The Evolution of Material Lines in Chaotic Flows

Application to Mixing

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Research Interests

- Dynamical systems. [A. H. Boozer]
- Chaotic and turbulent mixing. [A. H. Boozer, D. Lazanja]
- Dynamo theory. [A. H. Boozer, S. Childress]
- Hamiltonian description of fluids and plasmas. [P. J. Morrison]
- Weakly nonlinear theory. [N. J. Balmforth, P. J. Morrison]
- Kinetic theory of rarefied gases. [E. A. Spiegel]

Today: focus on chaotic mixing.

Material Lines in Flows

How do material lines embedded in a chaotic flow evolve?

$\Rightarrow$ Stretch, Fold, Twist

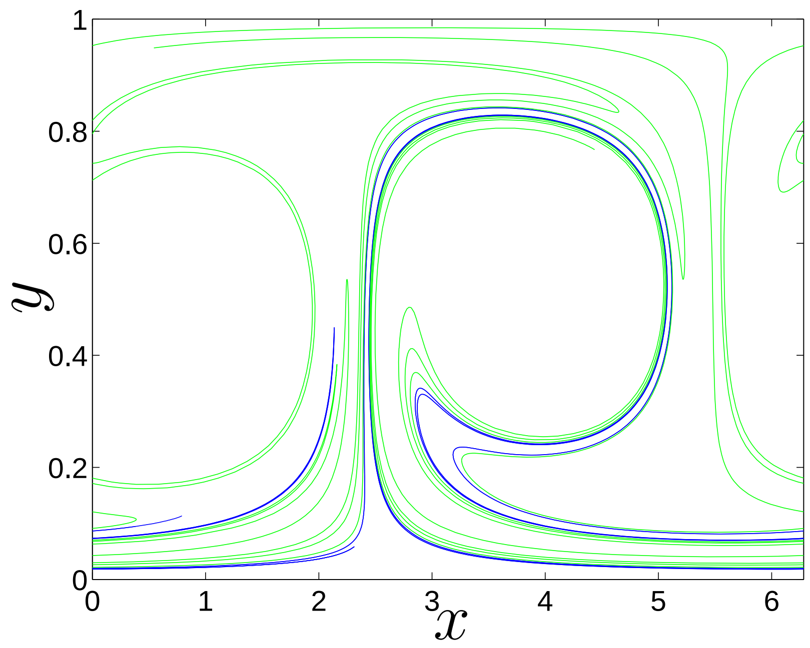
Relevance:

- Dynamo problem: evolution of magnetic field in a plasma.
- Chemical mixing: creation of intermaterial contact area.
- Identification of transport barriers.
- Much is known about stretching, but less about the bending of material lines (generation of curvature and torsion).

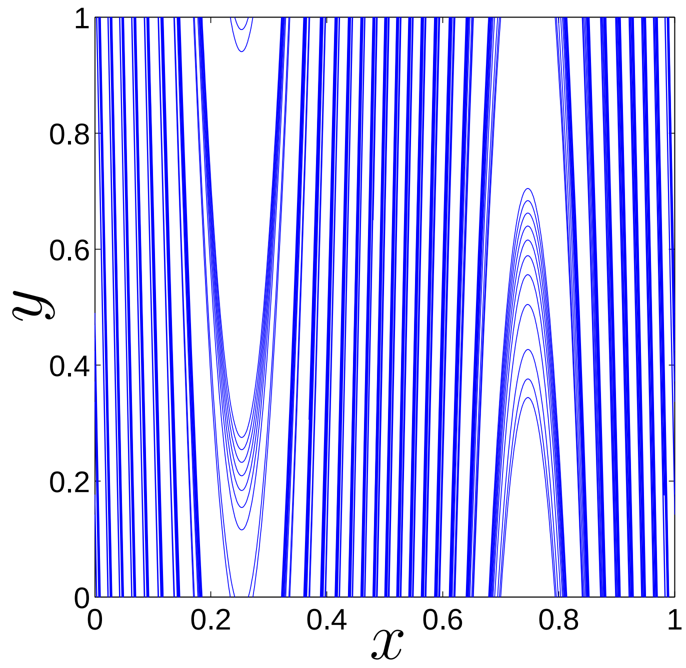
Some interesting regularities occur, such as a close anticorrelation between stretching and curvature.

Stretching and Folding

Traces out the unstable foliation of the flow.
Note the sharp folds that develop.



