Bubbles and Filaments:
Stirring a Cahn–Hilliard Fluid

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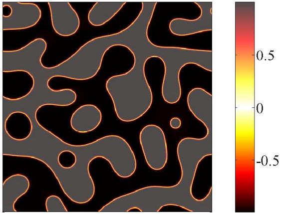
The classical Cahn–Hilliard equation

In the absence of flow, the Cahn–Hilliard equation models phase separation,

$$\frac{\partial c}{\partial t} = D \nabla^2 \left(c^3 - c - \gamma \nabla^2 c \right)$$

where c is the concentration field, D is the diffusion coefficient and $\sqrt{\gamma}$ is the hyperdiffusion length.



The solution is $c = \pm 1$ in domains with transition regions of thickness $\sqrt{\gamma}$ in between. The domains grow in time. The constant solution $c = 0$ is a well-mixed state but it is unstable.

The stirred Cahn–Hilliard equation

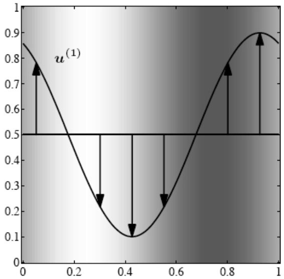
- The passive stirring a phase separted fluid is modelled by an advective term in the CH equation,

$$\frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = D \nabla^2 \left(c^3 - c - \gamma \nabla^2 c \right) .$$

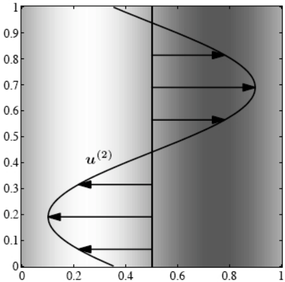
- This introduces competition between the stirring term, $\mathbf{v} \cdot \nabla c$, the desegregation terms, and the hyperdiffision $\gamma \nabla^4 c$ which limits the size of small scales.
- Two co-existing regimes are identified, depending on the strength of the stirring: Bubbles and filaments.

A model stirring flow

- Alternating horizontal and vertical sine shear flows:



first half-period



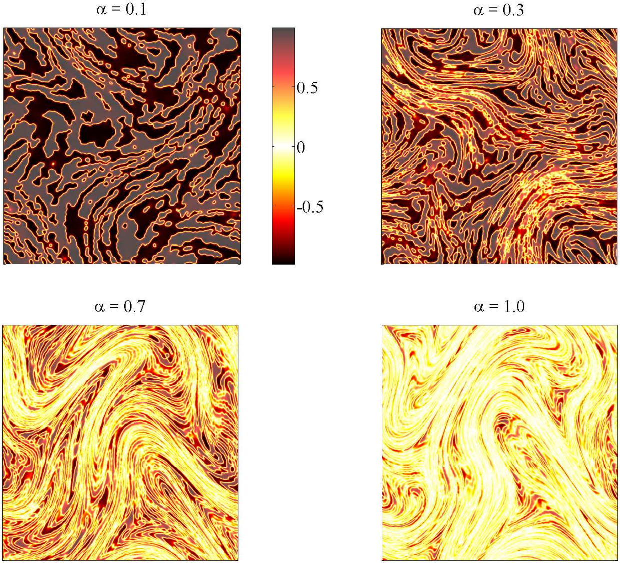
second half-period

- Mimics the effect of turbulence at large Prandtl number.
- Phase selected randomly for each period.
- The coefficient α measures the strength of stirring.
- The velocity field has a Lagrangian timescale given by the Lyapunov exponent λ, with $\lambda = 0.118\,\alpha^2$ for small α.

