Topology, pseudo-Anosov mappings, and fluid dynamics

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

Institute for Mathematics and its Applications
University of Minnesota – Twin Cities

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Collaborators:

Matthew Finn	University of Adelaide
Emmanuelle Gouillart	CNRS / Saint-Gobain Recherche
Erwan Lanneau	CPT Marseille
Toby Hall	University of Liverpool

The Taffy Puller

This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion.

[movie 1]

Experiment of Boyland, Aref, & Stremler

[movie 2] [movie 3] [movie 4]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

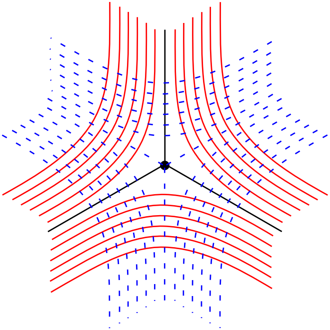
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

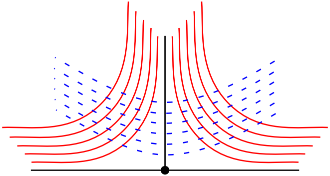
Number 3 is the one we want for good mixing

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

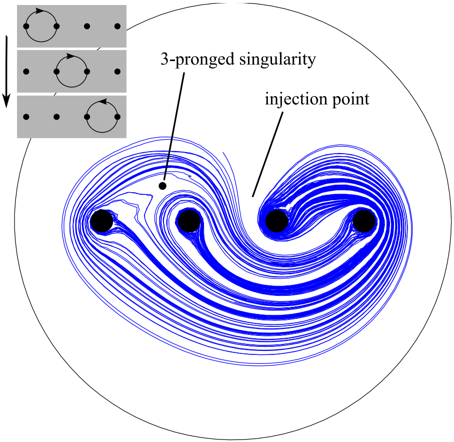


3-pronged singularity



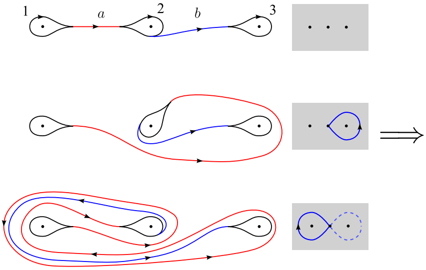
Boundary singularity

Visualising a singular foliation



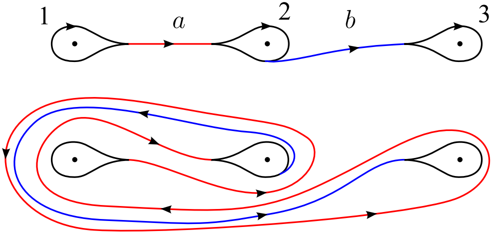
- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Train tracks



Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

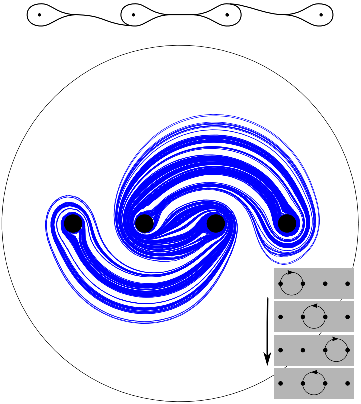
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of a and b, and write as matrix:

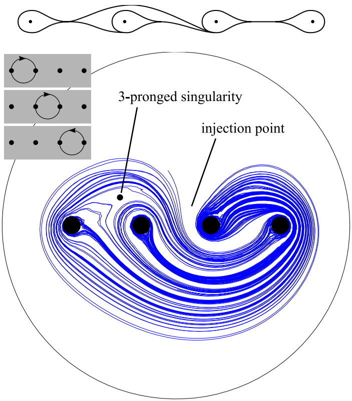
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



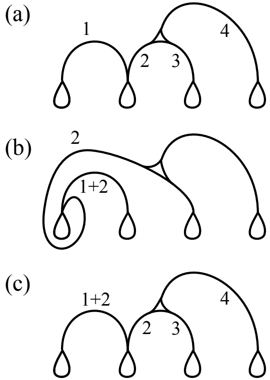
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

Pseudo-Anosovs involve 'folding' the foliation



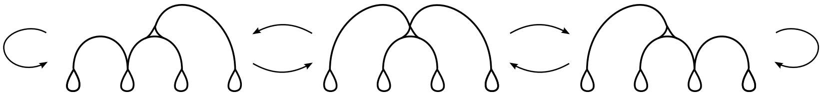
Build pA's 'in reverse,' by regarding them as a sequence of gluings or foldings of pieces of foliation.

