Topology of Chaotic Mixing Patterns

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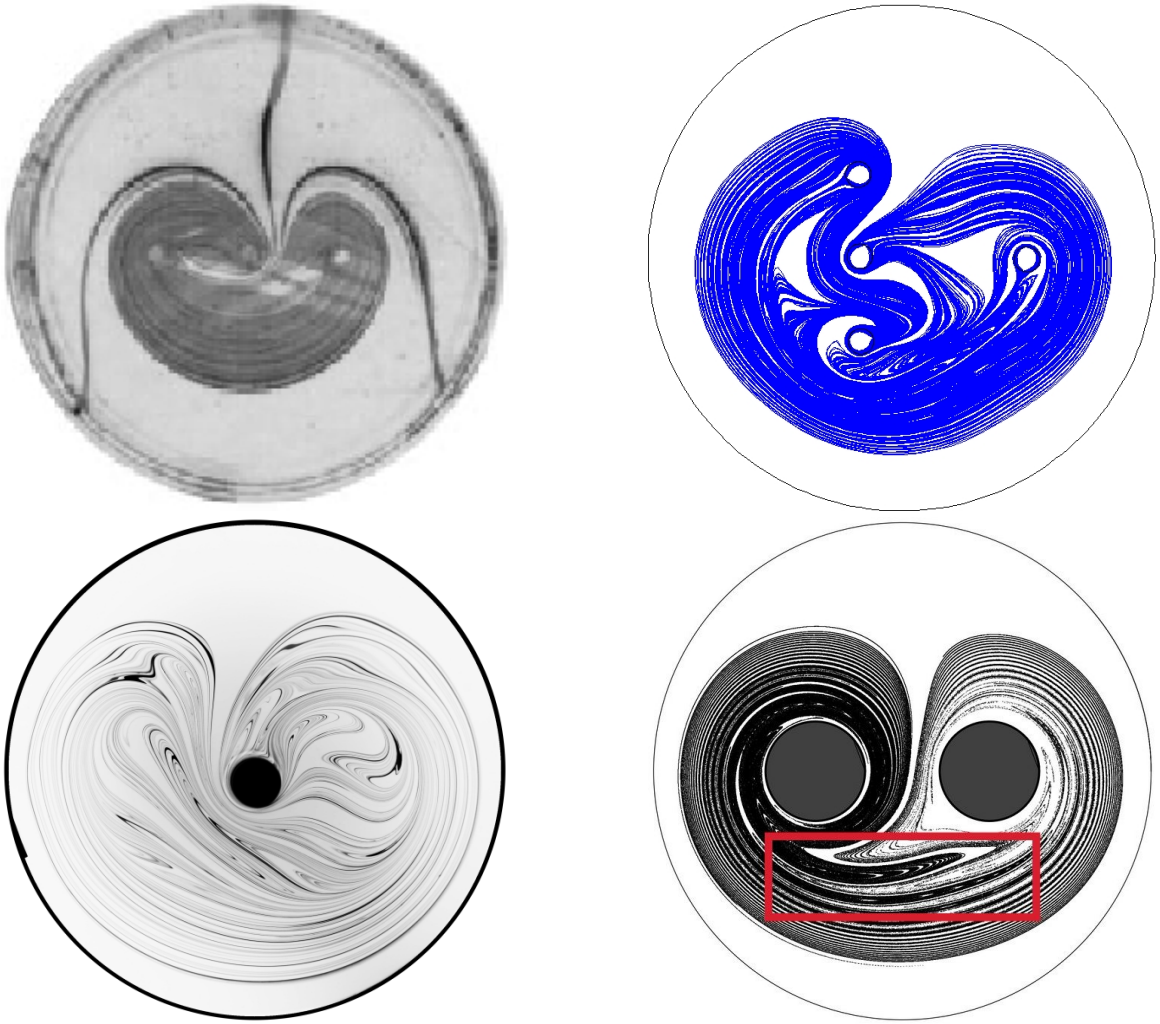
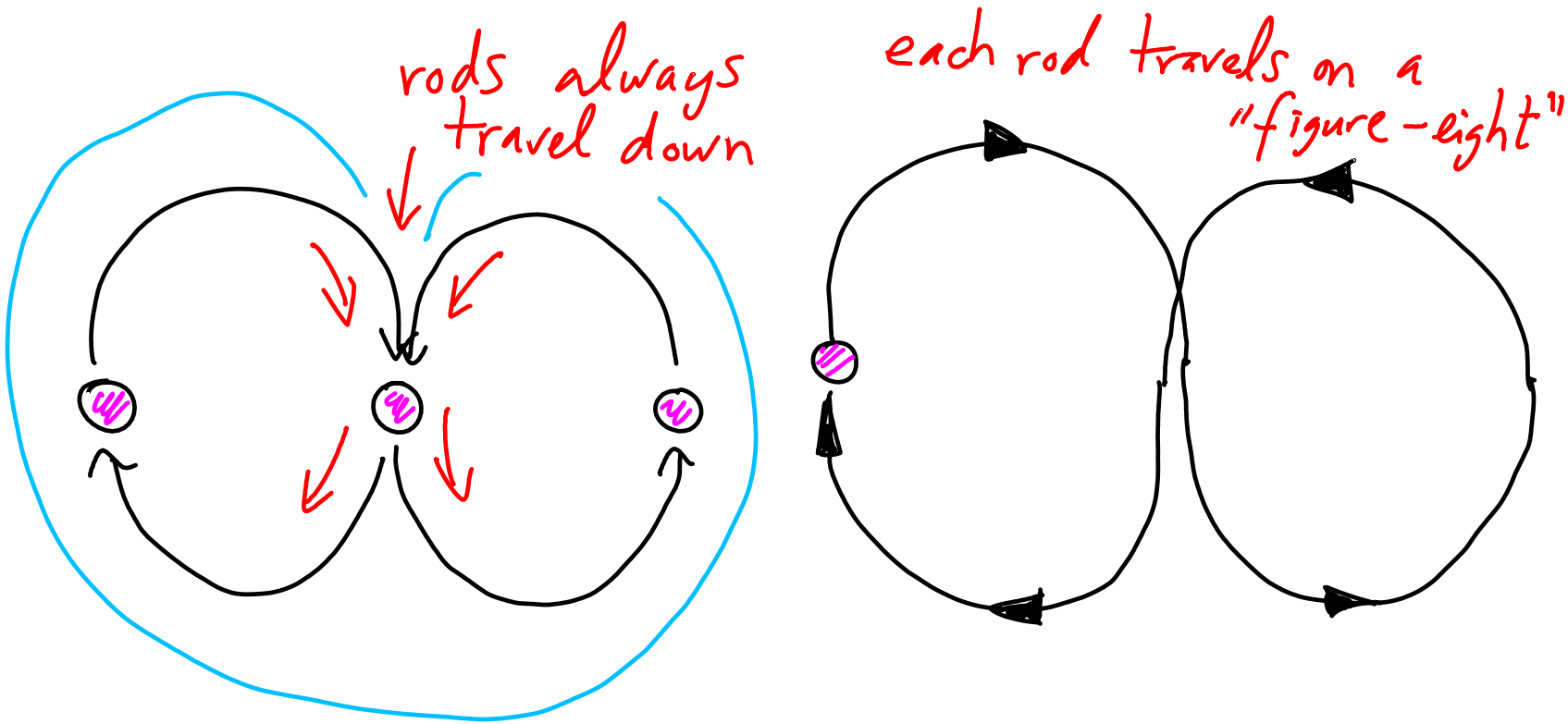
Heart-shaped stirring patterns