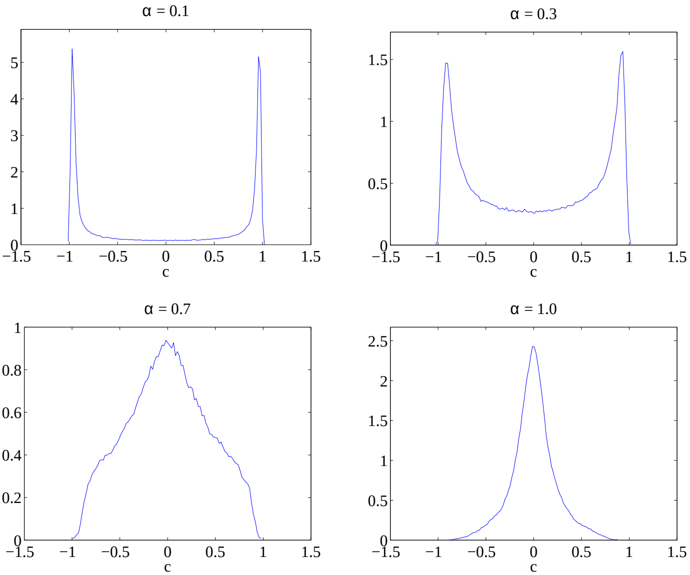
The Effect of Stirring

- A steady state is always achieved, owing to the balance between the advection and the CH terms in the equation.
- For small α the steady state comprises domains of constant size, while for larger α the mixed state is favoured.
- The domain growth always saturates — coarsening arrest.
- Previous work focused on arrest, but we study the breakup of domains and subsequent mixing due to vigorous stirring. [See for example Berti et al. (2005); Berthier et al. (2001).]

From bubbles to filaments



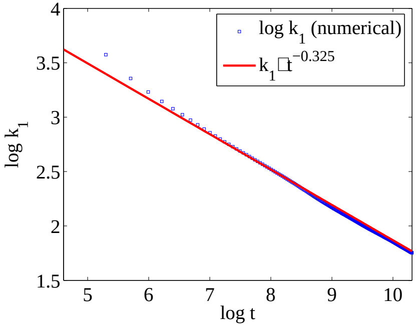
From bubbles to filaments: PDFs of concentration



Lifshitz–Slyozov Law

- The unstirred CH equation has late-time morphology that is independent of time when lengths are measured in terms of the typical bubble size, $R_{\mathrm{b}}(t)$.
- Theory (Lifshitz & Slyozov, 1961) and numerical simulations (Zhu et al., 1999) indicate the scaling law,

$$k_1^{-1} = R_{\mathrm{b}} \sim t^{1/3}.$$



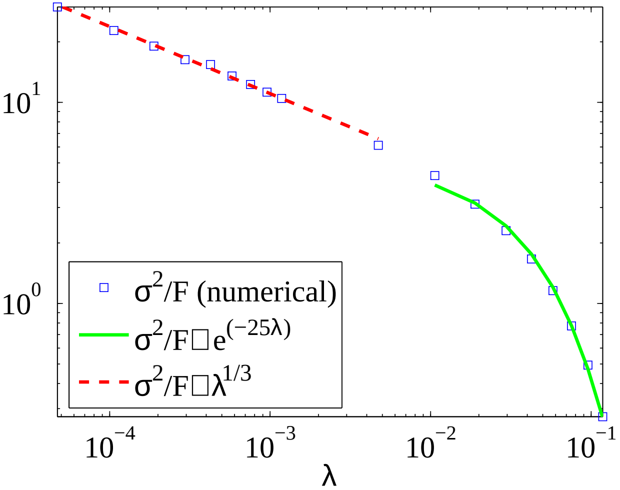
Scaling law for stirred fluids

- Simple arguments show that for moderate stirring amplitudes, the quantity σ^2/F is a proxy for R_{b}.
- Here σ^2 is the variance of the concentration field and F is the free energy, $F = \int d^2x \left[\frac{1}{4}\left(c^2-1\right)^2 + \frac{1}{2}\gamma\left|\nabla c\right|^2\right]$.
- Thus, for small α, we find

$$\sigma^2/F \sim \lambda^{1/3},$$

while for large α, σ^2/F falls off exponentially in λ, indicating the effectiveness of mixing at these amplitudes.

Scaling law for stirred fluids



- Performing the same simulation with a variable mobility (different LS exponents) gives a similar result.

Conclusions

- In a phase separating fluid, an imposed chaotic flow not only arrests domain formation, but overcomes it.
- For vigorous stirring, the phase separating fluid is therefore well-mixed.
- The morphology of the concentration field is characterized using the free energy and the variance.
- The numerical simulations suggest the existence of a critical stirring amplitude for mixing.
- However, in one-dimensional models, any amound of strain ($\lambda > 0$) destroys bubbles. Need a better theory to explain critical λ.

References

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