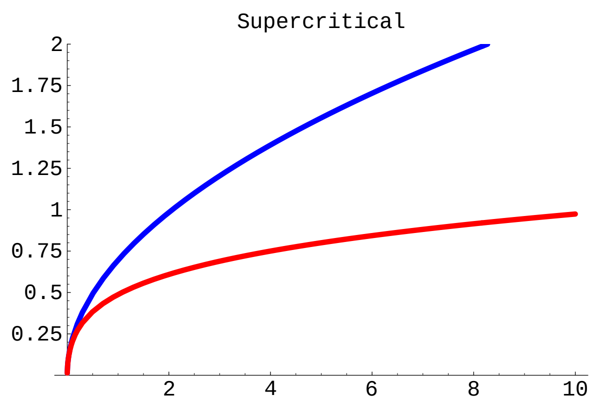
$$g(X,T) := \Sigma_{2,0}(X,T) - \frac{\Lambda_0}{\Xi_0}\,\Theta_{2,0}(X,T)$$

at this order. After rescaling to eliminate some parameters we have the coupled system

$$\begin{aligned}
f_T &= g_{XX} + \alpha\, f_{XX} + f_{XXXX} + \left(f_X^3\right)_X \\
g_T &= \lambda\, f_{XX} + \kappa\, f_{XXXX} - \gamma\, f_{XXXXXX} + \beta\, g_{XX} \\
&\quad - \rho\, g_{XXXX} + \xi\left(f_X^3\right)_X + \left(f_X^2\, g_X\right)_X \\
&\quad + \eta\left(f_X\, f_{XX}^2\right)_X - \zeta\left(f_X^3\right)_{XXX}
\end{aligned}$$

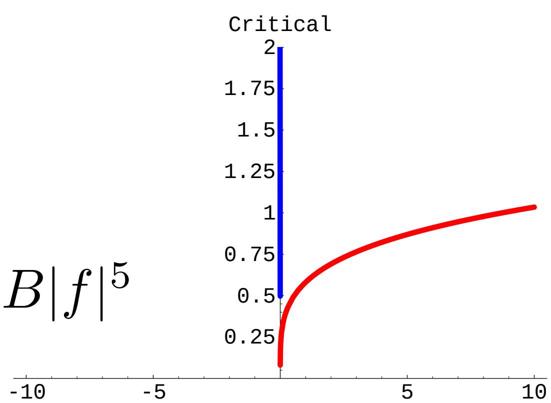
We fixed Le_0, Δ_0, and Λ_2. However, we are left with enough parameters to vary independently all the coefficients except η and ζ.

