the mathematics of burger flipping

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Consider a nice, tasty food object (∼meat) on a hot grill.

We will discuss strategies for perfect cooking, with an emphasis on flipping.

The internet provides a near-infinite number of opinions about the best way to grill meat:

- There seems to be a belief that you should flip *at most* once, to make a better charred surface.
- The Food Lab has a strong contrarian opinion on this:

  > *Flipping steak repeatedly during cooking can result in a cooking time about 30% faster than flipping only once. The idea is that with repeated flips, each surface of the meat is exposed to heat relatively evenly, with very little time for it to cool down as it faces upwards. The faster you flip, the closer your setup comes to approximating a cooking device that would sear the meat from both sides simultaneously.*

[J. Kenji López-Alt, The Food Lab: Flip Your Steaks Multiple Times for Better Results, July 2013]

$\tilde{z} = L$ — top — T_C (air)

food interior — $\widetilde{T}(\tilde{z}, \tilde{t})$

$\tilde{z} = 0$ — bottom — T_H (metal)
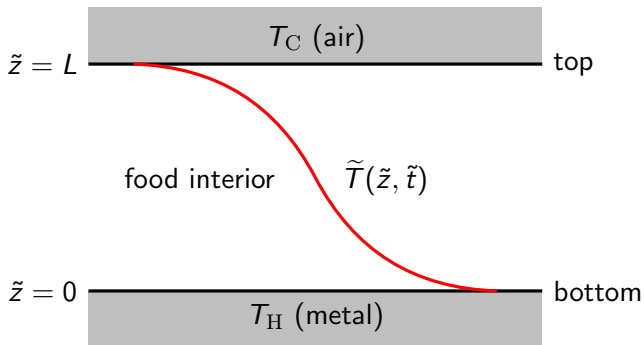
Plot temperature profiles with z on the vertical and T on the horizontal.

We solve the standard heat equation

$$\widetilde{T}_{\tilde{t}} = \kappa \widetilde{T}_{\tilde{z}\tilde{z}}, \qquad 0 < \tilde{z} < L, \quad \tilde{t} > 0,$$

with initial condition

$$\widetilde{T}(\tilde{z}, 0) = \widetilde{T}_0(\tilde{z}).$$

Newton's law of cooling gives the boundary conditions:

$$-k\, \widetilde{T}_{\tilde{z}}(0, \tilde{t}) = \tilde{h}_0 \left(T_{\mathrm{H}} - \widetilde{T}(0, \tilde{t}) \right), \qquad k\, \widetilde{T}_{\tilde{z}}(L, \tilde{t}) = \tilde{h}_1 \left(T_{\mathrm{C}} - \widetilde{T}(L, \tilde{t}) \right),$$

where $\tilde{h}_0 > 0$ and $\tilde{h}_1 \geq 0$ are heat transfer coefficients.

For $\tilde{h}_i \to \infty$ we recover a perfect conductor.

For $\tilde{h}_1 = 0$ we have a perfect insulator.

notation	value	d'less	description
L	$0.01\,\mathrm{m}$	1	food thickness (length scale)
κ	$1.205 \times 10^{-7}\,\mathrm{m^2/s}$	1	thermal diffusivity of food
L^2/κ	$830\,\mathrm{s}$	1	diffusive time (time scale)
ΔT	$175\,^\circ\mathrm{C}$	1	temperature diff. (temp. scale)
T_H	$200\,^\circ\mathrm{C}$	1	bottom temperature
T_C	$25\,^\circ\mathrm{C}$	0	top temperature (also initial)
T_cook	$70\,^\circ\mathrm{C}$	0.257	cooked temperature
h_0	$900\,\mathrm{W/m^2\,^\circ C}$	21.6	bottom heat transfer coeff.
h_1	$60\,\mathrm{W/m^2\,^\circ C}$	1.44	top heat transfer coeff.