Cellular Flow

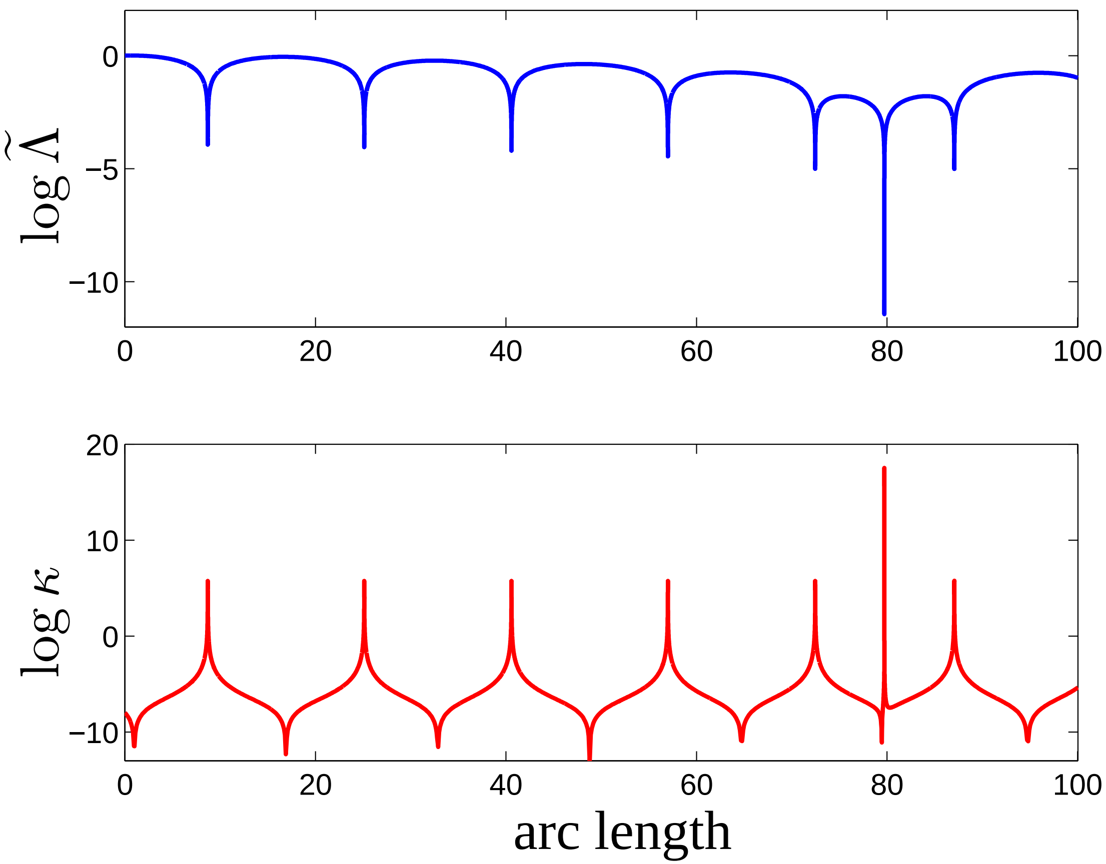


Standard Map

Can look surprisingly regular even in extremely chaotic cases.

Stretching along a Material Line

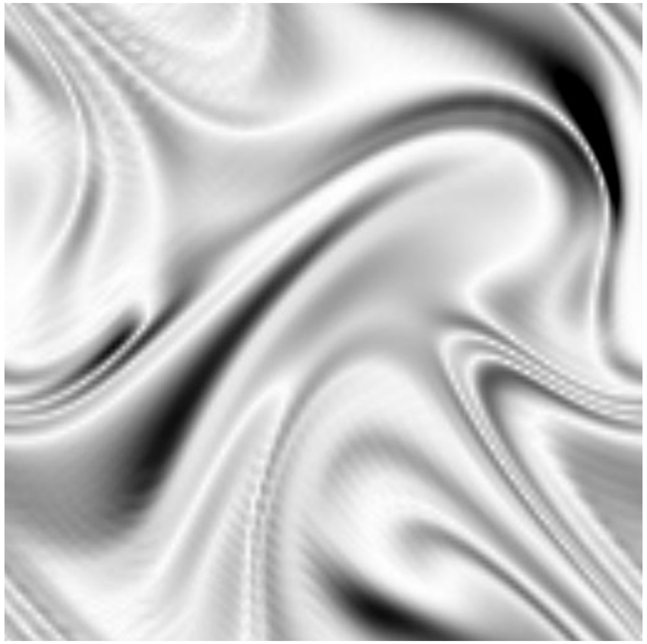
$\widetilde{\Lambda}$ is the deviation from mean stretching.



