optimizing transport in heat exchangers

a probabilistic approach

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advection–diffusion equation in a bounded region

Advection and diffusion of heat in a bounded region Ω, with Dirichlet boundary conditions:

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = D \Delta \theta, \qquad \mathbf{u} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = 0,$$

with $\nabla \cdot \mathbf{u} = 0$ and $\theta(\mathbf{x}, t) \geq 0$.

Write $\langle \cdot \rangle$ for an integral over Ω. The rate of heat loss is equal to the flux through the boundary $\partial\Omega$:

$$\partial_t \langle \theta \rangle = D \int_{\partial\Omega} \nabla \theta \cdot \hat{\mathbf{n}} \, \mathrm{d}S =: -F[\theta] \leq 0. \qquad *$$

Goal: find velocity fields $\mathbf{u}$ that maximize the heat flux.

Note that * is not so good for this, since velocity does not appear.

related problem: mean exit time

Take steady velocity $\mathbf{u}(\mathbf{x})$. The mean exit time $\tau(\mathbf{x})$ of a Brownian particle initially at $\mathbf{x}$ satisfies

$$-\mathbf{u} \cdot \nabla \tau = D \Delta \tau + 1, \qquad \tau|_{\partial\Omega} = 0,$$

This is a steady advection–diffusion equation with velocity $-\mathbf{u}$ and source 1.

Intuitively, a small integrated exit time $\langle \tau \rangle = \|\tau\|_1$ implies that the velocity is good at taking heat out of the system.

The exit time equation is much nicer than the equation for the contration: it is steady, and it applies for any initial concentration $\theta_0(\mathbf{x})$.

relationship between exit time and mean temperature

Recall that $\langle \cdot \rangle$ is an integral over space, and take $\langle \theta_0 \rangle = 1$. The quantity

$$\int_0^\infty \langle \theta \rangle \, \mathrm{d}t$$

is a cooling time. Smaller is better for transport.

We have the rigorous bounds

$$\int_0^\infty \langle \theta \rangle \, \mathrm{d}t \leq \|\tau\|_\infty \qquad \int_0^\infty \langle \theta \rangle \, \mathrm{d}t \leq \|\tau\|_1 \, \|\theta_0\|_\infty .$$

Thus, decreasing a norm like $\|\tau\|_1$ or $\|\tau\|_\infty$ will typically decrease the cooling time, as expected.

optimization problem

Advection–diffusion operator and its adjoint:

$$\mathcal{L} := \mathbf{u} \cdot \nabla - D\Delta, \qquad \mathcal{L}^\dagger = -\mathbf{u} \cdot \nabla - D\Delta \,.$$

Minimize $\langle \tau \rangle$ over steady $\mathbf{u}(\mathbf{x})$ with fixed total kinetic energy E.

The functional to optimize:

$$\mathcal{F}[\tau, \mathbf{u}, \vartheta, \mu, p] = \langle \tau \rangle - \langle \vartheta(\mathcal{L}^\dagger \tau - 1) \rangle + \tfrac{1}{2}\mu(\|\mathbf{u}\|_2^2 - 2E) - \langle p \nabla \cdot \mathbf{u} \rangle$$

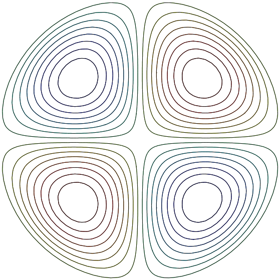
Here ϑ, μ, p are Lagrange multipliers.

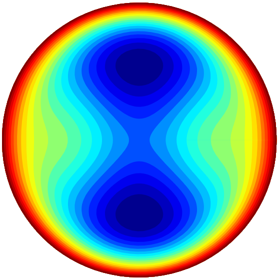
the two-dimensional disk for small E

Simple system: 2D disk. Think of the cross-section of a pipe.

For small energy E, exact solution in terms of Bessel functions $J_m(\rho_n)$, where ρ_n are zeros.

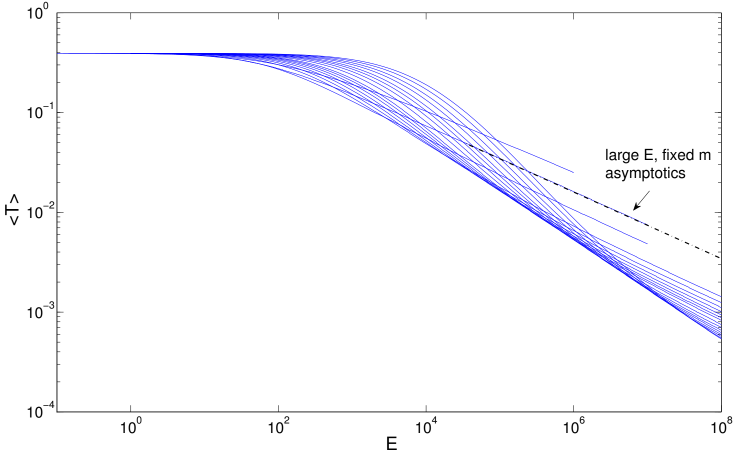
Pick the solution with largest transport: $m = 2, n = 1$:





asymptotics: large E case

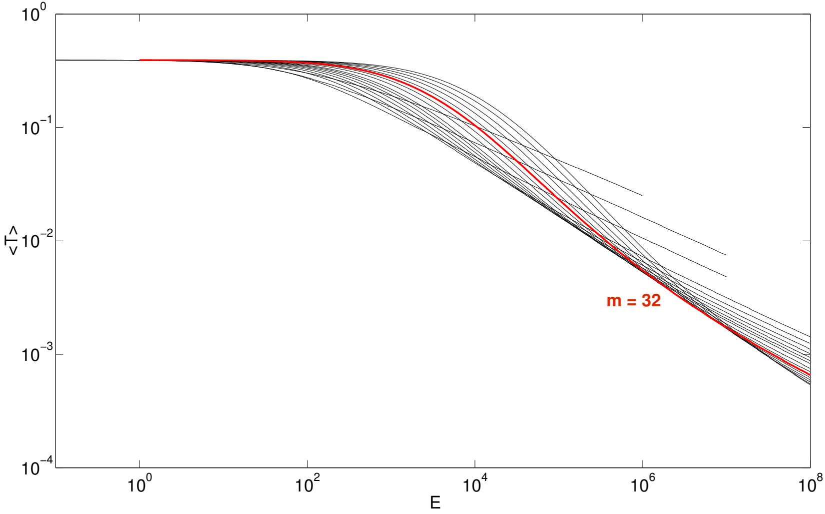
Numerical solution with **bvp5c** (Shampine, 2000), using a continuation method.



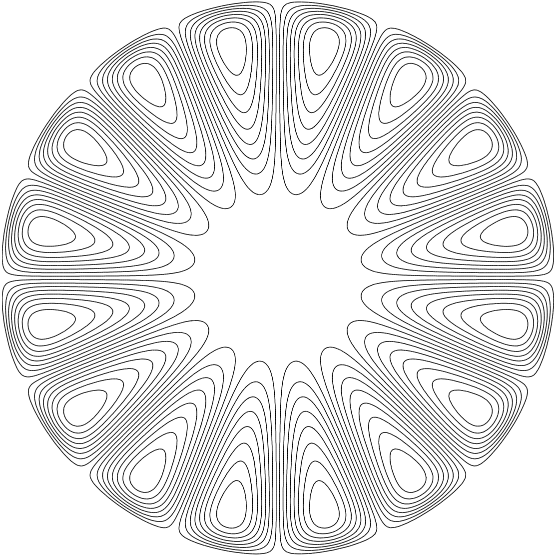
Asymptotics at large E, fixed m: $\langle \tau \rangle \sim m^{-2/3} E^{-1/3}$.

asymptotics: large E case (cont'd)

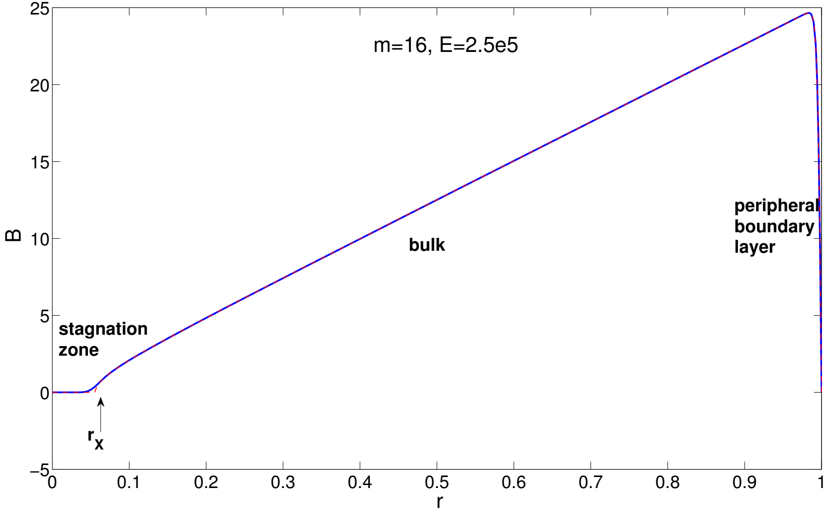
Optimal m at fixed energy E:



Penalty on large m: the "stagnation zone"



structure of the solution for large E



conclusions

- Transport in heat exchangers has a very different character than ‘freely-decaying’ problem.
- Using the probabilistic mean exit time formulation simplifies the problem. (Idea came from Iyer et al. 2010.)
- Optimal solutions for $\mathbf{u}$ are reminiscent of Dean flow.
- Optimal exit time at fixed flow energy shows increasing number of “cells” as energy increased.
- This is a pathology of fixing E. In future work we will fix viscous dissipation, which penalizes small structures.
- Generalizations: use different norms, spatial weight. . .