[1]Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65

$$T_t = T_{zz}, \qquad 0 < z < 1, \quad t > 0,$$
$$T(z, 0) = T_0(z), \qquad 0 < z < 1,$$
$$T_z(0, t) = -h_0\left(1 - T(0, t)\right), \qquad t > 0,$$
$$T_z(1, t) = -h_1\, T(1, t), \qquad t > 0.$$

Our model of cooking is very simplified: we assume a point z is cooked if

$$T(z, t) \geq T_{\text{cook}} \qquad \text{for any } t.$$

The food is declared cooked if every z is cooked. We ignore burning, drying out, phase changes, ...

[For much more complex models, see for example

Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65
Ou, D. & Mittal, G. S. (2007). *J. Food Eng.* **80**, 33–45
Chapwanya, M. & Misra, N. N. (2015). *Appl. Math. Model.* **39** (14), 4033–4043]

Let's solve the steady problem first ($T_t = 0$), which has a linear profile:

$$S(z) = \frac{h_0(1 + h_1 - h_1 z)}{h_0 + h_1 + h_0 h_1} = S(0) - \Delta S \, z, \qquad \Delta S := S(0) - S(1),$$

with

$$S(0) = \left(1 + \frac{h_1/h_0}{1 + h_1}\right)^{-1} \leq 1, \qquad S(1) = \left(1 + h_1 + \frac{h_1}{h_0}\right)^{-1} \leq 1.$$

For small h_1 and large h_0, we have

$$S(0) \simeq 1 - h_1/h_0, \qquad S(1) \simeq 1 - h_1,$$

so both temperatures are near 1, that is, the steady profile is nearly uniform.

Reformulate as a homogeneous problem for the temperature deviation

$$\theta(z,t) = T(z,t) - S(z):$$

$$
\begin{aligned}
\theta_t &= \theta_{zz}, & 0 < z < 1, \quad t > 0, \\
\theta(z,0) &= \theta_0(z) = T_0(z) - S(z), & 0 < z < 1, \\
\theta_z(0,t) &= h_0\,\theta(0,t), & t > 0, \\
\theta_z(1,t) &= -h_1\,\theta(1,t), & t > 0.
\end{aligned}
$$

$T_0(z) \equiv 0$ when the food starts at room temperature.

Write solution as

$$\theta(z, t) = \sum_{m=1}^{\infty} \hat{\theta}_{0m} \, e^{-\mu_m^2 t} \, \phi_m(z),$$

with L^2-normalized eigenfunctions

$$\phi_m(z) = C_m^{-1} \left(\sin \mu_m z + (\mu_m/h_0) \cos \mu_m z \right).$$

The eigenvalues μ_m^2 satisfy the transcendental relation

$$(h_0 + h_1)\mu = (\mu^2 - h_0 h_1) \tan \mu$$

which must be solved numerically for μ:

$$\mu_1 \approx 2.0803, \ \mu_2 \approx 4.7865, \ \mu_3 \approx 7.6966\ldots$$

Approximate by keeping only the first mode:

$$T(z, t) \approx S(z) - \widehat{S}_1\, \mathrm{e}^{-\mu_1^2 t} \phi_1(z).$$

The criterion for the food to have cooked through to $z = 1$ without needing to flip is

$$S(1) - \widehat{S}_1\, \mathrm{e}^{-\mu_1^2 t}\, \phi_1(1) = T_{\mathrm{cook}}$$

where $S(1)$ (top) is the coldest point of the steady solution.

The 'cookthrough time' for the food to cook without needing a flip is

$$t_{\mathrm{cookthrough}} \approx \frac{1}{\mu_1^2} \log\left(\frac{\widehat{S}_1 \phi_1(1)}{S(1) - T_{\mathrm{cook}}} \right), \qquad \mathrm{e}^{-(\mu_2^2 - \mu_1^2)\, t_{\mathrm{cookthrough}}} \ll 1.$$

We have $t_{\mathrm{cookthrough}} \approx 0.340$, or $283\,\mathrm{s}$ in dimensional terms.

(accurate to 0.07%)

Two mathematically equivalent approaches to deal with the 'flipping' of the food on the hot surface:

1. Flip the food (the $[0, 1]$ domain itself);
2. Flip the boundary conditions.

Respectively Eulerian (fixed frame) and Lagrangian (moving frame) pictures.