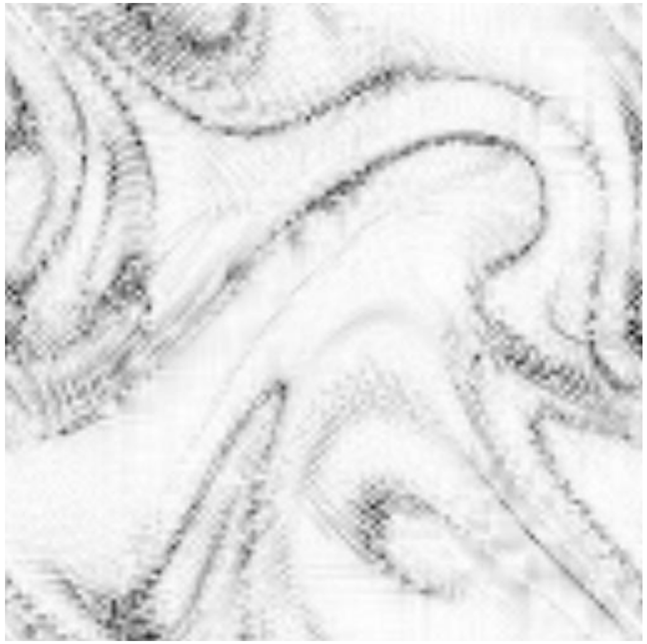
$\Rightarrow$ Suppression of stretching. [Drummond & Münch (1991)]

Stretching and Curvature: the Dynamo

A similar effect was recently observed for the magnetic dynamo.



Magnetic field, B



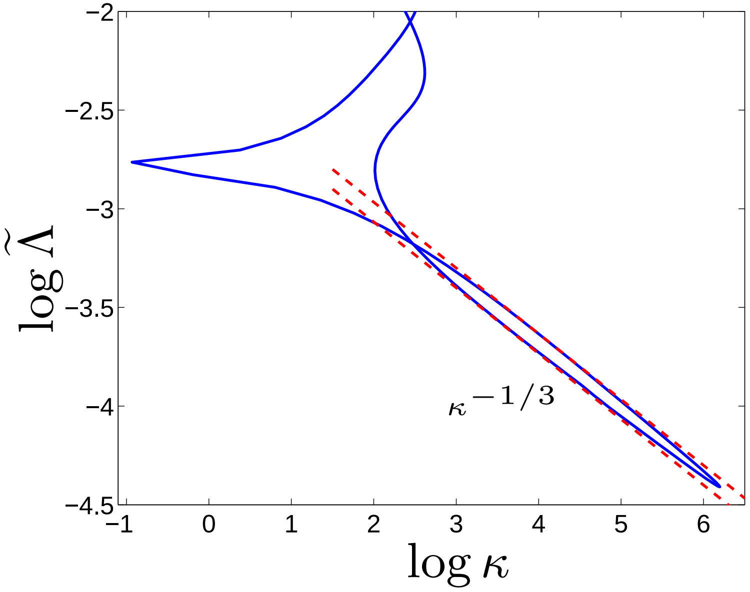
Curvature of B, κ

The magnetic field and its curvature are anticorrelated

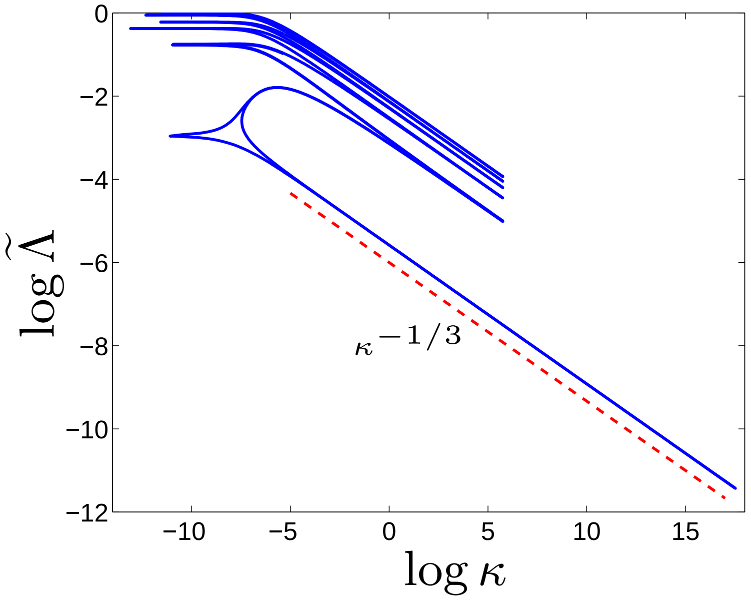
[Schekochihin, Cowley, Maron & Malyshkin, Phys. Rev. E (2002)]

