# Mixing in Thin Flows over a Curved Substrate

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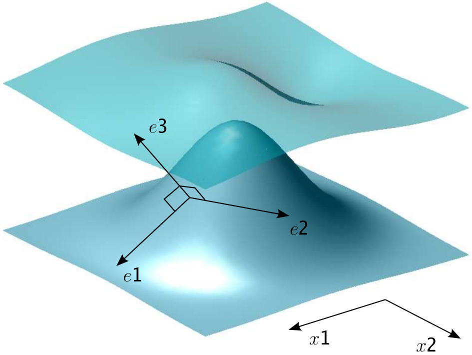
# Introduction

- A thin layer of fluid flowing down an inclined substrate.
- Reduce to two-dimensional problem by asymptotic expansion: PDE for the height field.
- But the velocity field is still three-dimensional, with a nontrivial vertical component.
- Steady three-dimensional flows can exhibit chaotic trajectories.
- This leads to fluid particles rapidly decorrelating: good for mixing.
- Can suitable substrate shapes lead to good horizontal mixing?

# Strategy

- Thin-layer expansion in the direction normal to the substrate.
- Similar derivation to [Roy, Roberts, and Simpson, *JFM* **454**, 235 (2002)].
- For simplicity, assume steady flow.
- Use non-orthogonal coordinates, since globally orthogonal coordinates are difficult to compute in general.
- Correct velocity field to satisfy kinematic constraints — this is crucial for particle advection.
- Integrate trajectories and make Poincaré sections in a spatially periodic domain.

# Coordinate System




Surface $y = \eta(x^1, x^2)$

Substrate $y = 0$
at position $\mathbf{X}(x^1, x^2)$

$$\mathbf{r}(x^1, x^2, y) = \mathbf{X}(x^1, x^2) + y\,\hat{\mathbf{e}}_3(x^1, x^2)$$

$$\mathbf{e}_\alpha = \frac{\partial \mathbf{X}}{\partial x^\alpha} = \partial_\alpha \mathbf{X}\,; \qquad \hat{\mathbf{e}}_3 = (\mathbf{e}_1 \times \mathbf{e}_2)/\,\|\mathbf{e}_1 \times \mathbf{e}_2\|$$

# Dynamical Equations

We now assume $\mathbf{u}$ satisfies the Stokes equation,

$$\Delta \mathbf{u} = \nabla p - \hat{\mathbf{g}}, \qquad \nabla \cdot \mathbf{u} = 0,$$

where $p$ is the pressure and $\hat{\mathbf{g}}$ is a unit vector in the direction of gravity. The velocity satisfies the boundary conditions

$$\begin{aligned} \mathbf{u} &= 0 && \text{at } y = 0 && \text{no-slip at substrate} \\ \mathbf{t}_\alpha \cdot \tau \cdot \hat{\mathbf{n}} &= 0 && \text{at } y = \eta && \text{tangential stresses at free surface} \\ -p + \hat{\mathbf{n}} \cdot \tau \cdot \hat{\mathbf{n}} &= \sigma \kappa_{\mathrm{surf}} && \text{at } y = \eta && \text{normal stress at free surface} \end{aligned}$$

where

$$\tau := \nabla \mathbf{u} + (\nabla \mathbf{u})^T$$

is the deviatoric stress, $\hat{\mathbf{n}}$ is the unit normal to the surface, $\mathbf{t}_\alpha$ are tangents to the surface, and $\kappa_{\mathrm{surf}}$ is the mean curvature of the surface. All quantities are dimensionless.

## Small-parameter Rescaling

The time has come to make the layer thin: we do this by assuming that horizontal scales vary slowly:

$$x^\alpha = \varepsilon^{-1}\, x^{\alpha *}, \quad v = \varepsilon\, v^*, \quad p = \varepsilon^{-1}\, p^*, \quad \sigma = \varepsilon^{-2}\, \sigma^*.$$

Everything else is of order unity, including vertical scales. We immediately drop the * superscripts, and expand the fields as

$$\begin{aligned} u^\alpha &= u^\alpha_{(0)} + \varepsilon\, u^\alpha_{(1)} + \ldots, \\ p &= p_{(0)} + \varepsilon\, p_{(1)} + \ldots. \end{aligned}$$

Note that we leave $v$ unexpanded (see later).