There's pros and cons to both choices; we choose to flip the food.

Careful: Make sure the z coordinate refers to the same point in the food when determining if it's cooked.

'Flipping operator':

$$\mathcal{F}f(z) = f(1-z).$$

Self-adjoint with respect to the standard inner product on $[0, 1]$.

Define the flip-heat map, where we take an initial heat profile $T(z, t)$, flip it, then allow it to evolve for a time Δt.[2]

The temperature profile at time $t + \Delta t$ is

$$T(z, t + \Delta t) = S(z) + \mathcal{H}_{\Delta t}(\mathcal{F}T(z, t) - S(z)).$$

where the heat operator $\mathcal{H}_{\Delta t}$ evolves a homogeneous profile by a time Δt.

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[2]Since our initial condition $T(z, 0) \equiv 0$ is $\mathcal{F}$-invariant, it is immaterial whether we flip at the beginning or the end of the interval $[t, t + \Delta t)$.

flipping at fixed intervals

We can solve this recurrence relation to obtain at time $t_k = k\Delta t$

$$T(z, t_k) = \mathcal{K}_{\Delta t}^k \, T_0(z) + \sum_{j=0}^{k-1} \mathcal{K}_{\Delta t}^j \, (1 - \mathcal{H}_{\Delta t}) S(z)$$

$$= \mathcal{K}_{\Delta t}^k \, T_0(z) + (1 - \mathcal{K}_{\Delta t})^{-1} (1 - \mathcal{K}_{\Delta t}^k) \, (1 - \mathcal{H}_{\Delta t}) S(z)$$

where the flip-heat operator is

$$\mathcal{K}_{\Delta t} := \mathcal{H}_{\Delta t} \mathcal{F}.$$

Since $\|\mathcal{K}_{\Delta t}\| < 1$, this converges as $k \to \infty$ to

$$\boxed{U_{\Delta t}(z) := (1 - \mathcal{K}_{\Delta t})^{-1} \, (1 - \mathcal{H}_{\Delta t}) \, S(z).}$$

This is the fixed point for the flip-heat map. The rate of convergence is determined by the modulus of the eigenvalues of $\mathcal{K}_{\Delta t}$.

Even this simple description highlights the two mathematical factors that will determine cooking time under repeated flipping:

- The spectrum of the flip-heat operator $\mathcal{K}_{\Delta t} = \mathcal{H}_{\Delta t}\mathcal{F}$.

  Smaller lead eigenvalue $\implies$ faster approach to equilibrium.

- The structure of the equilibrium profile $U_{\Delta t}(z)$.

  If the profile itself is warmer, then cooking will be faster.

The eigenvalues of $\mathcal{H}_{\Delta t}$ are $e^{-\mu_m^2 \Delta t}$.

Denote the eigenvalues of $\mathcal{K}_{\Delta t}$ by σ_m, and define

$$\nu_m^2 = -\frac{1}{\Delta t} \log |\sigma_m|$$

so we can compare the inverse timescale μ_1^2 to ν_1^2 (larger is faster).

We find numerically a residual value

$$\lim_{\Delta t \to 0} \nu_1 \approx 2.685$$

so that

$$\frac{\nu_1^2}{\mu_1^2} \approx \frac{(2.685)^2}{(2.0803)^2} \approx 1.67, \qquad \Delta t \to 0.$$

That is, the 'infinitely-fast flipping' limit is 67% faster. We should regard this as an upper bound (but no proof yet — working on it!).