Stretching vs Curvature along a Material Line

Power law relation around sharp folds: The "$-1/3$" law.



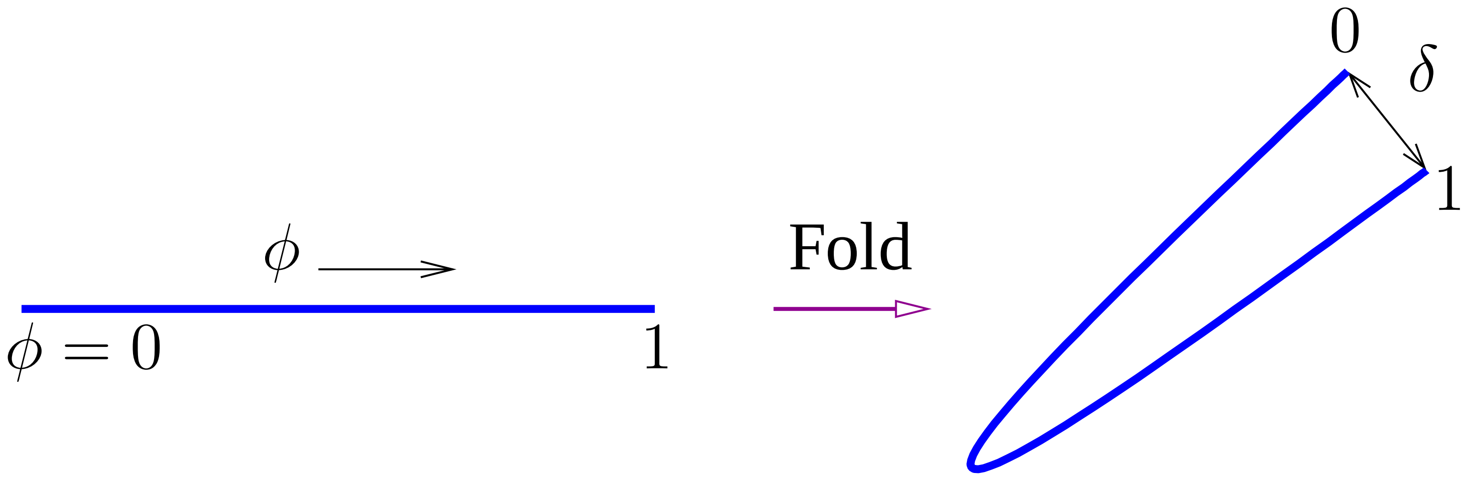
Cellular Flow



Standard Map

The law is very robust even with varying degree of chaos and different flows (2D and 3D).

Enhancement to Gradients by Folding



- Assume linear gradient of ϕ varying from 0 to 1;
- The endpoints of the line are brought to a distance δ;
- Enhancement in $\nabla\phi$ proportional to δ^{-1};
- Fluid elements in the crest of the bend do not benefit.
- Can explain $-1/3$ law with this simple model. [Thiffeault, 2002]