(Boyland et al., 2000; Gouillart et al., 2007, 2008; Thiffeault et al., 2008)

Hearts arise from "figure-eight" motion

For the Boyland et al. (2000) stirring protocol:



The figure-eight motion imposes a minimum amount of "complexity" in the flow — topological chaos.

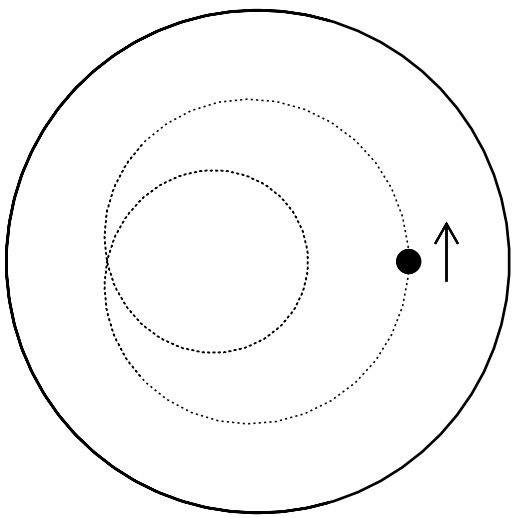
But where's the "figure-eight" here?

Eggbeater

Viscous blinking vortex (Aref, 1984; Jana et al., 1994)

Ghost rods: Periodic orbits that stir

When trying to explain the stretching observed in a simulation, physical rods are usually not enough:



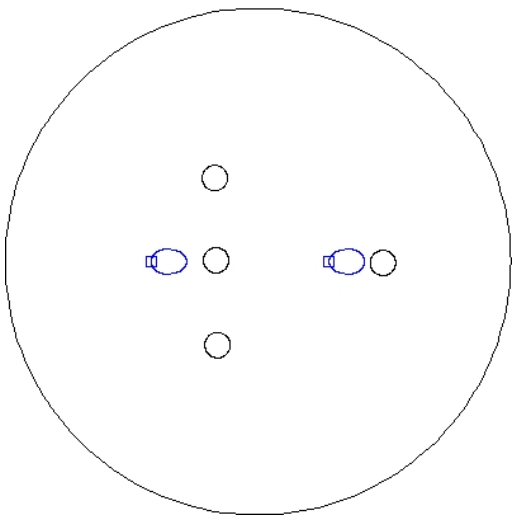
'ghost rods'

solid rods

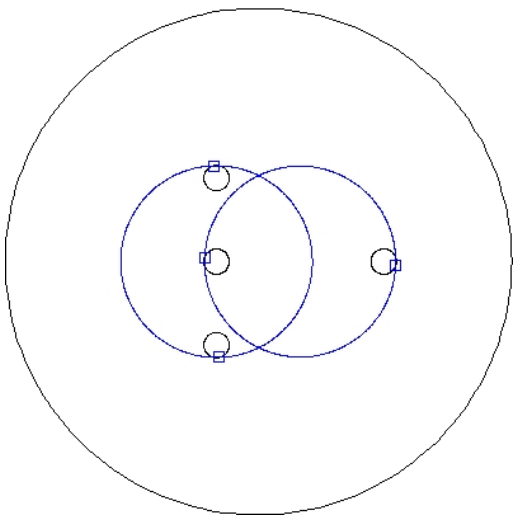
(Gouillart et al., 2006; Stremler & Chen, 2007; Binder & Cox, 2008; Thiffeault et al., 2008)

So where are the ghost rods?