$$U_{\Delta t}(z) = (1 - \mathcal{K}_{\Delta t})^{-1} \, (1 - \mathcal{H}_{\Delta t}) \, S(z)$$
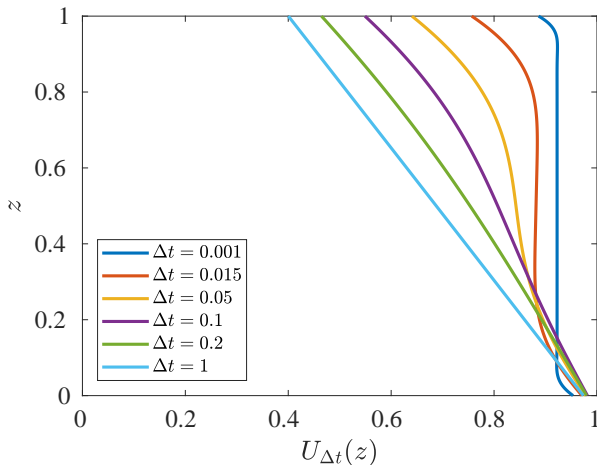
Rapid flipping gives constant interior temperature with boundary layers.
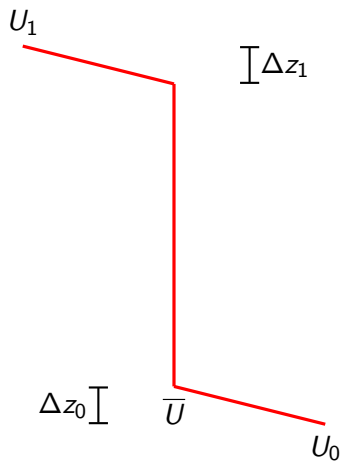
Goal: find $\overline{U}$, the limiting interior temperature.

The boundary conditions require

$$\frac{\overline{U} - U_0}{\Delta z_0} \approx -h_0 (1 - U_0), \qquad \frac{U_1 - \overline{U}}{\Delta z_1} \approx -h_1 U_1.$$

If we want the limit $\Delta z_i \to 0$ to exist, we demand

$$\overline{U} - U_0 = \mathrm{O}(\Delta z_0) \qquad U_1 - \overline{U} = \mathrm{O}(\Delta z_1), \qquad \Delta z_i \to 0.$$

Subtracting these, we have

$$\overline{U} = \tfrac{1}{2}(U_0 + U_1) + \mathrm{O}(\Delta z_i).$$

Insert this back into the BCs:

$$-\frac{\Delta U}{\Delta z_0} \approx -h_0 \, (1 - U_0), \qquad -\frac{\Delta U}{\Delta z_1} \approx -h_1 \, U_1.$$

The fluxes $\Delta U / \Delta z_i$ must balance, since otherwise the equilibrium profile would have a net gain or loss of heat during an interval Δt.

This means that $\Delta z_0 = \Delta z_1$ (the boundary layers have the same thickness), and we find

$$\boxed{\overline{U} \approx \frac{h_0}{h_0 + h_1}, \qquad \Delta t \to 0.}$$

For our parameters this gives $\overline{U} \approx 0.9375$, in good agreement with the numerics.

The fact that this is close to 1 (the heating temperature) reflects the possibility of much faster cooking.

Things simplify dramatically when the material properties are the same at the top and bottom (e.g., metal press). In that case

$$\mathcal{F}\mathcal{H}_t = \mathcal{H}_t\mathcal{F},$$

so that $\mathcal{F}$ and $\mathcal{H}_t$ share eigenfunctions, and $\nu_m = \mu_m$.

This case is fairly inefficient (a good conductor is bad on top), but some analytical progress can be made.

Some questions / observations that may eventually become theorems:

- Is $z = 1/2$ always the last point to be cooked? No!
- If it is, then the total cooking time is independent of Δt and of how many flips. Strange!

A single flip at Δt_1:
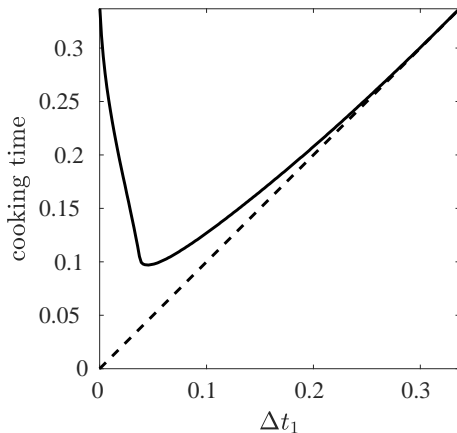
Notice the asymmetry of the minimum: it is better to err on the right.

