Stretching and Curvature in Chaotic Flows

Jean-Luc Thiffeault

Department of Applied Physics and Applied Mathematics

Columbia University

http://plasma.ap.columbia.edu/~jeanluc

6 January 2002

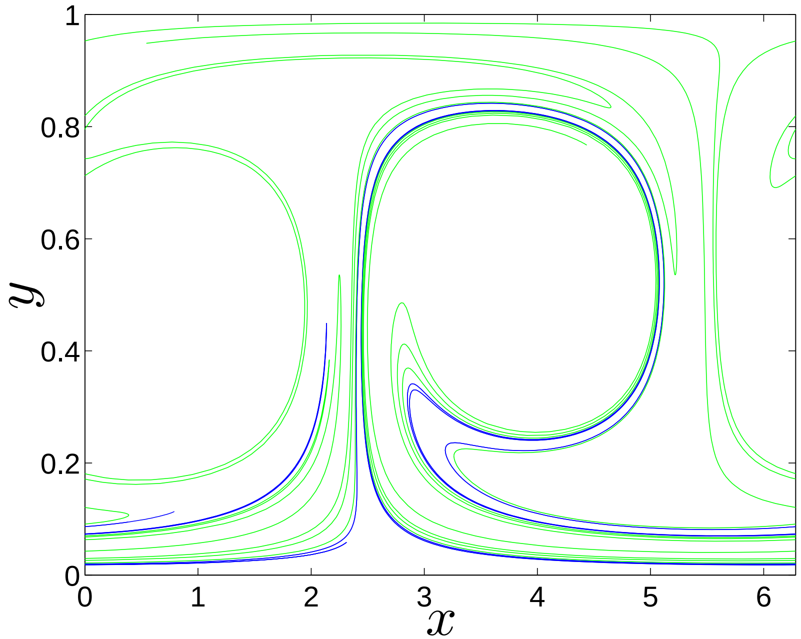
with David Lazanja and Allen Boozer

Chaotic Stirring

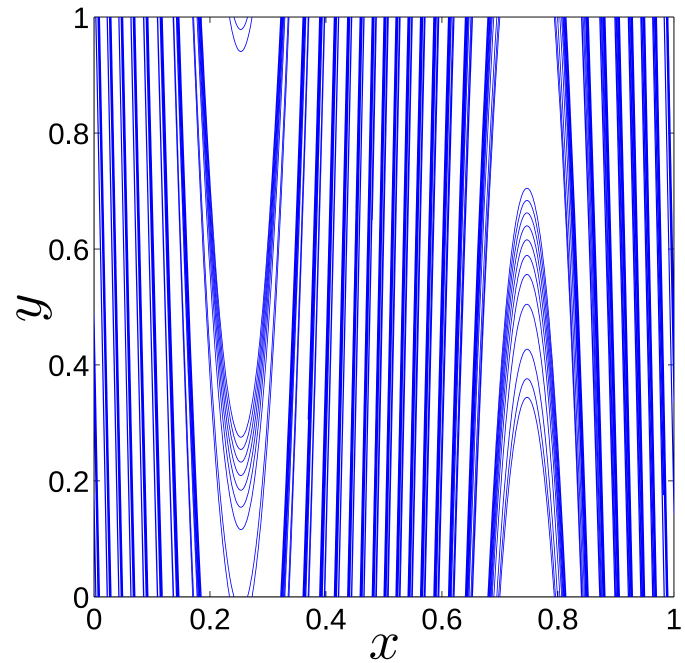
- Chaotic trajectories of fluid particles generates small scales, even in non-turbulent flows.
- Material lines advected by a flow develop very sharp folds, corresponding to regions of large curvature.
- Important to have a clear picture of how these folds form, and how they are correlated with stretching.
- Observe power law relation between amplitude of stretching and curvature.
- Can be explained by a simple model of folding of material lines and compression of fluid elements.

Stretching and Folding

Typical material lines advected by chaotic flows: note the sharp folds that develop.