material line

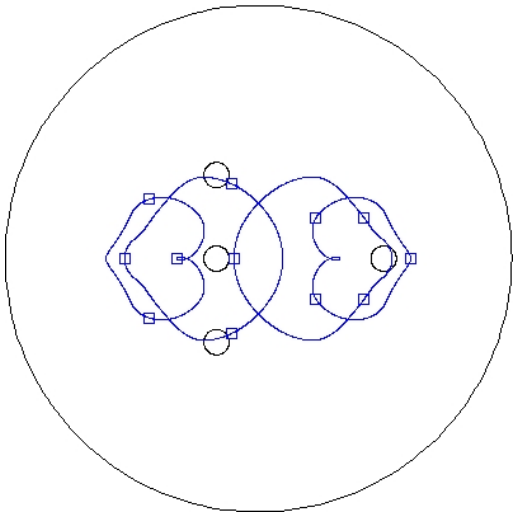


2 period-1 orbits

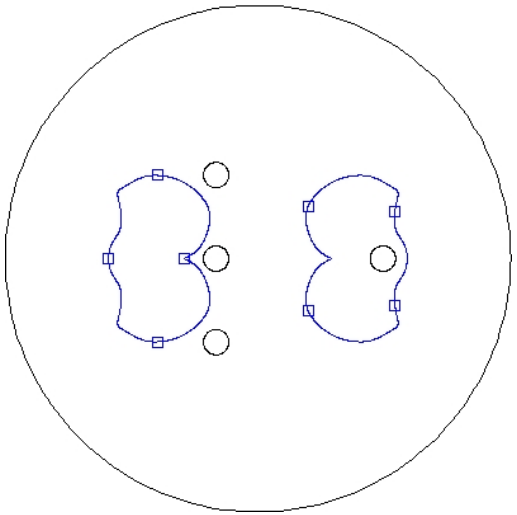


2 period-2 orbits

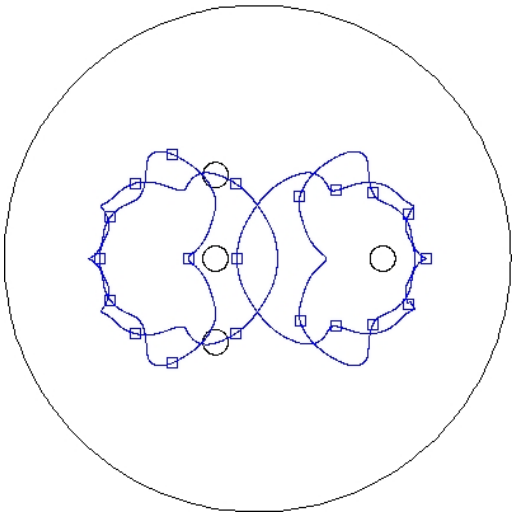
From period 3 to 8



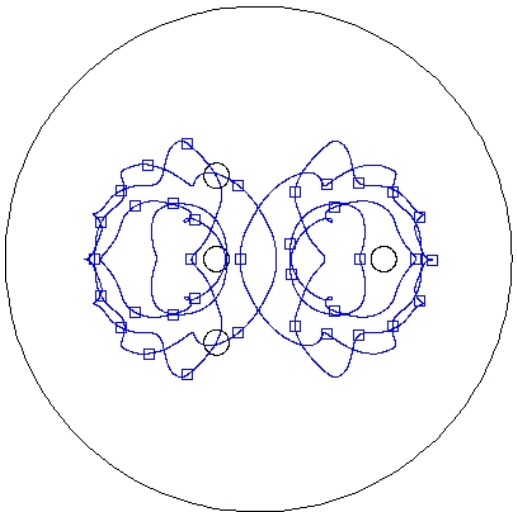