Minimum t_{cook} is 0.0970 (80.5 s) with a flip at 0.0436 (36 s).
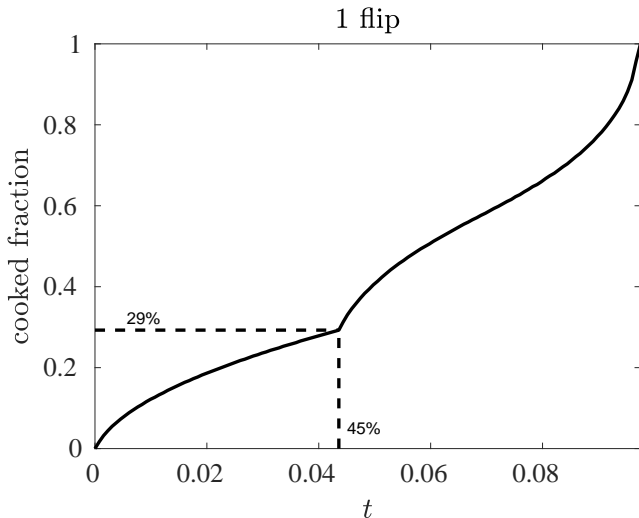
1 flip

Optimal solutions end in a 'tangency.' Note the longer final phase.

Two flips of duration Δt_1 and Δt_2: 8% improvement over 1 flip.
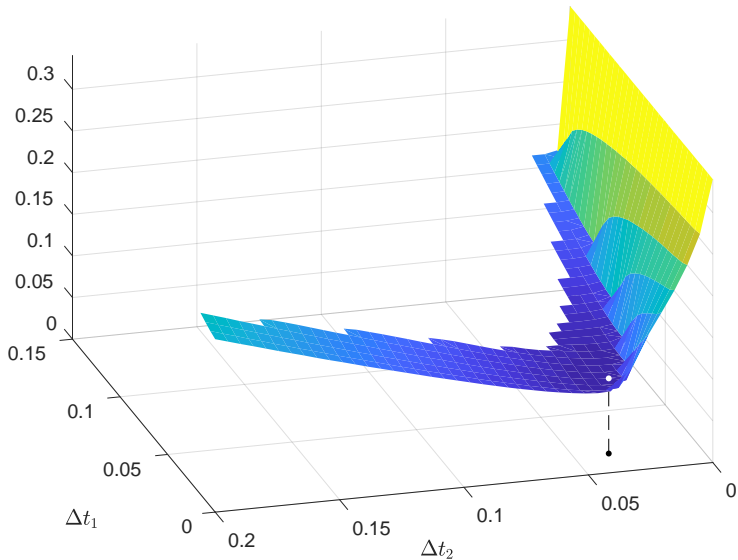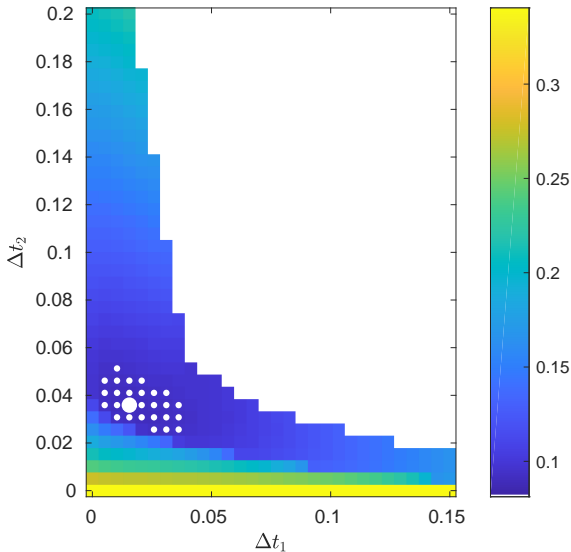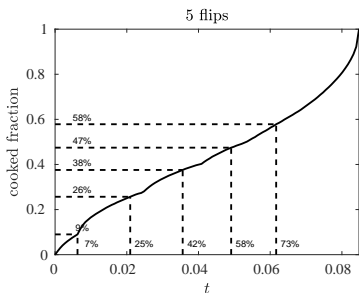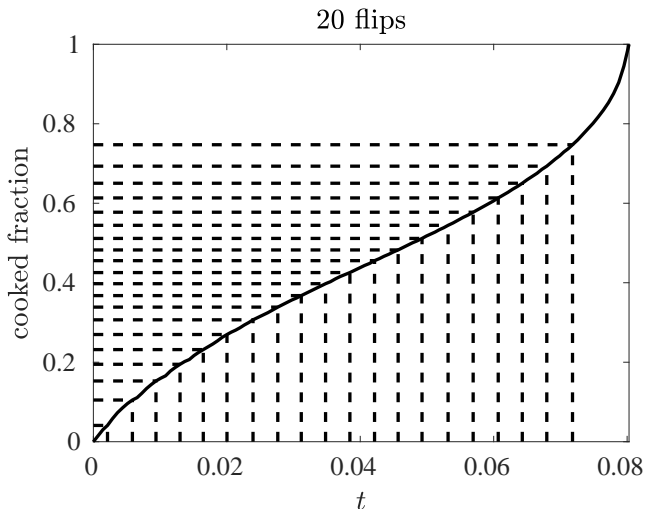
White dots have cooking times within 5% of min.
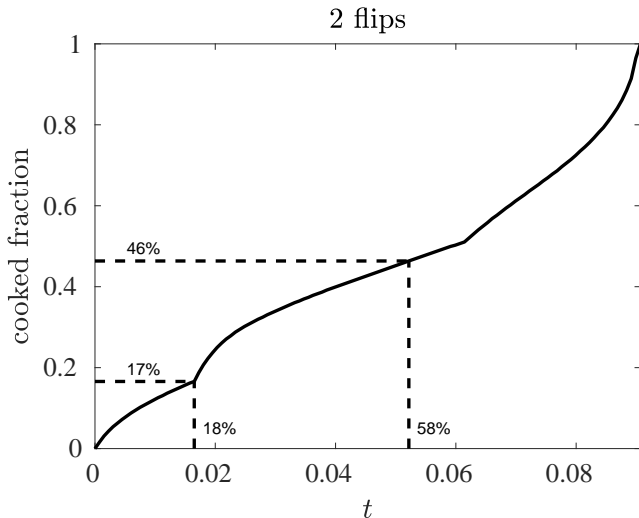
2 flips