# The Mass Flux

The horizontal velocity field is sufficient to find the mass flux,

$$q^\alpha = \int_0^\eta \omega\, u^\alpha dy = \int_0^\eta (1 - \varepsilon\, \kappa\, y) u^\alpha\, dy + \mathcal{O}\left(\varepsilon^2\right),$$
$$= q^\alpha_{\mathrm{grav}} + q^\alpha_{\mathrm{surf}}\,,$$

$$q^\alpha_{\mathrm{grav}} = \tfrac{1}{3}\eta^3 \left\{ \hat{g}^\alpha_{\mathrm{s}} - \varepsilon\, \hat{g}^\beta_{\mathrm{s}} \left( \kappa\, \delta_\beta{}^\alpha + \tfrac{1}{2}\, \mathbb{K}_\beta{}^\alpha \right) \eta + \varepsilon\, \hat{g}_y\, \partial^\alpha \eta \right\}$$
$$+ \varepsilon^2 \tfrac{1}{120}\, \eta^4 \kappa \left\{ \eta\, \hat{g}^\beta_{\mathrm{s}} \left( 9\kappa\, \delta_\beta{}^\alpha + 11\, \mathbb{K}_\beta{}^\alpha \right) - 25\, \hat{g}_y\, \partial^\alpha \eta \right\} + \mathcal{O}\left(\varepsilon^2\right),$$

$$q^\alpha_{\mathrm{surf}} = \tfrac{1}{3}\sigma\eta^3 \left\{ \partial^\alpha \kappa_{\mathrm{surf}} - \varepsilon\, \eta\, \kappa\, \partial^\alpha \kappa + \tfrac{1}{2} \varepsilon\, \eta\, \mathbb{K}_\beta{}^\alpha\, \partial^\beta \kappa \right\}$$
$$+ \varepsilon^2 \tfrac{1}{120}\, \sigma\, \eta^4 \kappa \left\{ 9\eta\, \kappa\, \partial^\alpha \kappa - 14\, \eta \mathbb{K}_\beta{}^\alpha \partial^\beta \kappa - 25\, \partial^\alpha (\kappa_2 \eta + \Delta \eta) \right\} + \mathcal{O}\left(\varepsilon^2\right),$$

Note that we've kept some second-order terms but not others. The above fluxes are only asymptotic to order $\varepsilon^1$, but they preserve the free-surface kinematic BC to all orders. . .

# The Vertical Velocity

The vertical velocity is obtained from mass conservation:

$$v = -\frac{1}{1-\varepsilon\kappa y}\int_0^y \partial_\alpha\left((1-\varepsilon\kappa y)\,u^\alpha\right)\,dy\,,\quad \text{not expanded in } \varepsilon.$$

Mass conservation follows from using this form for $v$, and the free-surface kinematic boundary condition is satisfied exactly if the second-order terms are included in the flux.

The exact kinematic constraints are crucial for particle advection:

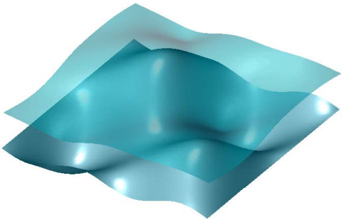
- Mass preservation prevents the existence of attractors in the flow where particles bunch up.
- The kinematic boundary condition prevents particles escaping from the top surface of the flow.

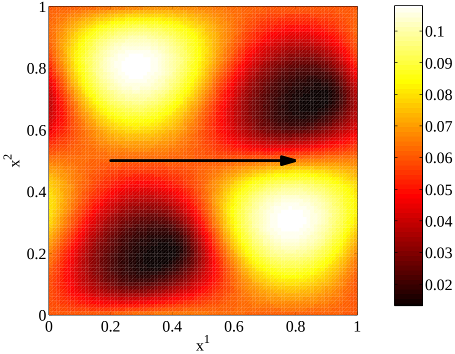
These are only exact to the extent that $\nabla_\alpha q^\alpha = 0$ is satisfied numerically, but this is a much smaller error than $\varepsilon^2$.

# Numerical Solution

We now solve $\nabla_\alpha q^\alpha = 0$ for the height field $\eta(x^1, x^2)$. The pictures below are for a substrate shape given by

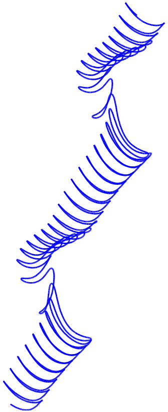
$$f(x^1, x^2) = 0.05\,\{\sin(2\pi x^1)\sin(2\pi x^2) + 0.2\sin(4\pi x^2)\}$$

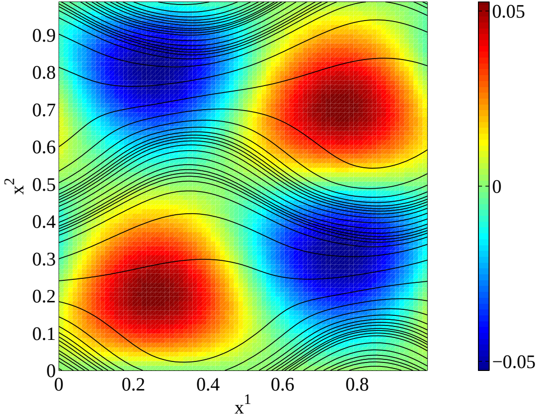





Parameters: $\varepsilon = 0.06$ (thickness), $\theta = 0.1$ (tilt), $\phi = 0$, $\sigma = 0$.