4 period-3 orbits



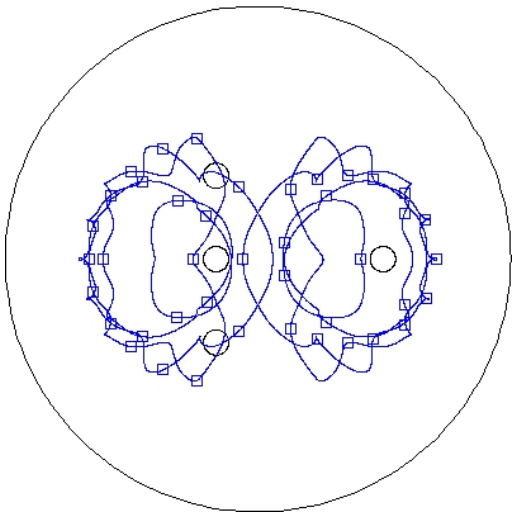
2 period-4 orbits



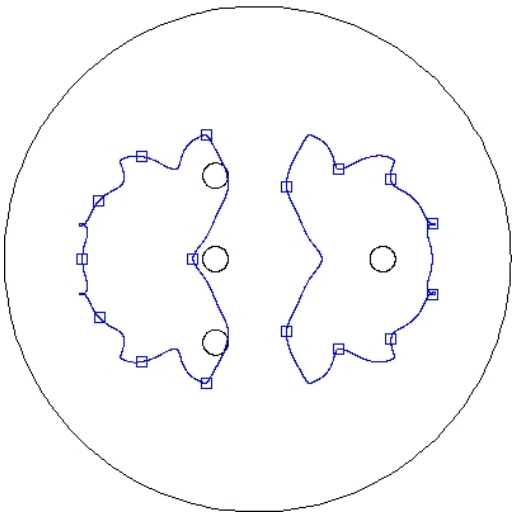
4 period-5 orbits



6 period-6 orbits



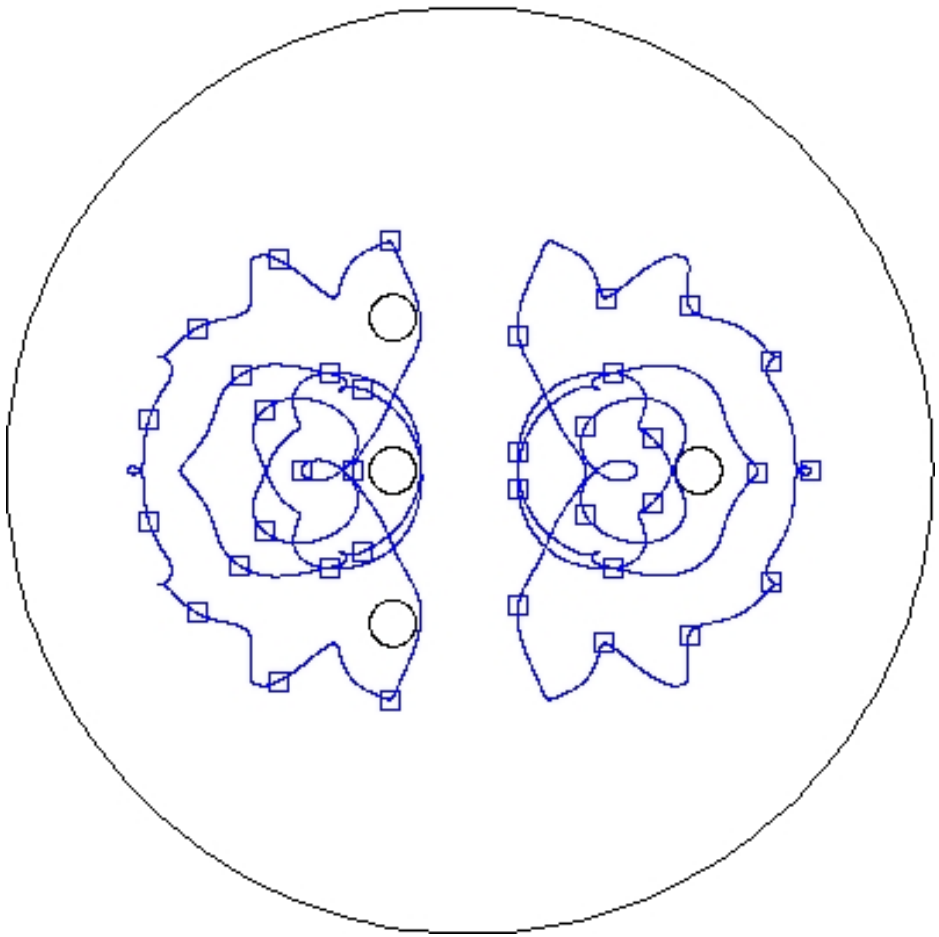