Note the nonsmooth points aren't always at flips!

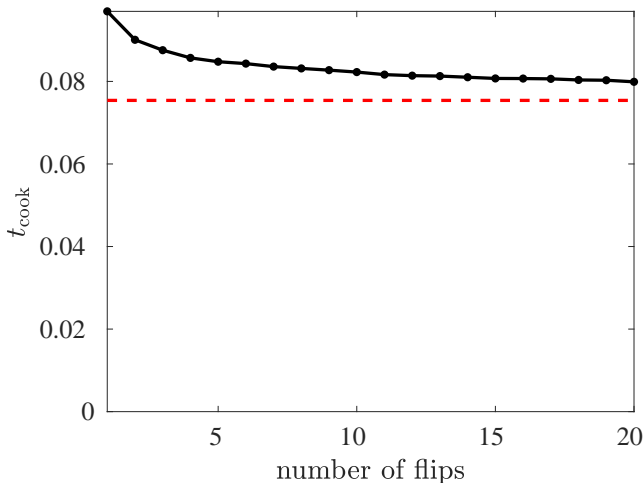MATLAB's nonlinear optimizer chews through these:

20 flips
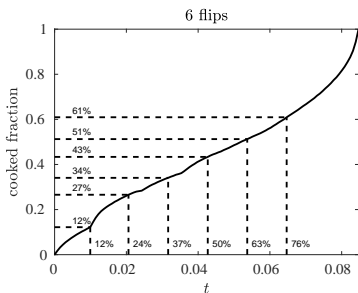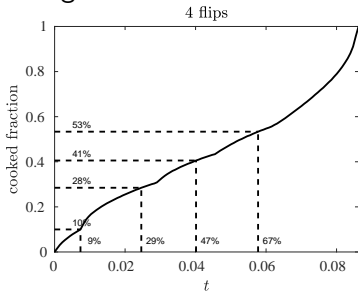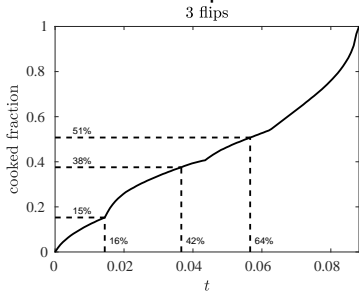
For many flips, the intervals are fairly equal, except the last one.