A Simple Model

Very sharp bend in a material line,

$$y = f(x) = \tfrac{1}{2}\kappa_0\, x^2 + \mathrm{O}(x^3)$$

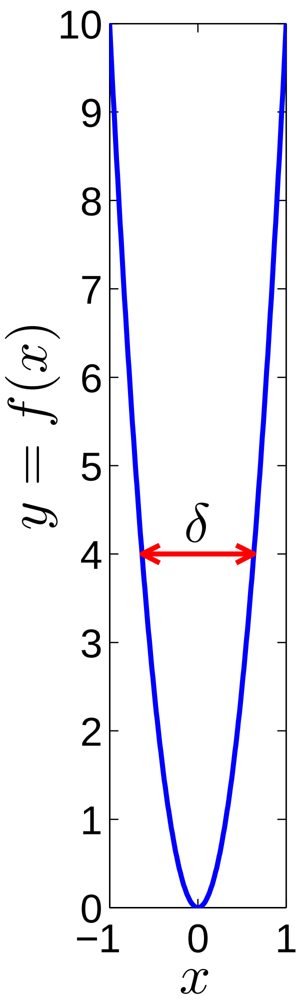
where $\kappa_0 = f''(0)$ is the curvature at the tip. $f(x) \gg x$ away from the tip. Approximate the arc length τ from $(0,0)$ to $(x, f(x))$ by

$$\tau(x) \simeq f(x).$$

Enhancement to gradients:

$$\widetilde{\Lambda}(x) = \tau(x)/x \simeq f(x)/x.$$

$\Rightarrow$ Measure of stretching (incompressible)



The curvature is $\kappa \equiv |(\hat{\mathbf{t}} \cdot \boldsymbol{\nabla})\hat{\mathbf{t}}|$, where $\hat{\mathbf{t}}$ is the unit tangent to f. To leading order this is

$$\kappa(x) = \kappa_0^{-2}\,x^{-3} + \mathrm{O}(x^{-2}), \qquad \widetilde{\Lambda}(x) = \kappa_0\,x + \mathrm{O}(x^2).$$

Solve for x in terms of κ,

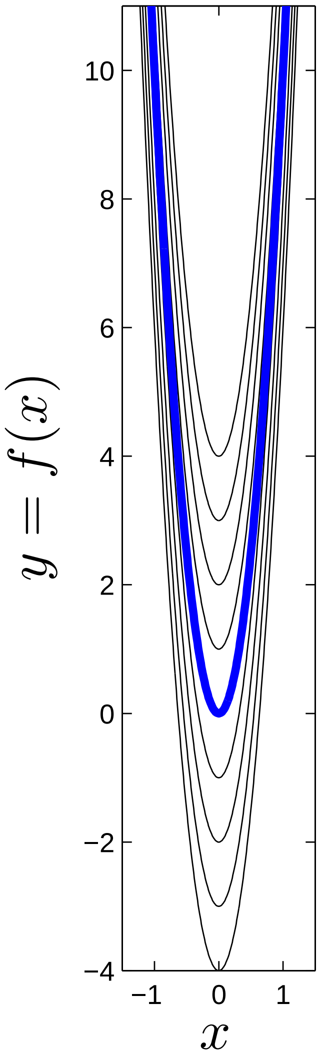
$$\boxed{\widetilde{\Lambda} \sim \kappa^{-1/3}}$$

Problem: the $-1/3$ law works much better than predicted by this simple model.

(Predicts breakdown near the tip, works perfectly in 3D. . .)

A Foliation of Bends

- Material lines are not isolated objects.
- Continuum of other material lines.
- Standard map resembles a foliation of bends.
- Extend the tangent of the quadratic bend to a vector field.
- Distance between lines is not constant: Compression is not uniform.
- How do we relate to stretching?



Conservation Law for Lyapunov Exponents

The tangent $\hat{\mathbf{t}}$ to the material line aligns with the unstable direction of the flow, $\hat{\mathbf{u}}$, the direction of maximum stretching. That direction satisfies

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{u}} + \hat{\mathbf{u}}\cdot\boldsymbol{\nabla}\log\widetilde{\Lambda} \longrightarrow 0, \qquad \text{(exponentially)}$$

[Thiffeault, 2002], based on earlier work by [Tang & Boozer, 1996] and [Thiffeault & Boozer, 2001].

This is a "constraint" on the variation of $\widetilde{\Lambda}$ along the unstable manifold.

$$\frac{\partial}{\partial\tau}\log\widetilde{\Lambda} + \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} = 0, \qquad \tau \equiv \text{arc length along } \hat{\mathbf{u}}$$

Convergence of $\hat{\mathbf{u}}$ $\Rightarrow$ increase in $\widetilde{\Lambda}$.