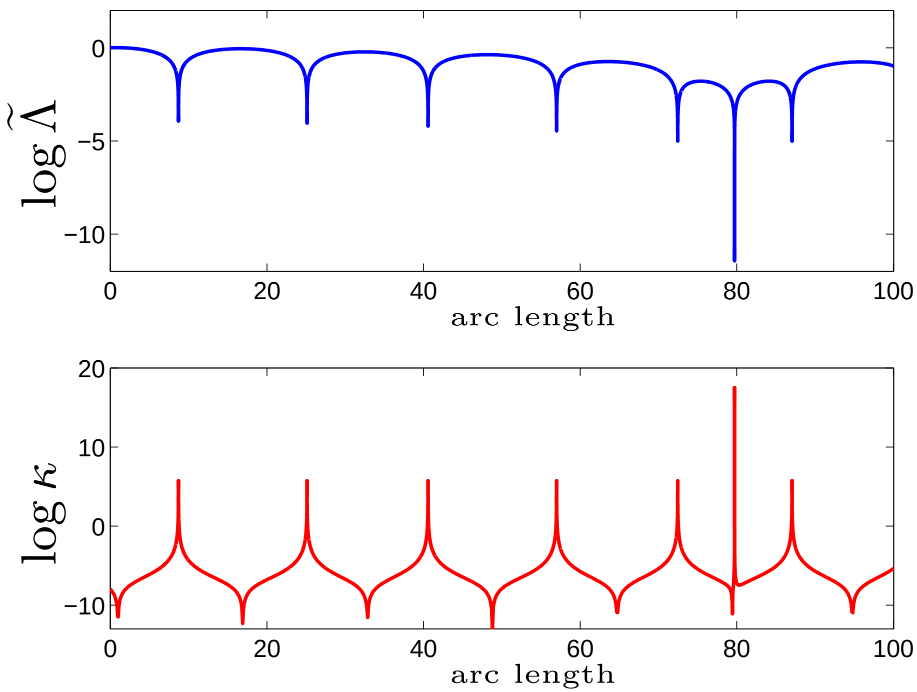
Cellular Flow



Standard Map

Stretching vs Curvature along a Material Line

Compare magnitude of curvature (κ) and stretching ($\widetilde{\Lambda}$) as a function of distance along a material line (standard map).