Make a transition matrix showing how edges 1–4 are folded:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

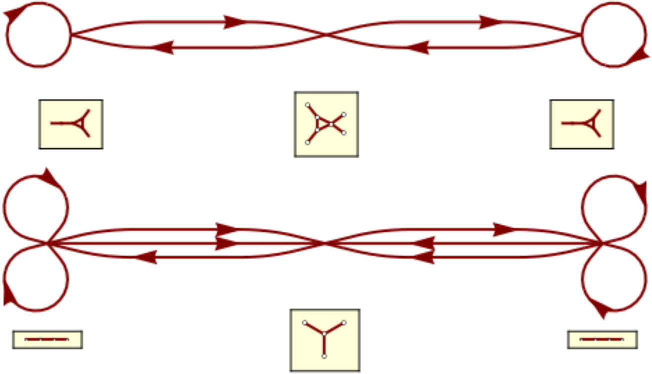
A train track folding automaton

The result is a folding automaton (a graph of train tracks):

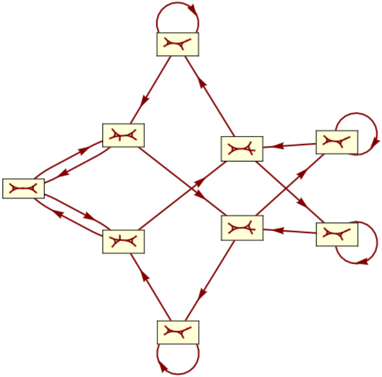


- Each arrow represents a folding of an edge onto another.
- A transition matrix is associated with each arrow.
- pA's are closed paths in this automaton, since they should leave the foliation invariant.
- All pA's are contained therein (up to conjugacy).

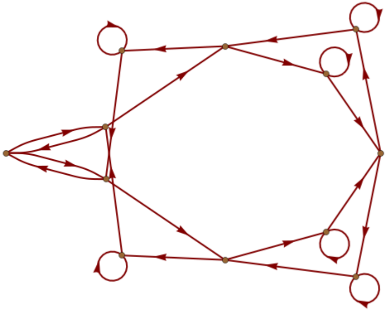
Automata can be simple. . .



Or elegant. . .

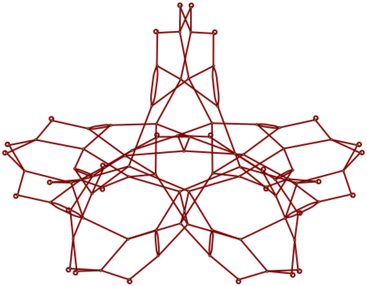


$n = 5, 2 \times$ 3-prong

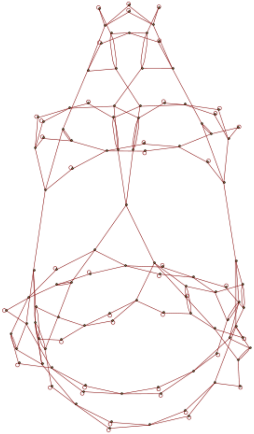


$n = 7, 2 \times$ 4-prong

Or pretty. . .

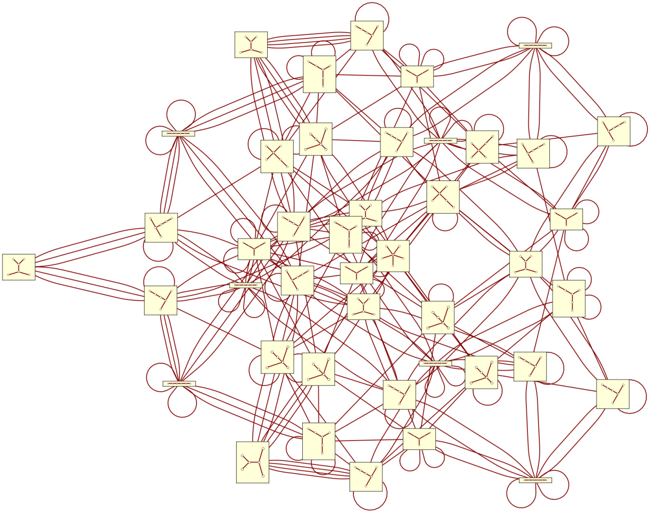


$n = 7, 4 \times 3$-prong
"The maple leaf"



