Topological Optimization of Rod Mixers

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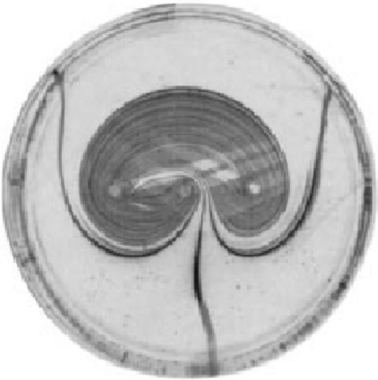
APS-DFD Meeting, 20 November 2006

Experiment of Boyland, Aref, & Stremler

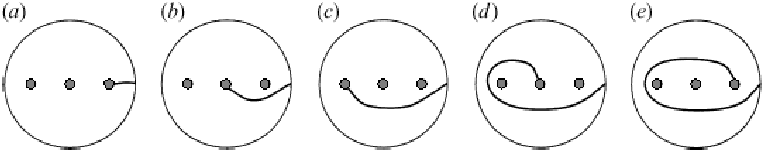


- Three rods, only 'moves' allowed is interchange.
- Two protocols have radically different properties.

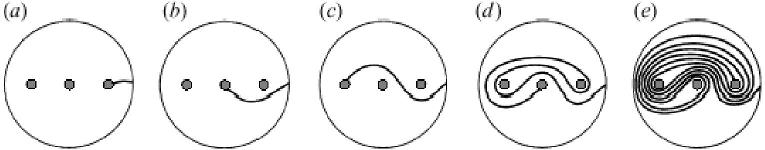
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol

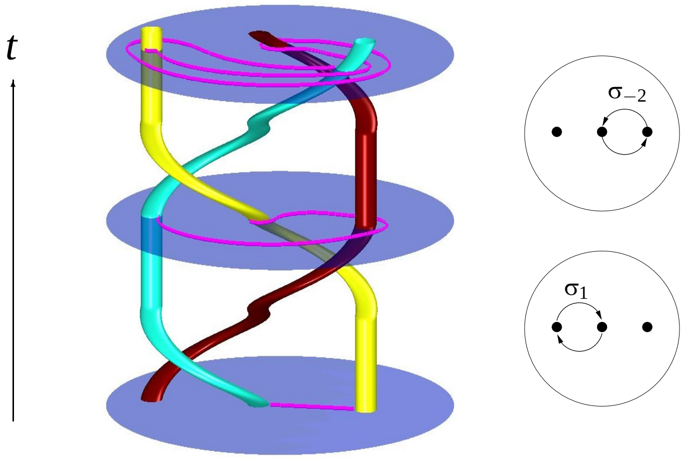


$\sigma_1^{-1}\sigma_2$ protocol



[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

The Connection with Braids



Picture from [E. Gouillart, M. D. Finn, and J.-L.Thiffeault, *Phys. Rev. E* **73**, 036311 (2006)]

Optimal Braids

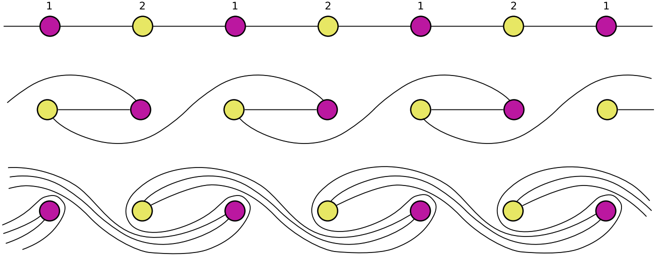
- The stretching of material lines is bounded from below by the braid's topological entropy.
- D'Alessandro et al. (1999) showed that $\sigma_1\,\sigma_2^{-1}$ is optimal for 3 rods.
- This means that it has the most entropy per generator, in this case equal to $\log\phi$, where ϕ is the Golden Ratio.
- For $n > 3$ rods, all we have are conjectures (Thiffeault & Finn, 2006; Moussafir, 2006):
    - For $n = 4$, the optimal braid is $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$, also with entropy per generator $\log\phi$;
    - For $n > 4$, the entropy per generator is always less than $\log\phi$.