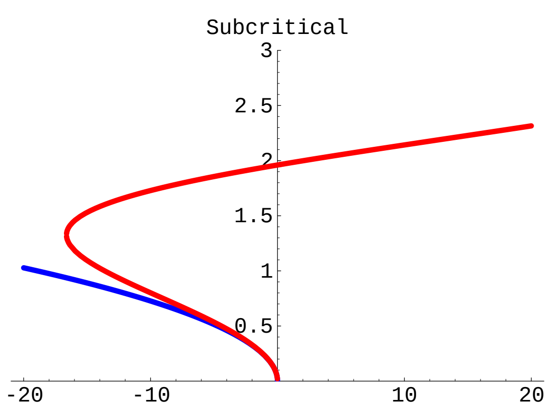
Captures Turnaround



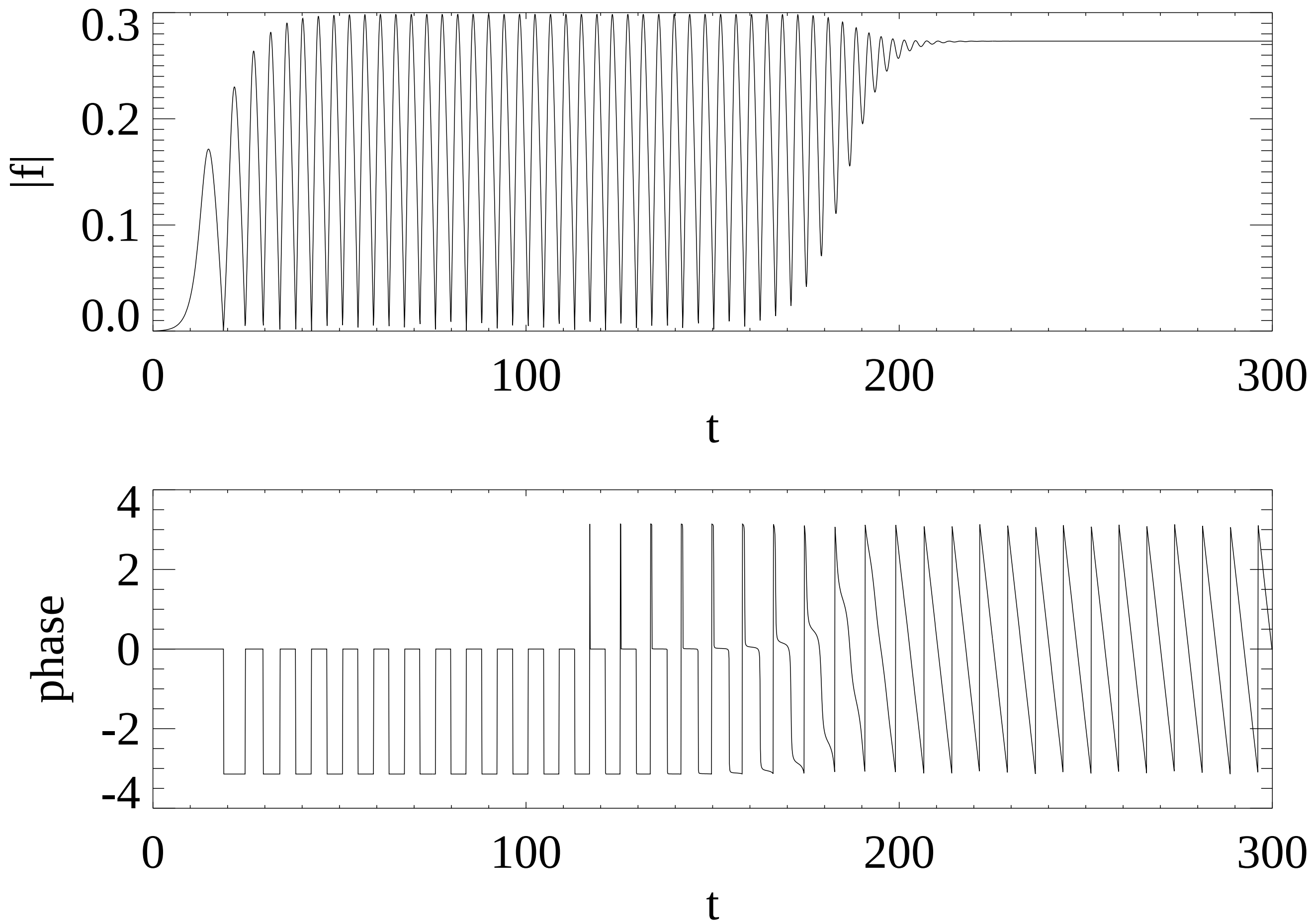
Steady state

Blue – $A|f|^3$
Red – $A|f|^3 + B|f|^5$





Numerical Solution



Traveling waves stable.

Conclusions

- For anisotropic double-diffusion in long-wave theory, we have shown that an extended system equation can be asymptotically derived.

- The equation contains several known equations as limits (Chapman&Proctor 1980, Childress&Spiegel 1981, Knobloch 1989).

- Compare reconstituted result.

- Explore numerical solutions.

- Make connection with physics.