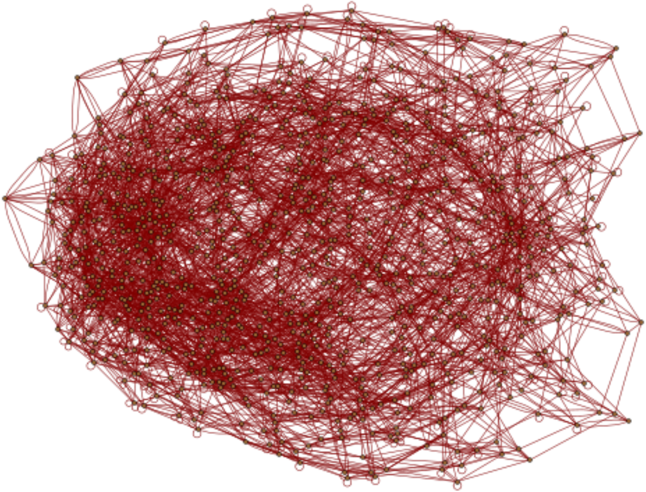
$n = 7, 2\times 3$-prongs, 1×4-prong
"The scarab"

Or rather large...



$n = 6$

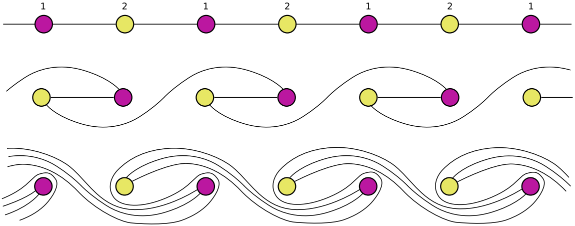
Or just ridiculous. . .



$n = 7, 2 \times 3$-prongs (977 train tracks!)

Optimization

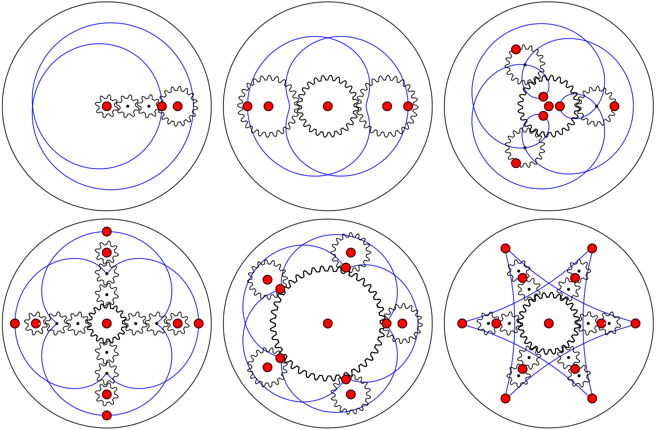
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1\,\sigma_2^{-1}$ with their neighbor (Boyland et al., 2000; Thiffeault & Finn, 2006).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

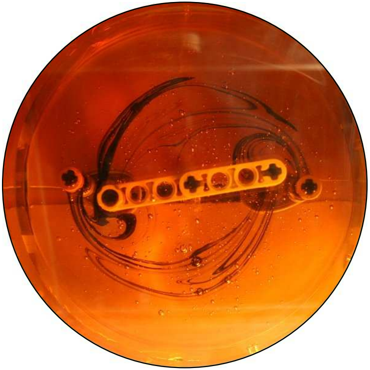
Silver Mixers!

- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



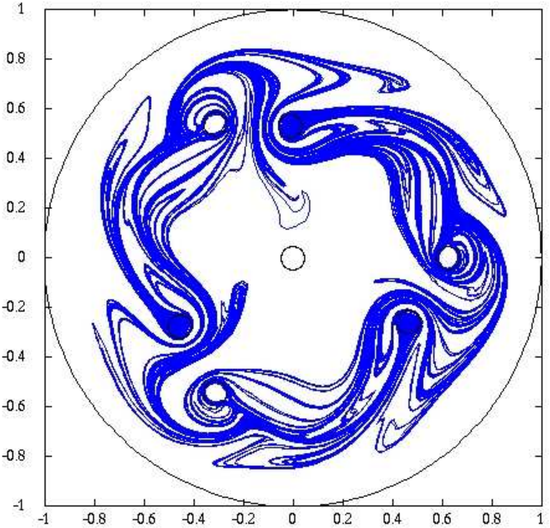
[movie 5]

Four Rods



[movie 6] [movie 7]

Six Rods



[movie 8]

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy (stretching of material lines).
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs.
- We have an optimal design, the silver mixers.
- Need to also optimise other mixing measures, such as variance decay rate.
- Holy grail: Three dimensions! (though current work applies to many 3D situations. . .)

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