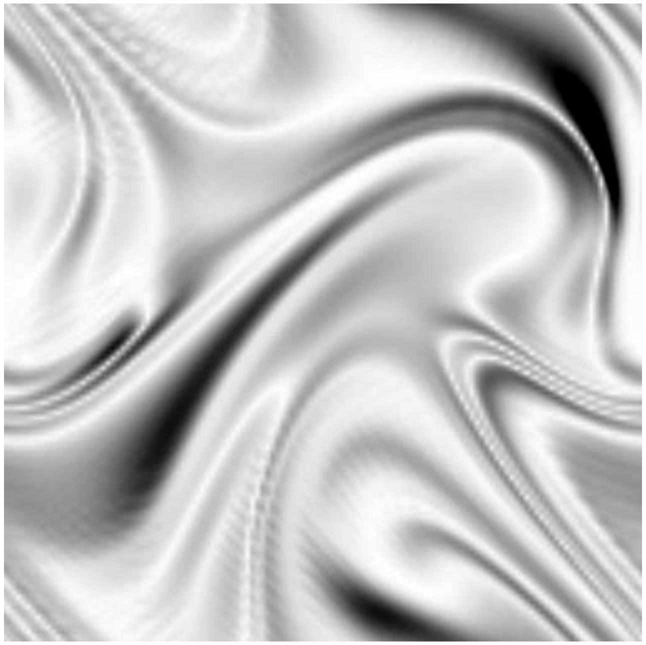


$\widetilde{\Lambda}$ is small whenever the curvature is large

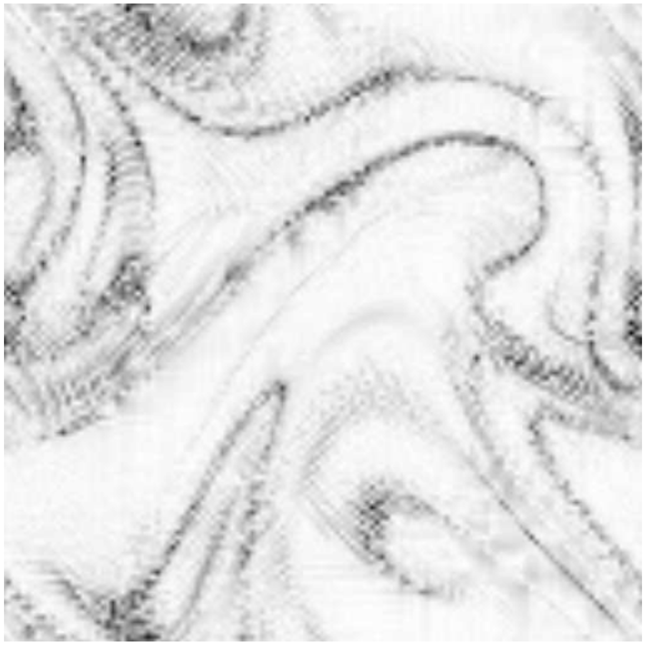
$\Rightarrow$ Suppression of stretching. [Drummond and Münch (1991)]

Stretching and Curvature: the Dynamo

A similar effect was recently observed for the magnetic dynamo.



Magnetic field, B



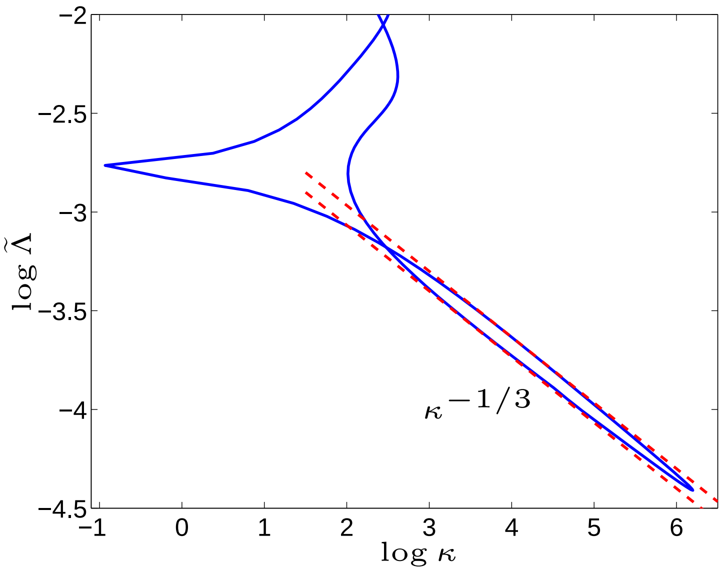
Curvature of B, κ

Observe that the magnetic field is large whenever curvature is small, and vice versa $\Rightarrow$ Anticorrelated

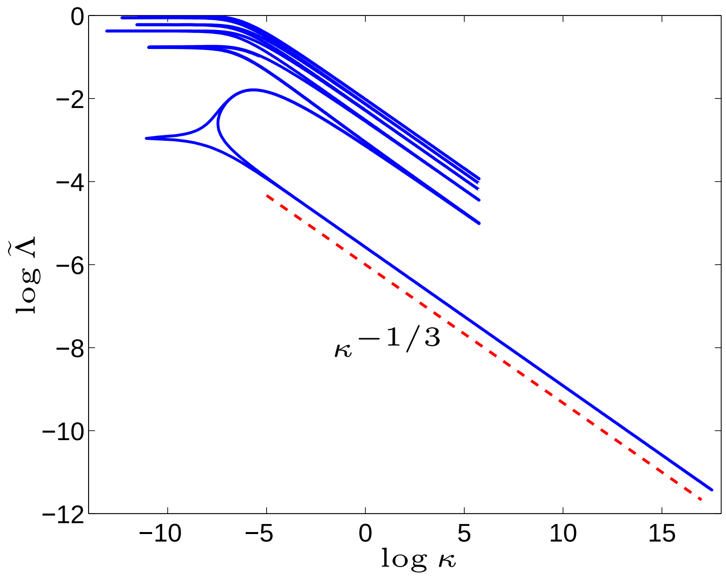
[Schekochihin, Cowley, Maron, and Malyshkin, Phys. Rev. E (2002)]