Assuming our foliation of quadratic bends, the divergence of $\hat{\mathbf{u}}$ is easily computed,

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{u}} \simeq \boldsymbol{\nabla} \cdot \hat{\mathbf{t}} = \frac{\partial \hat{t}_x}{\partial x} = -\frac{f' f''}{(1 + f'^2)^{3/2}} \,.$$

Derivative of $\widetilde{\Lambda}$ along $\hat{\mathbf{u}}$:

$$\frac{\partial}{\partial \tau} \log \widetilde{\Lambda} = \hat{\mathbf{u}} \cdot \boldsymbol{\nabla} \log \widetilde{\Lambda} = \frac{1}{(1 + f'^2)^{1/2}} \frac{\partial}{\partial x} \log \widetilde{\Lambda} \,,$$

Equate and integrate to yield

$$\widetilde{\Lambda} \sim \left(1 + f'^2\right)^{1/2}.$$

To exhibit the relationship between stretching and curvature, we use

$$\kappa(x) = |f''(x)|/(1 + f'^2)^{3/2}$$

for the magnitude of the curvature and obtain finally

$$\boxed{\widetilde{\Lambda} \sim |f''(x)|^{1/3}\,\kappa^{-1/3}}$$

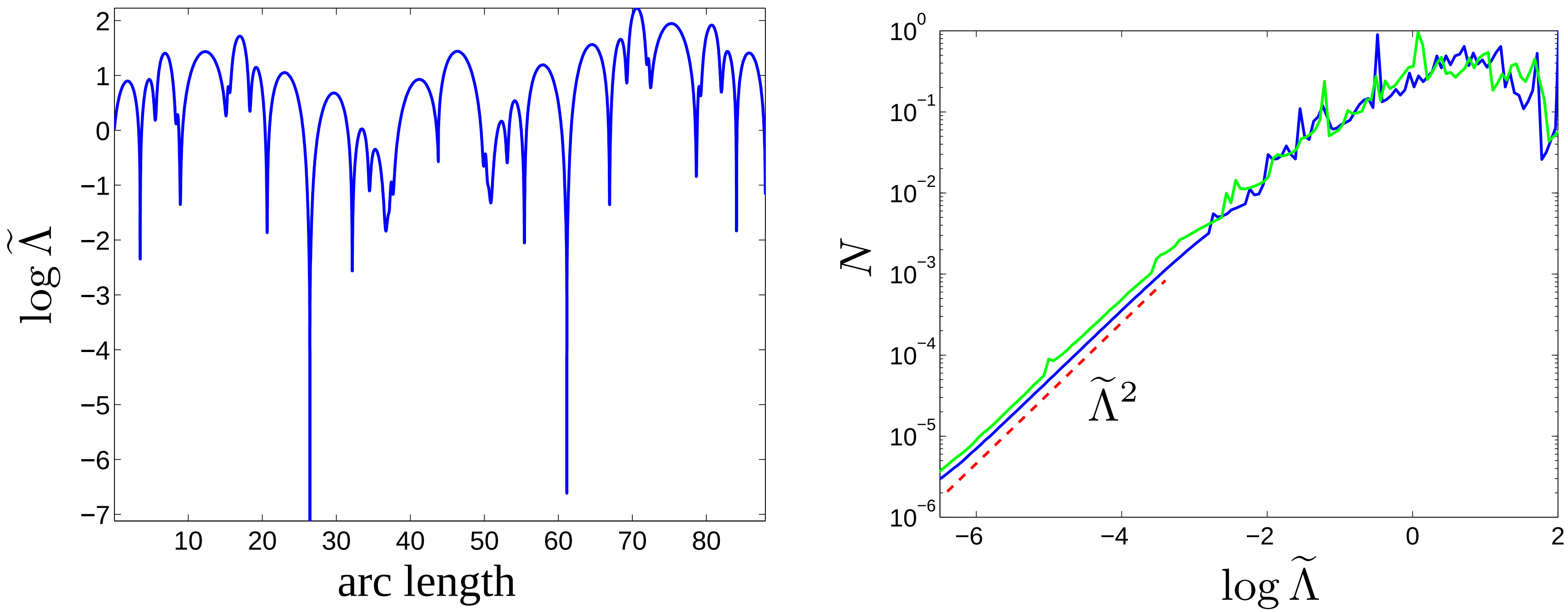
For quadratic f,

$$\widetilde{\Lambda} \sim (\kappa/\kappa_0)^{-1/3}\,,$$

so that the power-law relation holds exactly.