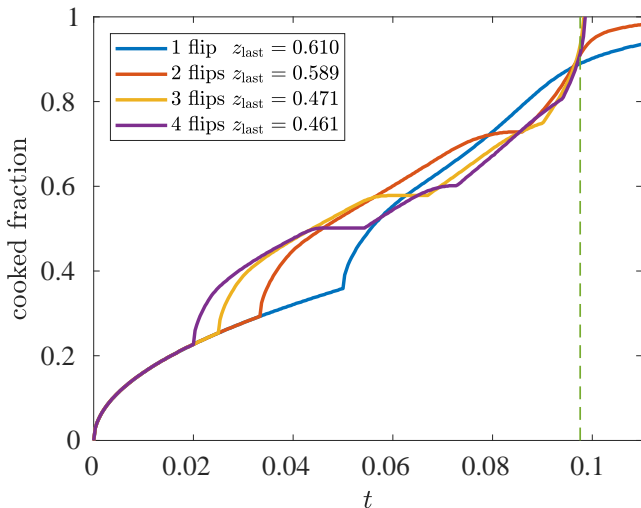[Smoothness of curve suggests an effective continuum description.]

The optimal time converges to ≈ 0.0754 (63 s), vs 0.0970 (80.5 s) for a single flip.

This is a theoretical maximum increase of 29% (same as Food Lab prediction!).

The vertical line is the optimal time 0.0976.

None of the curves are optimized (equal Δt).

a different approach

There is a different approach to the whole problem. Instead of regarding the process as a piewise map, use Duhamel's principle to rewrite as a 'kicked' system:

$$T_t - T_{zz} = \sum_{k=1}^{\infty} \delta(t - k\Delta t)(\mathcal{F}T(z,t) - T(z,t)), \qquad 0 < z < 1, \quad t > 0.$$

The quantity $\mathcal{E}(z)\,\Delta t = \mathcal{F}T(z,t) - T(z,t)$ is the jump at each flip. For periodic solutions, we can show that the jump must satisfy

$$-\tfrac{1}{2}\mathcal{E}(z)\,\Delta t = \mathcal{P}_z S(z) + \int_0^1 [\mathcal{P}_z \mathcal{P}_{z_0} K(z, z_0)]\,\mathcal{E}(z_0)\,\mathrm{d}z_0$$

which is a Fredholm integral equation for $\mathcal{E}(z)$ with a kernel K. ($\mathcal{P}_z$ is a projection onto odd modes.)

- Highly-simplified model of cooking allows a fair bit of analytic progress. Richer than expected.

- Many mysteries remain: for instance, relationship between last cooked point and cooking time for general BCs.

- Also related to work on heat exchangers.
  [Marcotte, F., Doering, C. R., Thiffeault, J.-L., & Young, W. R. (2018). *SIAM Journal on Applied Mathematics,* **78** (1), 591–608]

- Future work: Add some randomness to the flipping process, to model uncertainties or ensembles (the original problem with Diaconis & Holmes).

Chapwanya, M. & Misra, N. N. (2015). *Appl. Math. Model.* **39** (14), 4033–4043.

Marcotte, F., Doering, C. R., Thiffeault, J.-L., & Young, W. R. (2018). *SIAM Journal on Applied Mathematics,* **78** (1), 591–608.

Ou, D. & Mittal, G. S. (2007). *J. Food Eng.* **80**, 33–45.

Stroshine, R. (1998).

Zorrilla, S. E. & Singh, R. P. (2003). *J. Food Eng.* **57**, 57–65.