6 period-7 orbits



2 period-8 orbits

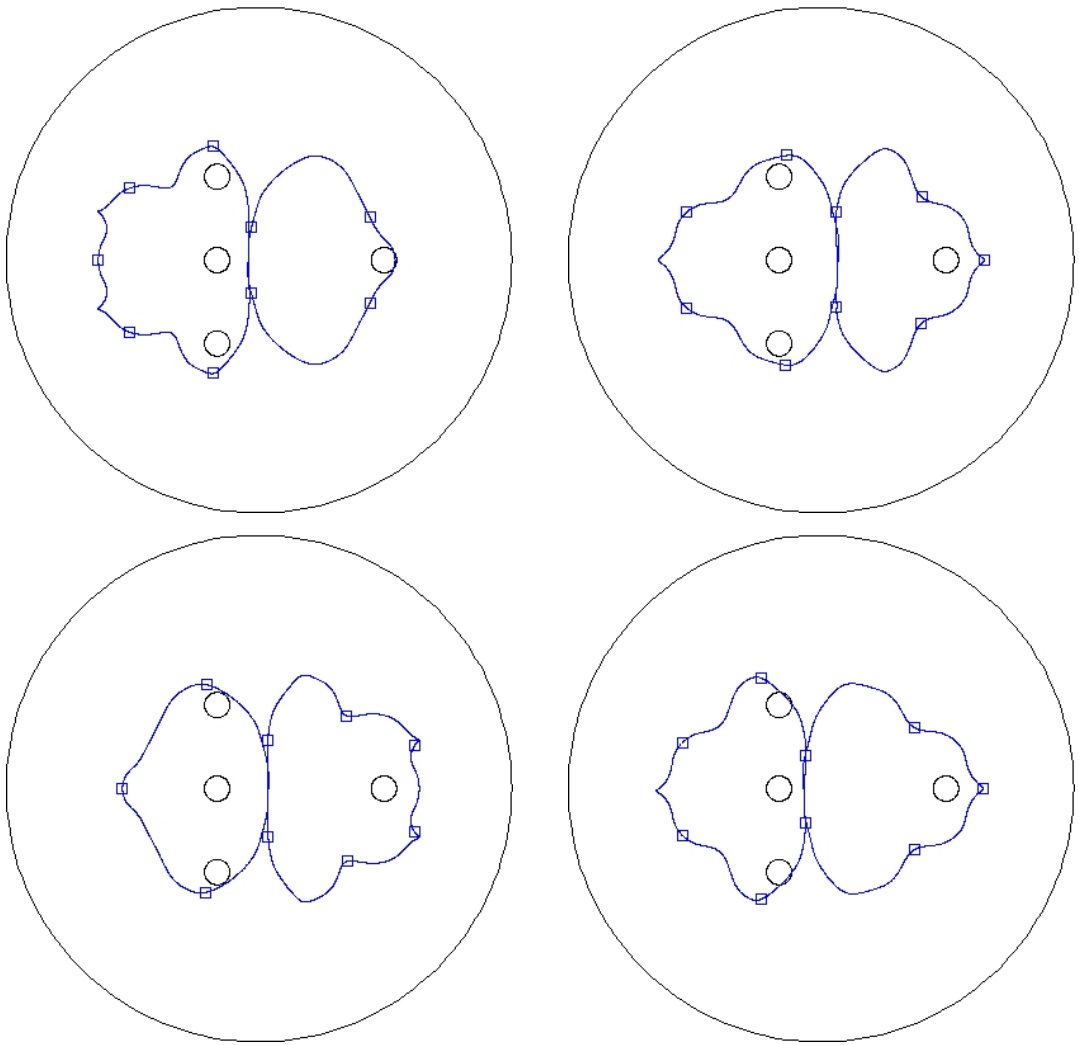
Period 9

8 period-9 orbits: 4 of the same type as before. . .



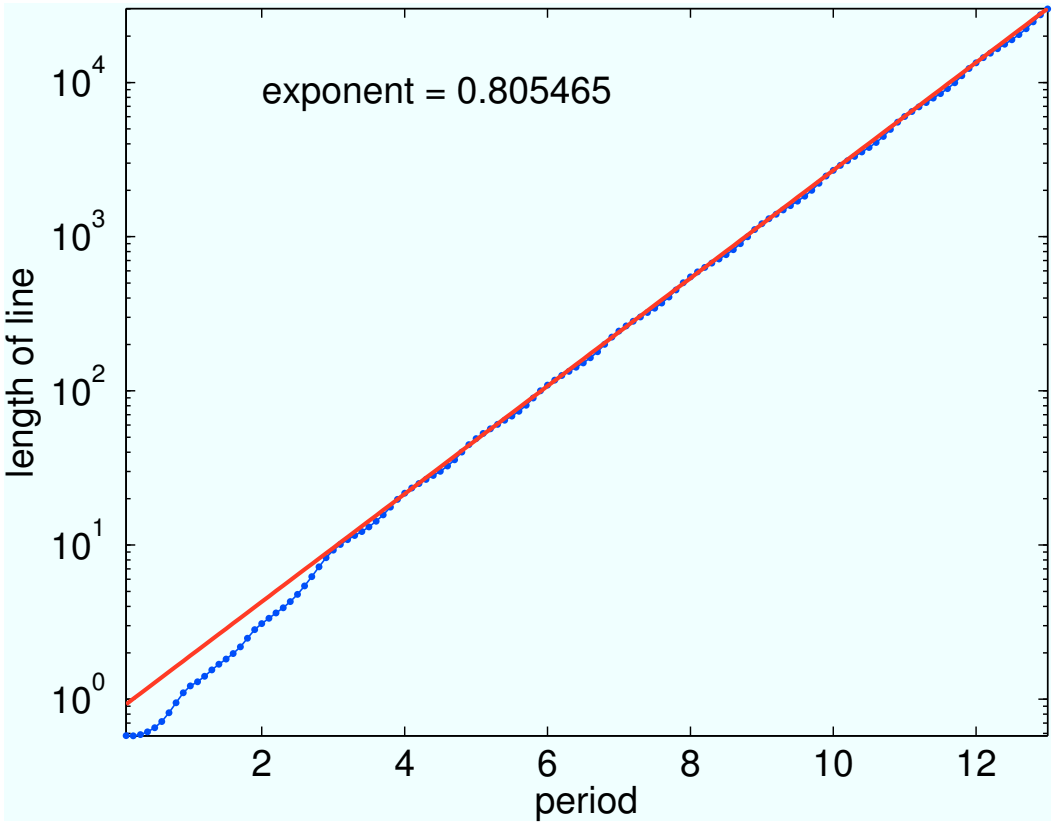
Period 9: Figure-eight orbits!