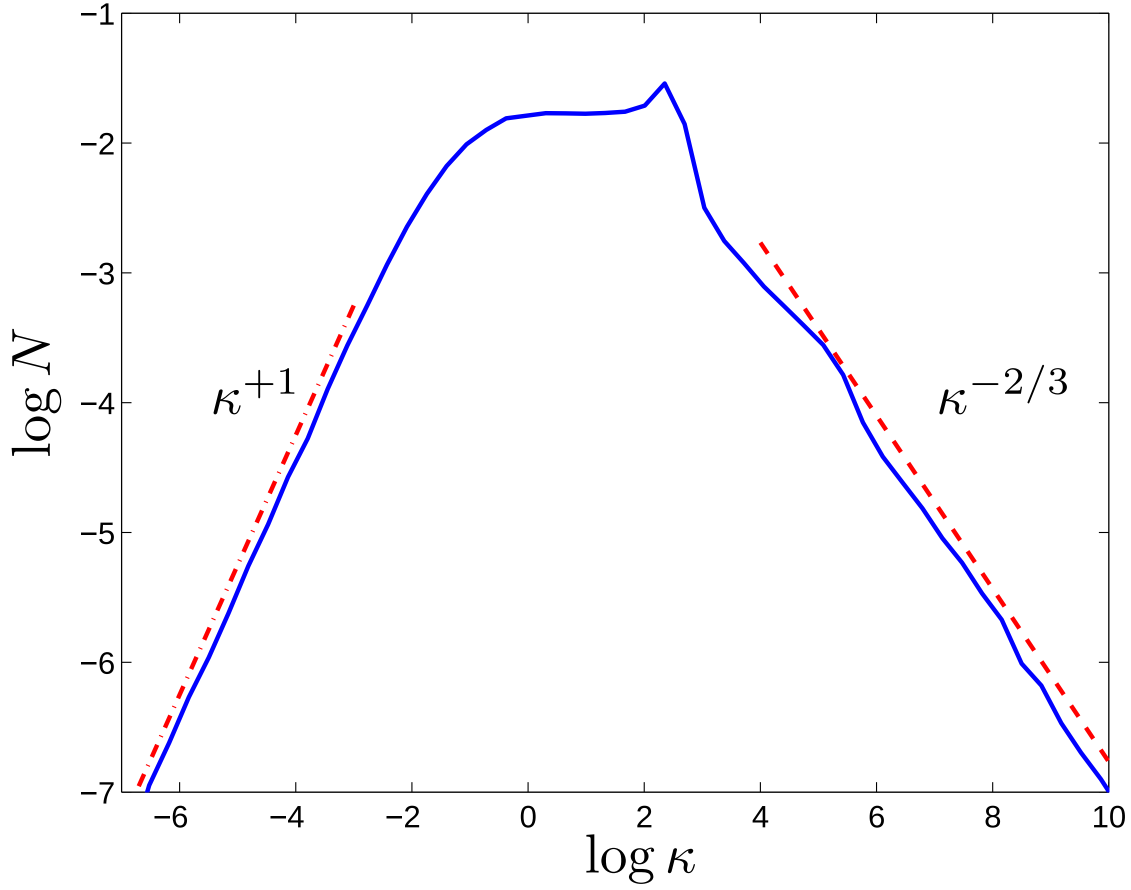
The shape of the bend and y-dependence of the tangent vector field will cause deviations from the $1/3$ law.

PDF of Stretching along a Material Line



The "folding" model predicts the $\widetilde{\Lambda}^2$ tail of the probability of extremely low stretching events. Exponential ("fat") tail: large fluctuations from the mean stretching.

PDF of Curvature



Stationary distribution. Tails seem independent of specific flow.

Chaotic Mixing

- Chaotic trajectories of fluid particles generates small scales, even for non-turbulent flows (small diffusivity);
- Huge gradients of advected scalar are created;
- Makes enhanced diffusion possible:
  For heat in a room, turns a diffusion time of months into minutes (exponential)
- Very difficult to simulate directly : scale separation $\sim 10^{10}$;
- Lagrangian (comoving) coordinates are very convenient because the chaos gets “hidden” in the coordinate transformation.
- Can write a one-dimensional diffusion equation along the stable manifold in Lagrangian coordinates (“backwards” material lines). [Thiffeault, 2002]

The Advection–Diffusion Equation

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \phi = \frac{1}{\rho} \boldsymbol{\nabla} \cdot (\rho \, \mathbf{D} \, \boldsymbol{\nabla} \phi)$$

Péclet number:

$$\mathrm{Pe} = vL/D \gg 1$$

Typical values of Pe:

Earth's core	10^3
Heat in a room	10^5
Solar corona	10^{12}
Galaxy	10^{19}

Singular limit: Even a tiny amount of diffusivity matters.