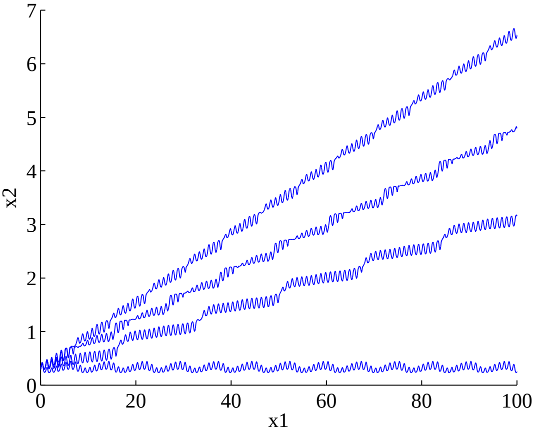
# A Typical Regular Trajectory





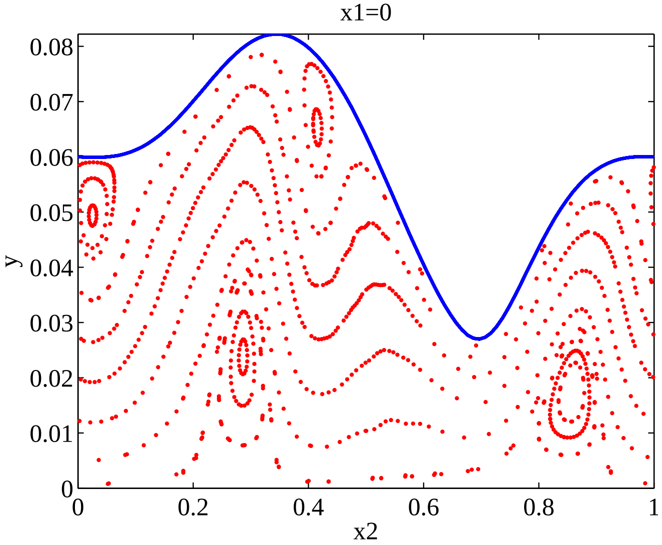

- The particle explores the top and bottom of the layer.
- Nonchaotic, but regularly 'jumps' laterally in $x^2$.

# Two Types of Trajectories

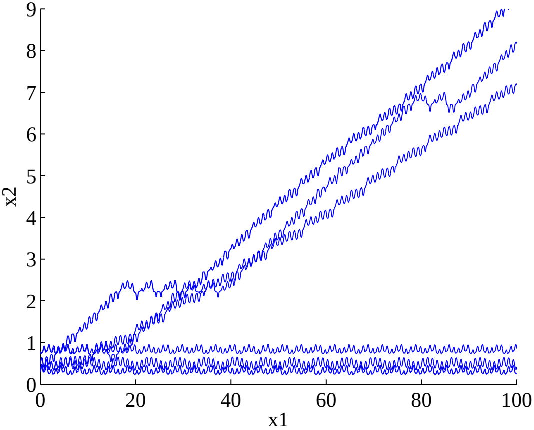



- Trapped and untrapped trajectories.
- The effective diffusivity is strongly anisotropic.

# Regular Poincaré Section

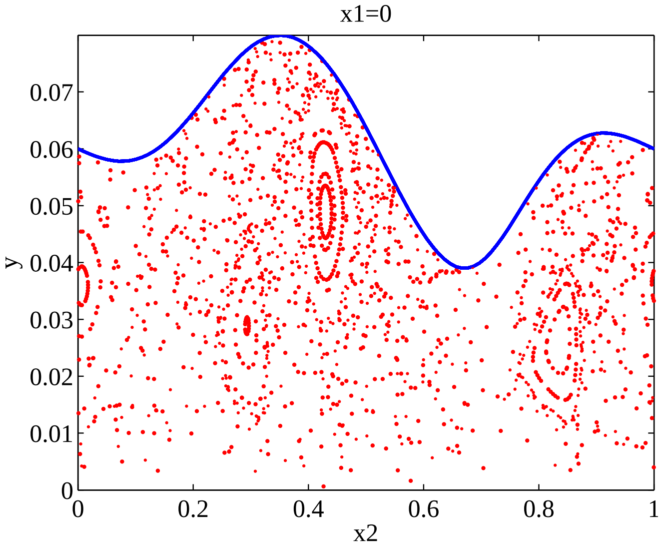


# Decrasing the Tilt Angle: Chaotic Trajectories




- The trajectories chaotically ‘jump’ between channels.
- Sequence of Lévy flights and trapped orbits, analogous to Solomon, Weeks, & Swinney (1993).

# Chaotic Poincaré Section



# Conclusions

- Wide range of behaviour of particle trajectories.
- This will affect mixing properties: strongly anisotropic effective diffusivity.
- Important for coating applications?
- Since the flow is 3D, can have chaos. Three factors:
  1. Break discrete symmetries of the substrate;
  2. Make the fluid deeper (but still thin);
  3. Make the tilt angle $\theta$ smaller.
- Relate substrate properties (curvature tensor) to chaotic features.
- Experiments!
- Time-dependence: induce chaotic mixing by vibrating the substrate or sending waves through it.