Shallow fluids meet Einstein

An experimental geodesic flow on a curved space

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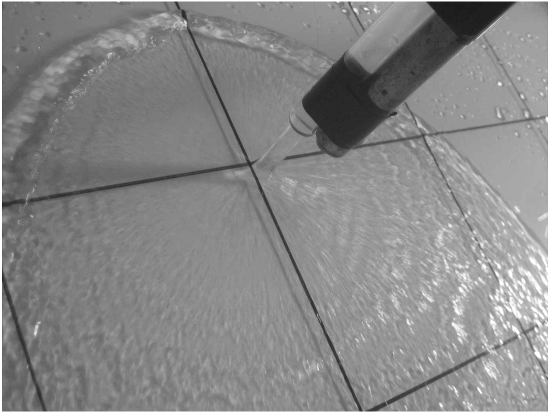
A jet hitting an inclined plane



Plane inclined at 45°. The flow rate is $Q \simeq 120\ \mathrm{cm}^3\,\mathrm{s}^{-1}$.

[with Andrew Belmonte in Claudia Cenedese and Karl Helfrich's lab at Woods Hole, GFD 2008]

Inviscid theory

Try steady potential flow: $\mathbf{u} = \nabla\varphi$, with

$$\begin{aligned} \nabla^2\varphi &= 0, && \text{mass conservation;} \\ \tfrac{1}{2}\left|\nabla\varphi\right|^2 + \frac{p}{\rho} - \mathbf{g}\cdot\mathbf{r} &= H, && \text{Bernoulli's law;} \end{aligned}$$

Boundary conditions:

$$\begin{aligned} \partial_z\varphi &= 0 && \text{at } z = 0, && \text{no-throughflow at substrate;} \\ \nabla\varphi\cdot\nabla h &= \partial_z\varphi && \text{at } z = h, && \text{kinematic condition at free surface;} \\ p &= 0 && \text{at } z = h, && \text{constant pressure at free surface.} \end{aligned}$$

Here z is normal to the substrate, x_1 and x_2 are parallel to it.

Small-parameter expansion

Expand Bernoulli's law in the small fluid depth ε:

$$\sum_{j=1}^{2}(\partial_j\varphi)^2 + \varepsilon^{-2}(\partial_z\varphi)^2 + \frac{2p}{\rho} - 2\mathbf{g}\cdot(\mathbf{X} + \varepsilon\, z\,\hat{\mathbf{e}}_3) = 2H,$$

where $\mathbf{X} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2$. Also expand φ:

$$\varphi(x_1, x_2, z) = \varphi_{(0)} + \varepsilon\,\varphi_{(1)} + \varepsilon^2\,\varphi_{(2)} + \dots,$$

to obtain at leading order $\partial_z\varphi_{(0)} = 0$, so that

$$\varphi_{(0)} = \Phi(x_1, x_2).$$

Small-parameter expansion II

At next order:

$$\sum_{j=1}^{2}(\partial_j\Phi)^2 + (\partial_z\varphi_{(1)})^2 + \frac{2p}{\rho} - 2\mathbf{g}\cdot\mathbf{X} = 2H,$$

Evaluate at $z = h$ and use the boundary conditions:

$$\sum_{j=1}^{2}(\partial_j\Phi)^2 - 2\mathbf{g}\cdot\mathbf{X} = 2H,$$

Solution in terms of characteristics

Differentiate to get rid of constant:

$$\sum_{j=1}^{2} \partial_j \Phi \, \partial_{ij} \Phi = \mathbf{g} \cdot \partial_i \mathbf{X} \,, \qquad i = 1, 2.$$

Introduce the characteristics $x_1(\tau)$, $x_2(\tau)$:

$$\dot{x}_1 = \partial_1 \Phi(\mathbf{x}), \qquad \dot{x}_2 = \partial_2 \Phi(\mathbf{x}),$$

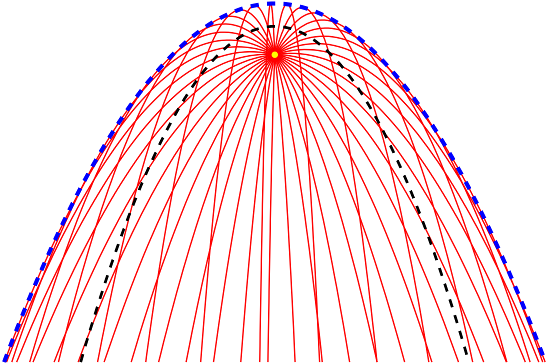
We have $\partial_{ij}\Phi = \partial_i \dot{x}_j$ and $\ddot{x}_i = (\partial_j \dot{x}_i)\dot{x}_j = \partial_j \Phi \, \partial_{ij} \Phi$, so that

$$\ddot{x}_i = \mathbf{g} \cdot \hat{\mathbf{e}}_i \,, \qquad i = 1, 2.$$

[Rienstra (1996)]