Stretching vs Curvature along a Material Line

If we make a parametric plot of the stretching and curvature as we march along a material line, we find they obey a perfect power law relation around sharp folds:



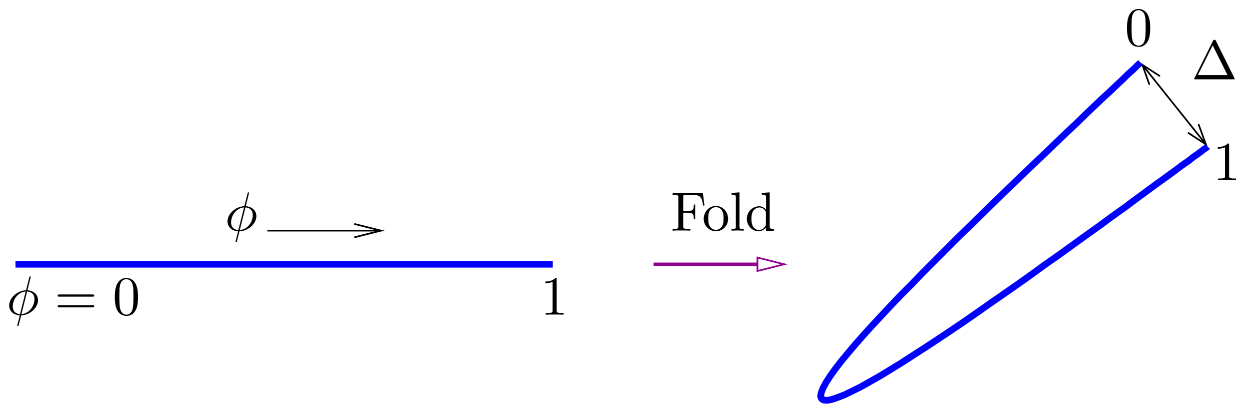
Cellular Flow



Standard Map

Material Line Folded by a Flow

To get a sense of why this anticorrelation, it is helpful to examine how gradients of a solute ϕ are enhanced by folding in a chaotic flow (corresponds to compression of fluid elements).



- Assume linear gradient of ϕ varying from 0 to 1;
- The endpoints of the line are brought to a distance Δ;
- Enhancement in $\nabla\phi$ proportional to Δ^{-1};
- Points in the crest of the fold do not benefit.

A Simple Model

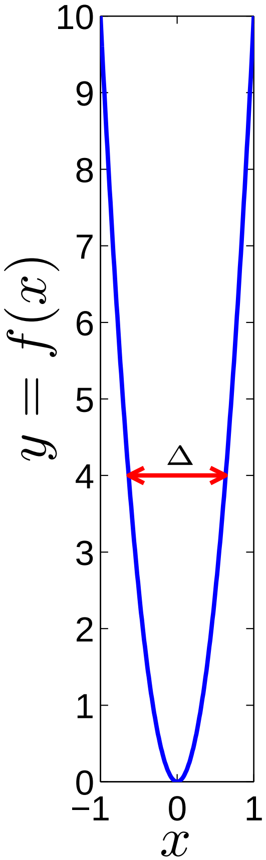
Consider a very sharp fold in a material line, parametrized in the x–y plane by $y = f(x)$. We Taylor expand about the minimum,

$$f(x) = \tfrac{1}{2}\kappa_0\, x^2 + \mathrm{O}(x^3)$$

where $\kappa_0 = f''(0)$ is the curvature at the tip, assumed large.

