Stirring and Mixing

Topology, Optimization, and those Pesky Walls

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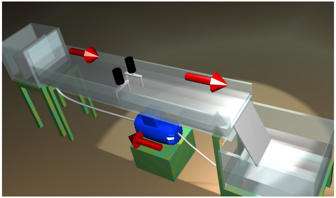
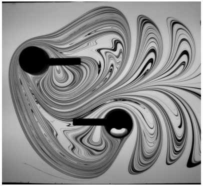
Stirring and Mixing of Viscous Fluids

- Viscous flows ⇒ no turbulence! (laminar)
- Open and closed systems
- Active (rods) and passive





Understand the mechanisms involved.

Characterise and optimise the efficiency of mixing.

Stirring and Mixing: What's the Difference?

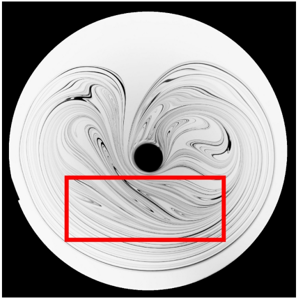
- Stirring is the mechanical motion of the fluid (cause);
- Mixing is the homogenisation of a substance (effect, or goal);
- Two extreme limits: Turbulent and laminar mixing, both relevant in applications;
- Even if turbulence is feasible, still care about energetic cost;
- For very viscous flows, use simple time-dependent flows to create chaotic mixing.
- Here we look at rod stirring and the impact of
  - the vessel walls on mixing rates;
  - the topology of the rod motions.

A Simple Example: Planetary Mixers

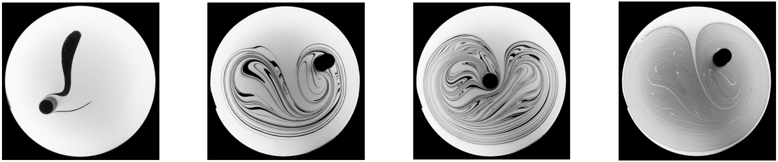
In food processing, rods are often used for stirring.

[movie 1] ©BLT Inc.

The Figure-Eight Stirring Protocol



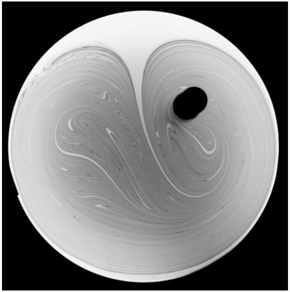
- Circular container of viscous fluid (sugar syrup);
- A rod is moved slowly in a ‘figure-eight’ pattern;
- Gradients are created by stretching and folding, the signature of chaos.

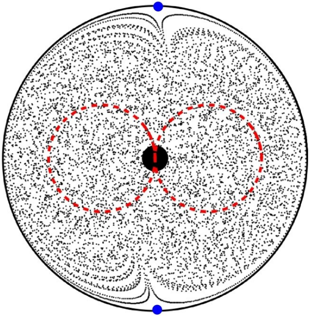


[movie 2] Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

The Mixing Pattern

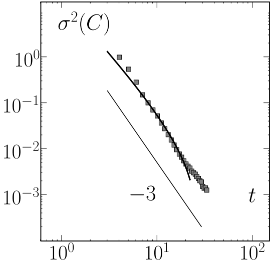
- Kidney-shaped mixed region extends to wall;
- Two parabolic points on the wall, one associated with injection of material;
- Asymptotically self-similar, so expect an exponential decay of the concentration ('strange eigenmode' regime).
(Pierrehumbert, 1994; Rothstein et al., 1999; Voth et al., 2003)



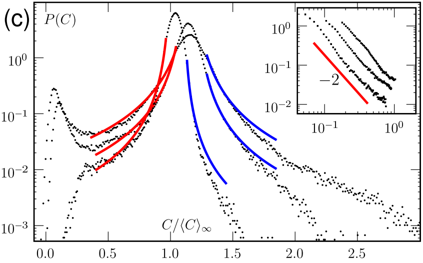


Mixing is Slower Than Expected

Concentration field in a well-mixed central region



Variance $= \int |\theta|^2 dV$

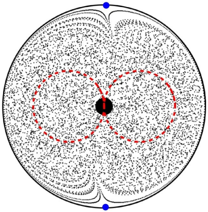


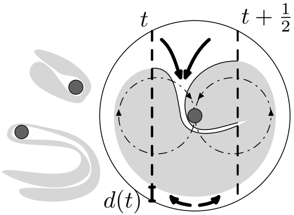
Concentration PDFs

$\Rightarrow$ Algebraic decay of variance $\neq$ Exponential

The ‘stretching and folding’ action induced by the rod is an exponentially rapid process (chaos!), so why aren’t we seeing exponential decay?

Walls Slow Down Mixing





- Trajectories are (almost) everywhere chaotic
  $\Rightarrow$ but there is always poorly-mixed fluid near the walls;
- Re-inject unmixed (white) material along the unstable manifold of a parabolic point on the wall;
- No-slip at walls $\Rightarrow$ width of "white stripes" $\sim t^{-2}$ (algebraic);
- Re-injected white strips contaminate the mixing pattern, in spite of the fact that stretching is exponential in the centre.

