Topological detection of Lagrangian coherent structures

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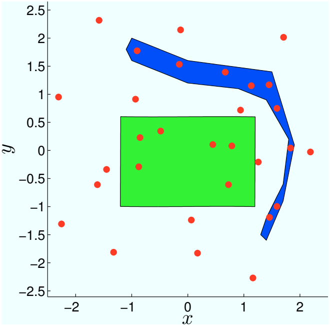
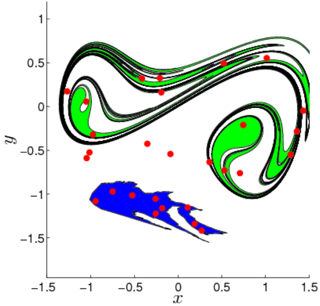
Sparse trajectories and material loops





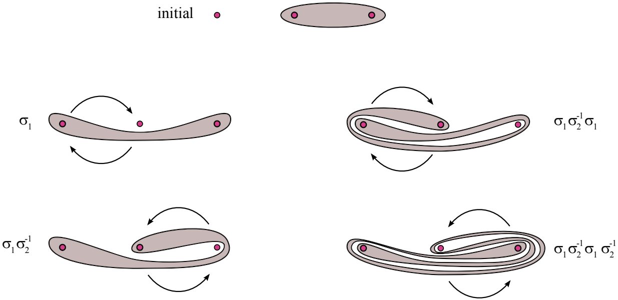
How do we efficiently detect trajectories that ‘bunch’ together?

This is the central problem for the detection of barriers to transport, or Lagrangian coherent structures (LCS).

[movie 1]

Growth of curves with moving obstacles

With 3 obstacles (floats), we can also look at the growth of curves:



The motion above is denoted $\sigma_1\sigma_2^{-1}$.

The rate of growth of the loop is called the topological entropy.

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

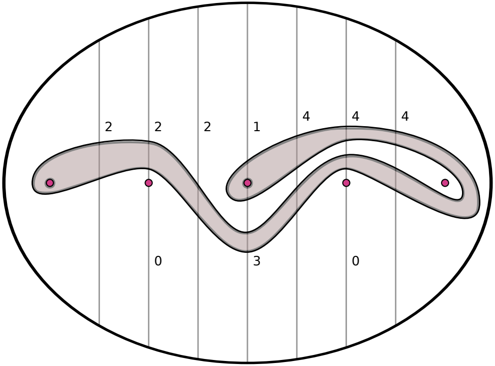
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

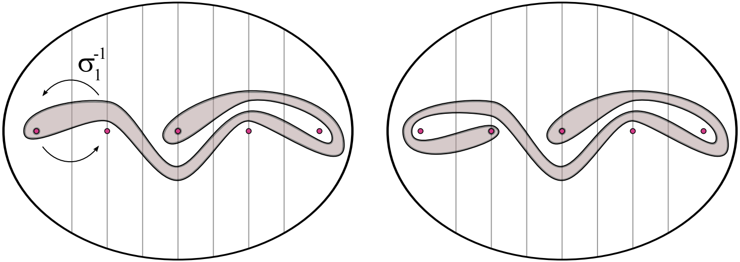
Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



Action on coordinates

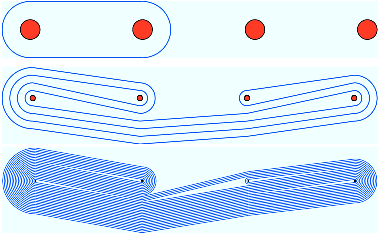
Moving the obstacles changes some crossing numbers:

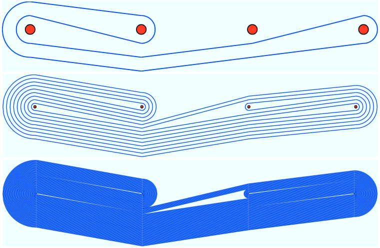


There is an explicit formula for the change in the coordinates!
(Dynnikov, 2002; Moussafir, 2006; Thiffeault, 2010)

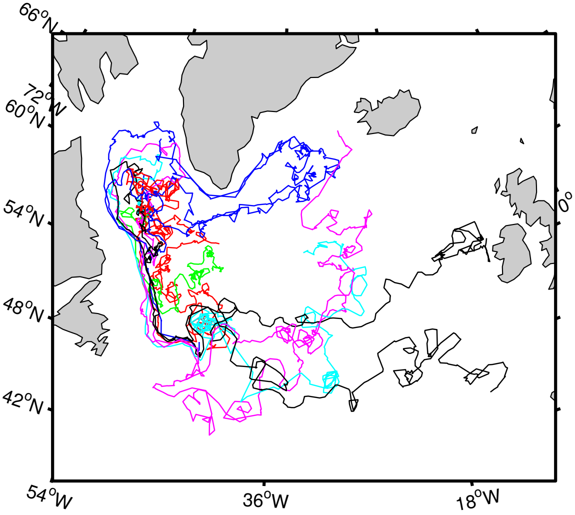
Growth of loop length

For a specific rod motion, we can easily see the exponential growth of L and thus measure the entropy:



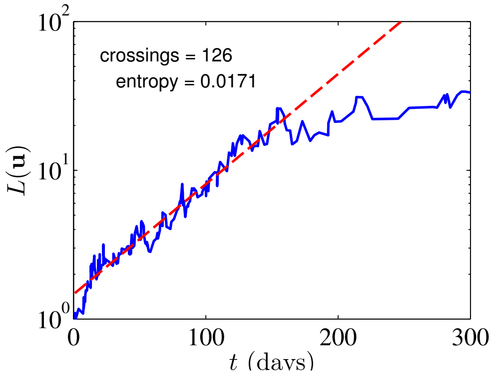


Oceanic float trajectories



Oceanic floats: Entropy

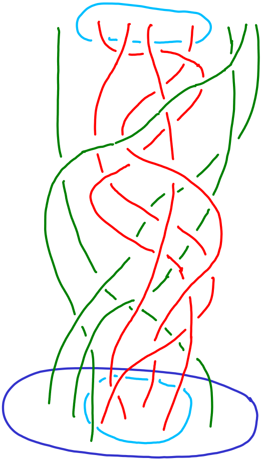
10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

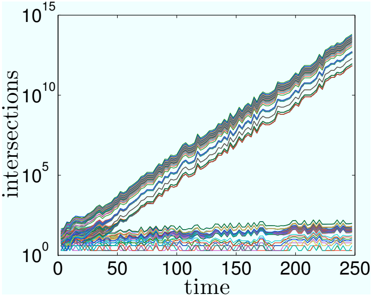
Source: WOCE subsurface float data assembly center (2004)

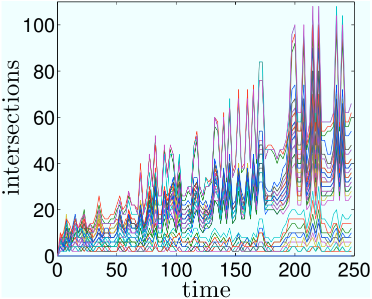
Lagrangian Coherent Structures



- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not 'leaky.'

Growth of a vast number of loops



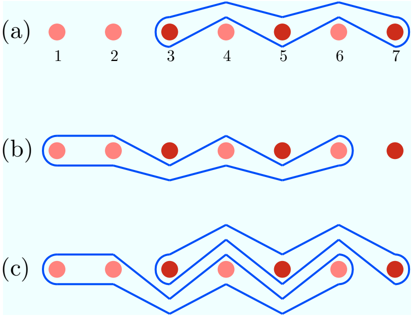


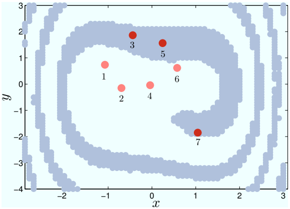
Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops: fast and slow-growing

What do the slowest-growing loops look like?

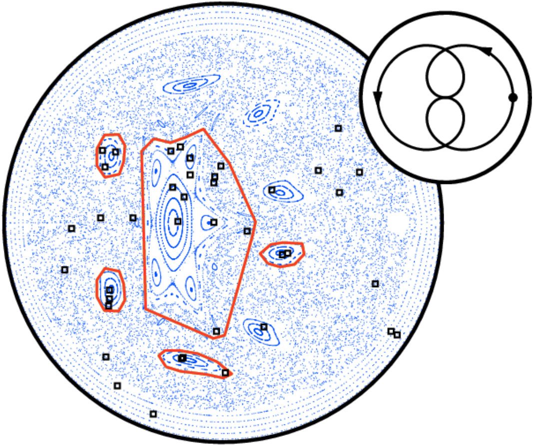
The slowest-growing loops surround bunches of trajectories that travel together (remain in the same ergodic component):





[(c) appears because the coordinates also encode 'multiloops.']

A physical example: Rod stirring device



[movie 2]

Conclusions

- Chaotic trajectories undergo 'braiding' motion that leads to growth of 'topological loops.' (crude Lyapunov exponent)
- Need a way to compute entropy fast: loop coordinates;
- There is a lot more information in this braid: extract invariant regions (related to Lagrangian coherent structures);
- Currently refining the technique, and applying to float data in the ocean as well as granular particle data (with K. Daniels, J. Puckett, and F. Lechenault).
- See Thiffeault (2005, 2010) and new paper by Allshouse & Thiffeault (*Physica D*, in press; arXiv:1106.2231).

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