Because $f(x) \gg x$ away from the tip of the fold, we can approximate the arc length τ from $(0,0)$ to $(x, f(x))$ by

$$\tau(x) \simeq f(x).$$



The stretching at x can be approximated by the difference in concentration of two fluid elements (proportional to arc length) divided by the distance of closest approach Δ of these elements (proportional to x); thus, we have up to a constant overall factor

$$\widetilde{\Lambda}(x) = \tau(x)/x \simeq f(x)/x.$$

The curvature is $\kappa \equiv |(\hat{\mathbf{t}} \cdot \boldsymbol{\nabla})\hat{\mathbf{t}}|$, where $\hat{\mathbf{t}}$ is the unit tangent to f. To leading order (away from the tip) this is

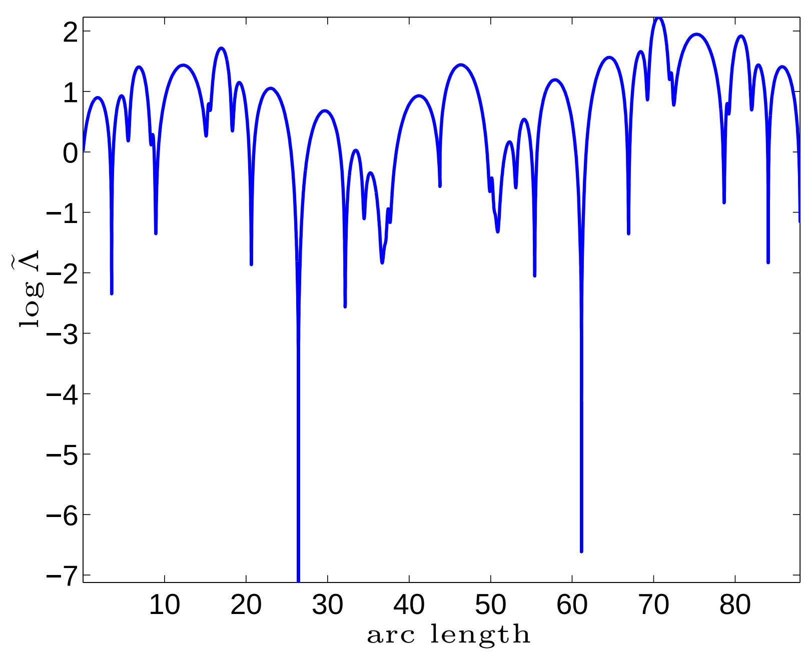
$$\kappa(x) = \kappa_0^{-2}\, x^{-3} + \mathrm{O}(x^{-2}), \qquad \widetilde{\Lambda}(x) = \kappa_0\, x + \mathrm{O}(x^2).$$

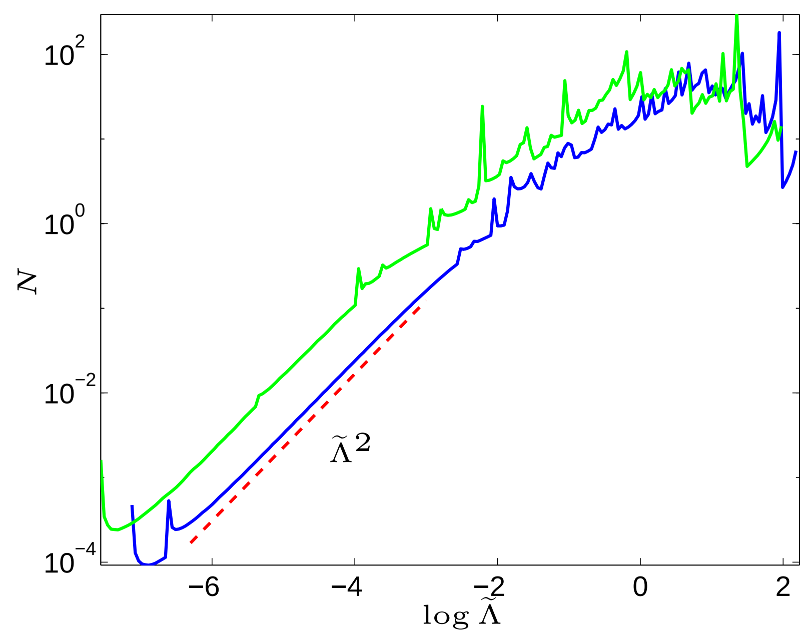
Solve for x in terms of κ,

$$\boxed{\widetilde{\Lambda} \sim \kappa^{-1/3}}$$

This simple law works remarkably well, even in flows where it is not so simple to "isolate" the folds such as the cellular flow.