Advection–Diffusion: Lagrangian Picture

In Lagrangian coordinates $\mathbf{a}$, the advection-diffusion equation is

$$\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{a}} = \sum_{p,q} D \frac{\partial}{\partial a^p} \left[g^{pq} \frac{\partial \phi}{\partial a^q} \right]$$

where $g^{pq} = (g^{-1})^{pq}$ characterizes the transformation from Eulerian to Lagrangian coordinates. Approximate by

$$\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{a}} \simeq \sum_{p,q} D \frac{\partial}{\partial a^p} \left[\Lambda^2 \, \hat{s}^p \, \hat{s}^q \frac{\partial \phi}{\partial a^q} \right]$$

where $\Lambda(\mathbf{a}, t) \gg 1$ is the coefficient of expansion (exponential). The diffusion favors the contracting direction $\hat{\mathbf{s}}(\mathbf{a}, t)$ of the flow. Not quite a 1–D diffusion equation…

A One-dimensional Equation

Analogous conservation law as before, but in Lagrangian coords:

$$\nabla_0 \cdot \hat{\mathbf{s}} + \hat{\mathbf{s}} \cdot \nabla_0 \log \widetilde{\Lambda} \longrightarrow 0,$$

where $\hat{\mathbf{s}}$ denotes the contracting direction.

Using this constraint yields a one-dimensional diffusion equation

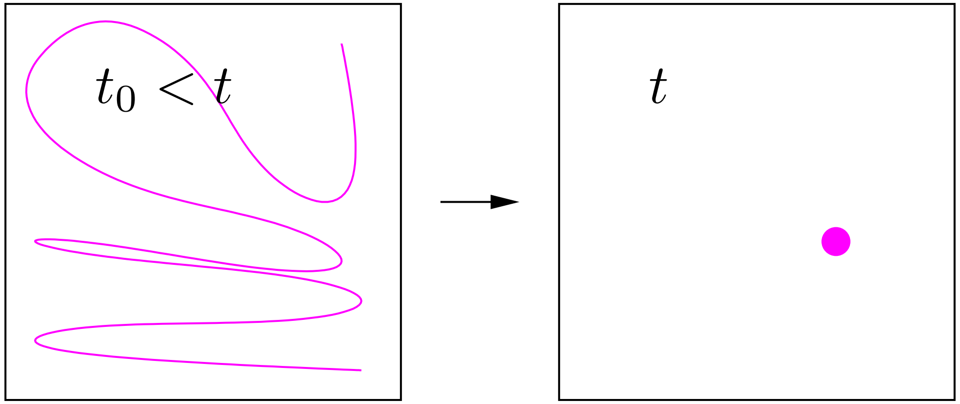
$$\boxed{\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{a}} = \widetilde{D}(t)\,\frac{\partial^2 \phi}{\partial s^2}} \qquad \text{where} \quad \frac{\partial}{\partial s} \equiv \widetilde{\Lambda}\;\hat{\mathbf{s}} \cdot \nabla_0$$

Exponentially-growing diffusion coefficient, $\widetilde{D}(t)$, constant along $\hat{\mathbf{s}}$.

Variation in stretching (and thus in mixing) along manifold given by the function $\widetilde{\Lambda}(\mathbf{a})$, analogous to the material line case.

So What?

- The 1D equation recovers local results about decay rates by averaging over trajectories [Antonsen *et al.*, 1996; Balkovsky & Fouxon, 1999].
- These rates are off by an order of magnitude from experiments [Gollub and Voth, private communication].
- Neglects the global aspects of mixing, as observed recently for the Baker's map [Fereday *et al.*, 2002].
- "Rate of exploration" of a material line when evolved backwards in time.



Future Projects

- The consequences of constraints (such as the conservation law) in physical applications (dynamo).
- Evolution of torsion. Constrained, like curvature?
- Physical applications: long polymers; “active” tracers; chemical reactions, biological mixing (plankton), excitable media. Coarse-graining effects (bugs).
- Some other ongoing research:
  - Kinetic theory of gases: derivation of improved equations for rarefied gases through modified asymptotic expansions.
  - Hamiltonian dynamics: properties of systems generated by nontrivial extensions of Lie–Poisson brackets (the Twisted Top).