The Right Optimality?

- Entropy per generator is interesting, but does not map to physical situations very well:
- Simple rod motions can correspond to too many generators.
- In practice, need generators that are more naturally suited to the mechanical constraints.
- Another problem is that in practical sitations it is desirable to move many rods at once.
- Energy constraint not as important as speed and simplicity.
- $\sigma_1 \sigma_2^{-1}$ not so easy to realize mechanically, though see Binder & Cox (2007) and Kobayashi & Umeda (2006).

Solution: Rods in a Circle

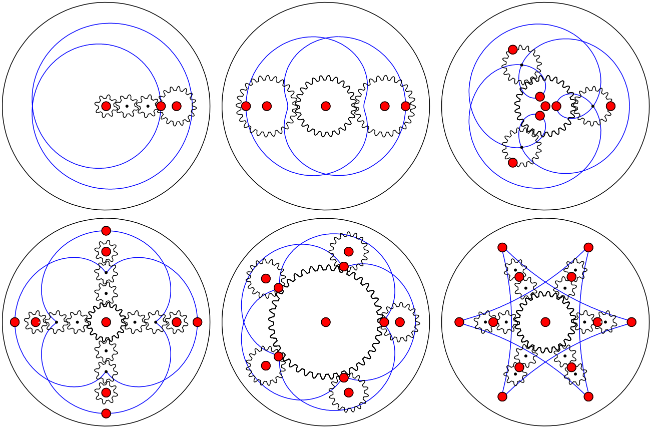
- A mixer design consisting of an even number of rods in a circle.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor.



- The entropy per 'switch' is $\log \chi$, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

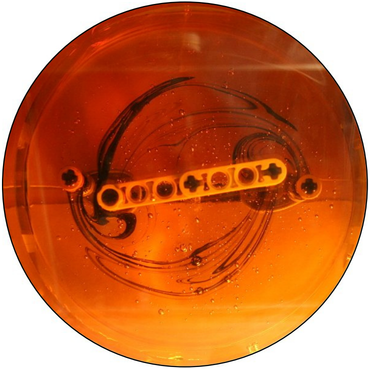
Silver Mixers!

- Even better: the designs with entropy given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



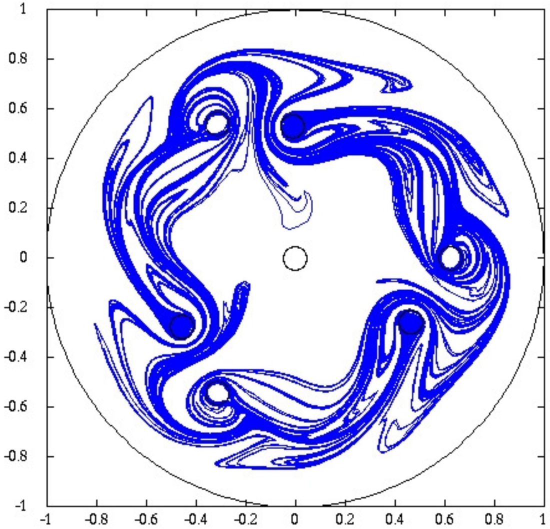
[movie 1]

Four Rods



[movie 2] [movie 3]

Six Rods



[movie 4]

Conclusions

- Topological entropy is a lower bound on the growth rate of material lines, a useful measure of mixing efficiency.
- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy.
- Design based on *n* rods arranged in a circle gives $n \log \chi$ entropy per full period, where χ is the silver ratio, $1 + \sqrt{2}$.
- …hence the name silver mixers.
- Can realize using simple gears; however, 4 and 6 rods work best.
- Need to tweak to optimize other mixing measures, such as variance decay rate.

References

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