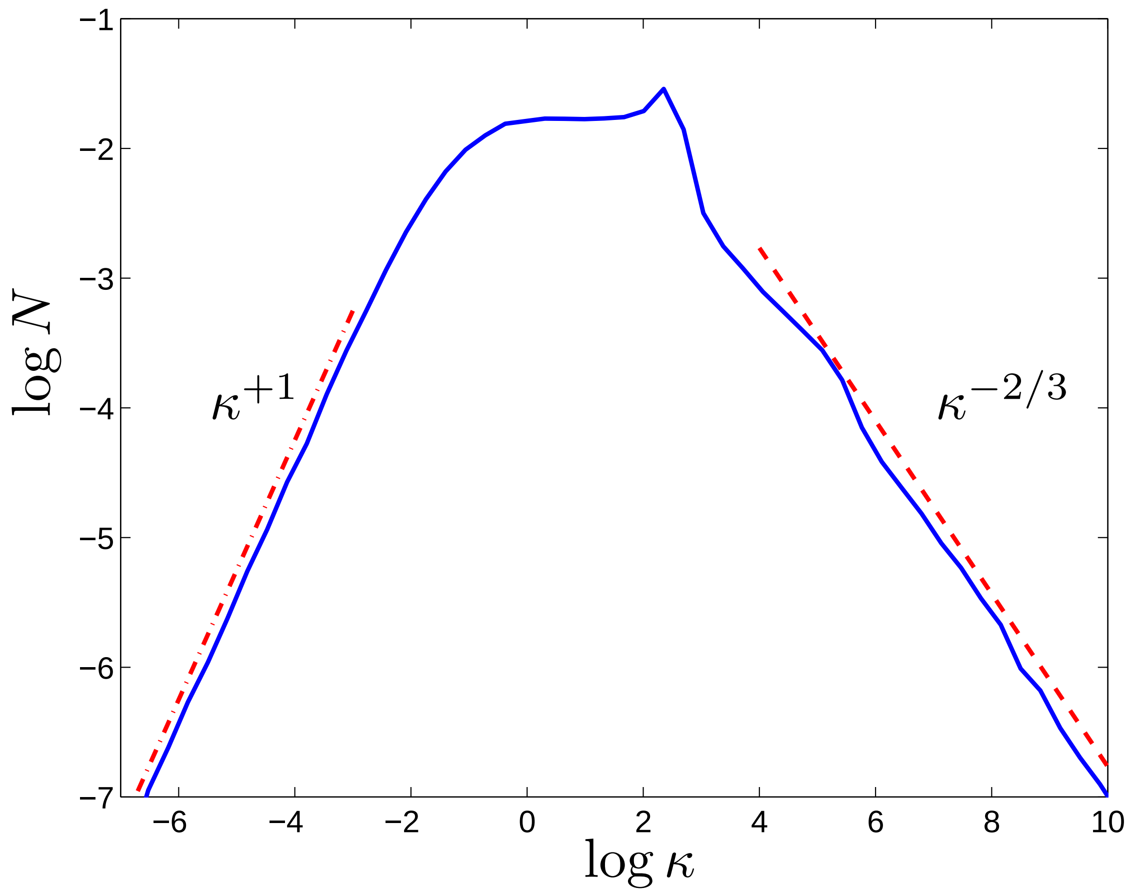
PDF of Stretching along a Material Line





The "fold" model predicts the $\widetilde{\Lambda}^2$ decay of the probability of extremely low stretching events. Exponential ("fat") tail: can have a tremendous impact on mixing.

PDF of Curvature



Stationary distribution. Tails seem independent of specific flow (even turbulent).

Summary and Other Work

- In smooth flows, important to understand the detailed manner in which gradients are enhanced through folding.
- Simple model captures the power-law behavior at sharp folds.
- Related work: Reduce the advection–diffusion equation to one dimension in Lagrangian coordinates. [Thiffeault, Physical Review E (2002)]
- The sharp folds are then local barriers to diffusion, because little chaotic enhancement in those regions.
- Requires the use of differential constraints [Thiffeault and Boozer, Chaos (2001); Thiffeault (2002)].