. . . and 4 new ones



[movie 1]

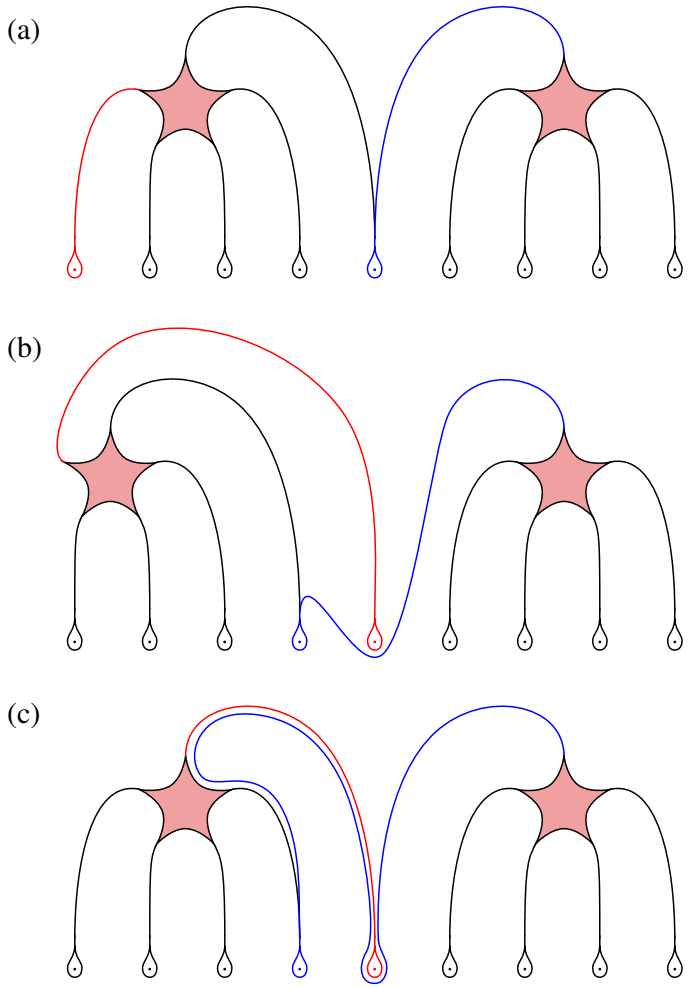
Growth rate of material lines



- The blue curve is the length of a material line;
- The red curve is the 'braid entropy' of the four period-9 figure-eight orbits.

The braid entropy is the minimum stretching rate imparted on material lines if the periodic orbits were 'rods'.

Calculating the braid entropy



The topological properties of a periodic orbit are computed using 'train tracks'.

On the left, the train track for a period-9 orbit is transformed according to the motion. The lines are proxies for material lines, and the stars are called pronged singularities.

When the dust settles, this braid's entropy is the log of the largest root of

$$1 - x - 4x^5 - x^9 - x^{10}$$

or $\simeq 0.4747$.

Conclusions

- Heart-shaped mixing regions prevalent;
- Can all be traced back to figure-eight stirring involving one or more rods;
- Usually these 'rods' are periodic orbits;
- Remarkably, such orbits of least period often capture all the topological properties of the device;
- A general topological theory of figure-eight orbits? What are the most desirable features for mixing?

References

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