Characteristics for a jet striking an inclined plane

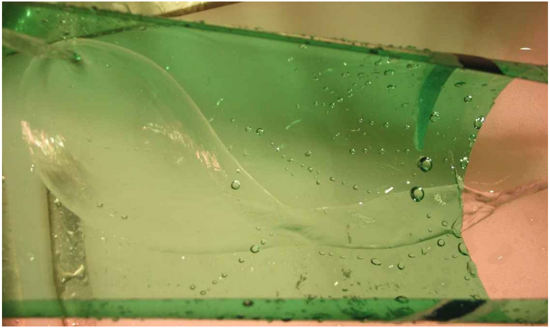
The characteristics have a parabolic envelope (blue dashed):



Edwards *et al.* (2008) used the 'delta-shock' framework to account for characteristics crossing: this lowers the rise distance by $5/9$, and the profile remains essentially parabolic (black dashes).

Curved substrates

Rienstra (1996) also applied his inviscid model to curved surfaces (spheres, cylinders). Here's my attempt at an experiment [Thiffeault & Kamhawi (2008)]:



Compare to characteristics on a cylinder:

General curved substrates

Rienstra (1996) treated surfaces with global orthogonal coordinates (plane, cylinder, sphere).

What about more general surfaces?

Write x^1, x^2 for general 2D coordinates that locate a point on the substrate. A small-thickness expansion similar to Rienstra's yields for the characteristics [Thiffeault & Kamhawi (2008)]:

$$\ddot{x}^\sigma + \Gamma^\sigma_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = \mathbf{g}\cdot\mathbf{e}^\sigma$$

where $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols for the shape of the substrate.

This is the geodesic equation with a gravitational forcing. The fluid particles (characteristics) are trying to follow straight lines, but their trajectories are bent by the substrate curvature and gravity.

Chaos?

The geodesic equations are actually a fourth-order autonomous system.

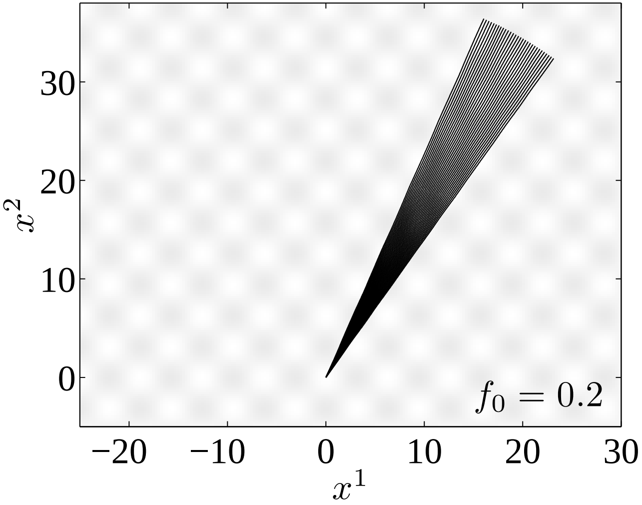
Hence, chaos is a possibility, as long as the substrate does not possess a continuous symmetry! (Ruled out for plane, cylinder, sphere.)

Consider a simple substrate shape parametrized by:

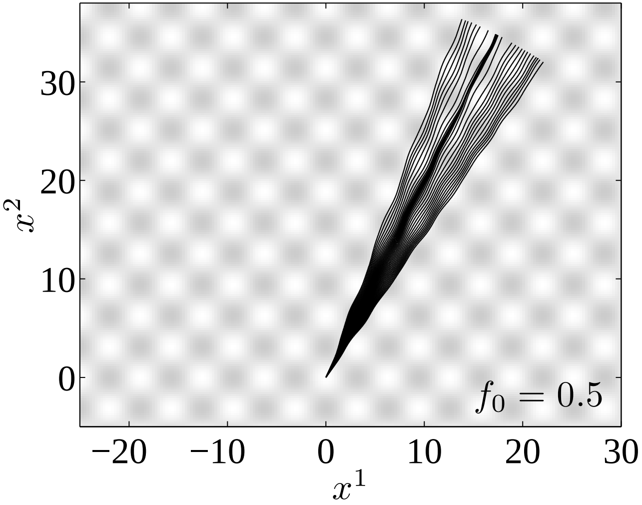
$$f(x^1, x^2) = f_0 \cos x^1 \cos x^2$$

Horizontal substrate: $f_0 = 0.2$

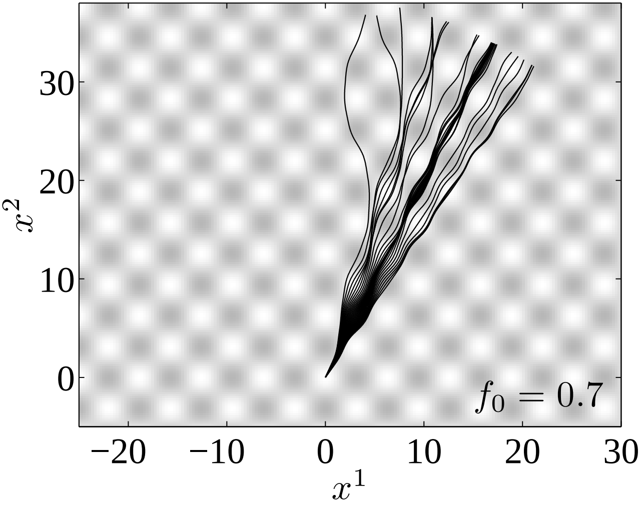
First take $g = 0$ and keep the surface horizontal.



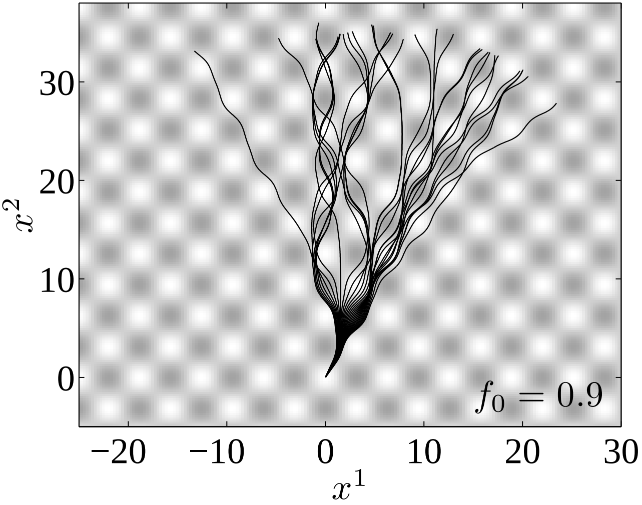
Horizontal substrate: $f_0 = 0.5$



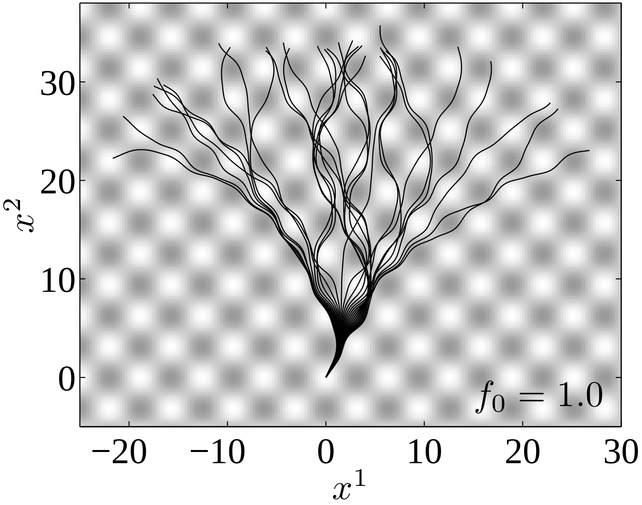
Horizontal substrate: $f_0 = 0.7$



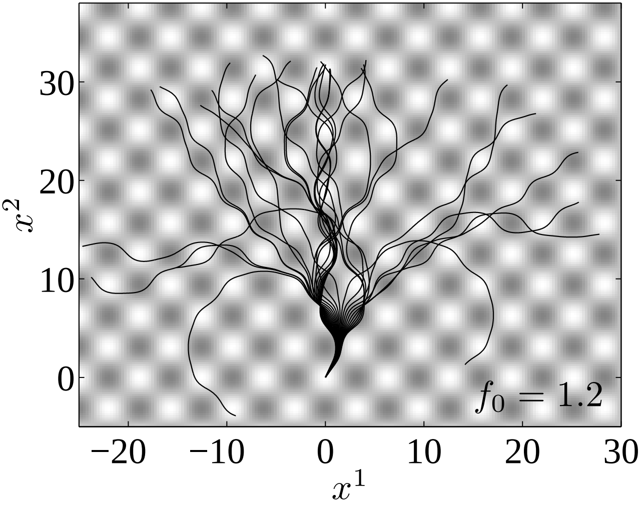
Horizontal substrate: $f_0 = 0.9$



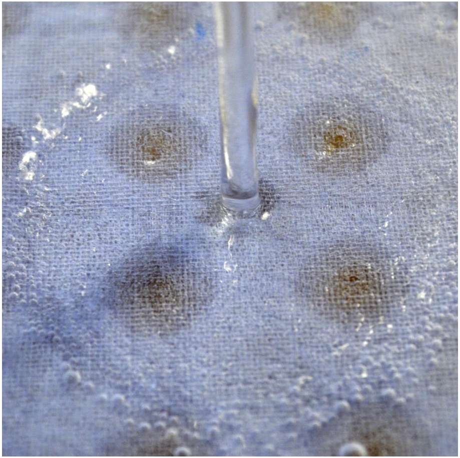
Horizontal substrate: $f_0 = 1.0$



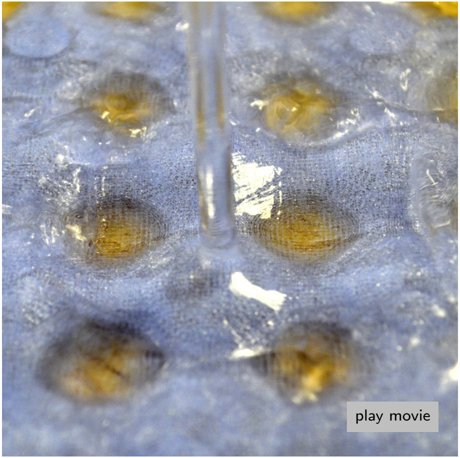
Horizontal substrate: $f_0 = 1.2$



Experiments with 3D-printed substrate



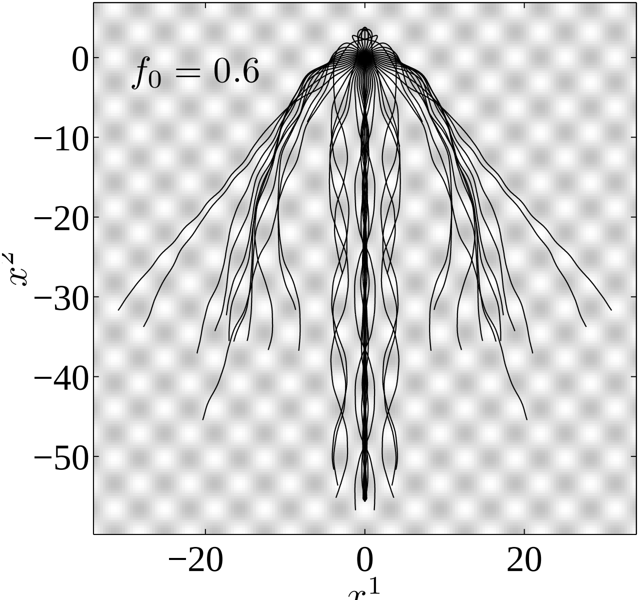
Flat substrate



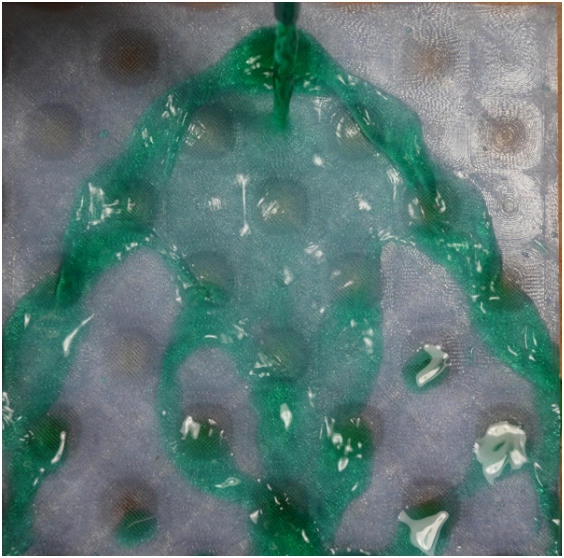
Patterned substrate

Jump is about 50% larger for a flat substrate. [Experiments with Jay Johnson.]

Inclined substrate: $f_0 = 0.6$



Experiments: an inclined substrate



The simple model correctly predicts the multiple 'paths'.

References

Edwards, C. M., Howison, S. D., Ockendon, H., & Ockendon, J. R. (2008). *IMA J. Appl. Math.* **73** (1), 137–157.

Rienstra, S. W. (1996). *Z. Angew. Math. Mech.* **76** (S5), 423–424.

Thiffeault, J.-L. & Kamhawi, K. (2008). In: *Chaos, Complexity, and Transport: Theory and Applications*, (Chandre, C., Leoncini, X., & Zaslavsky, G., eds) pp. 40–54, Singapore: World Scientific.