Hydrodynamics Near the Wall

We can characterize white strips in terms of hydrodynamics near the no-slip wall. $x_{\|}$ and $x_{\perp}$ denote respectively the distance along and $\perp$ to the wall. No-slip boundary conditions impose

$$v_{\|} \sim x_{\perp}, \qquad \text{near the wall:} \quad x_{\perp} \ll 1.$$

Incompressibility

$$\frac{\partial v_{\|}}{\partial x_{\|}} + \frac{\partial v_{\perp}}{\partial x_{\perp}} = 0,$$

implies

$$v_{\perp} \simeq -a\, x_{\perp}^2 \,.$$

Solve $\dot{x}_{\perp} = v_{\perp}$:

$$x_{\perp} \simeq \frac{x_0}{1 + at\, x_0}.$$

Hydrodynamics Near the Wall (continued)

Hence, the distance between the wall and a particle in the lower part of the domain (where $v_\perp < 0$) shrinks as

$$d(t) \simeq 1/at, \qquad t \gg 1.$$

This scaling was derived in Chertkov & Lebedev (2003), and we verified it experimentally.

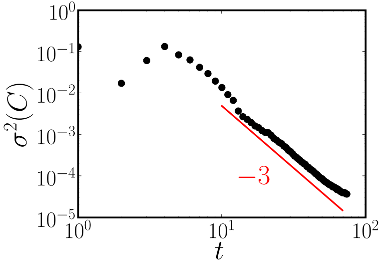
The amount of white that is 'shaved off' at each period is thus

$$\dot{d} \sim T/at^2, \qquad t \gg 1,$$

where T is the period. This is the origin of the power-law decay. Corrections due to the stretch/fold action are described in [Gouillart et al., *Phys. Rev. Lett.* **99**, 114501 (2007)].

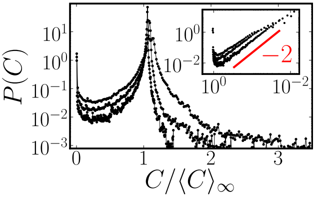
A Generic Scenario

- "Blinking vortex" (Aref, 1984) : numerical simulations



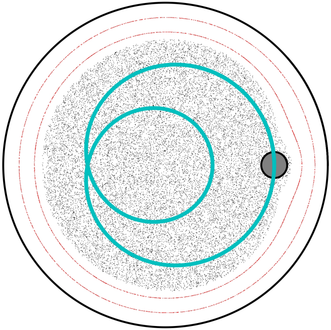
- 1-D Model: Baker's map + parabolic point

Reproduce statistical features of the concentration field;
Some analytical results possible.
(Gouillart et al., 2007)

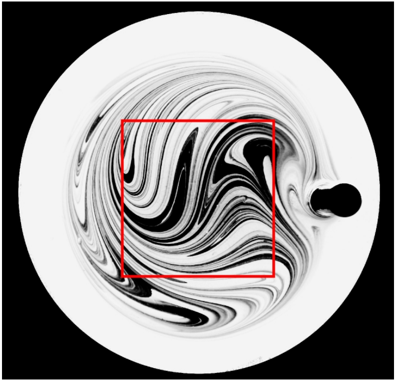


A Second Scenario

How do we mimic a slip boundary condition?

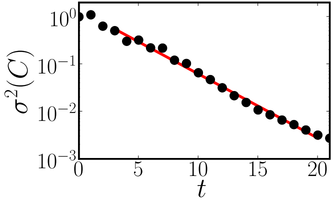


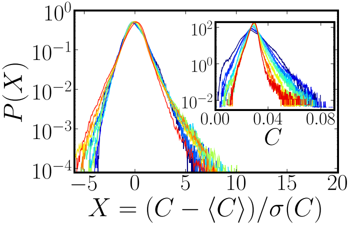
"Epitrochoid" protocol



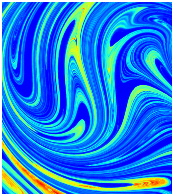
Central chaotic region + regular region near the walls.

Recover Exponential Decay

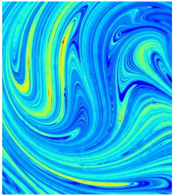




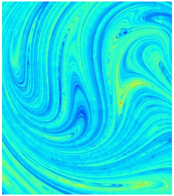
$t = 8$



$t = 12$

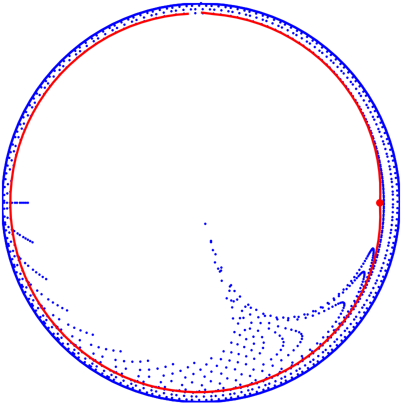


$t = 17$



. . . as well as ‘true’ self-similarity.

Another Approach: Rotate the Bowl!



The Taffy Puller

This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion!

[movie 3]

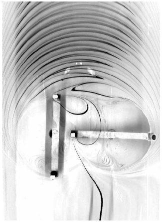
Experiment of Boyland, Aref, & Stremler

[movie 4] [movie 5]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Channel flow: Injection into mixing region

Injection against flow



Injection with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 6] [movie 7]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \rightarrow \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

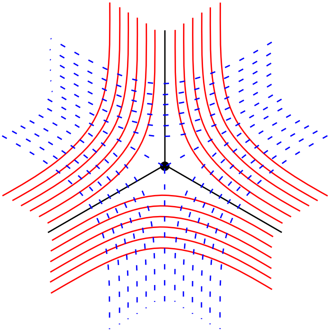
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

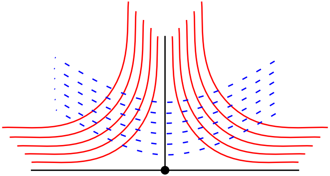
Number 3 is the one we want for good mixing

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

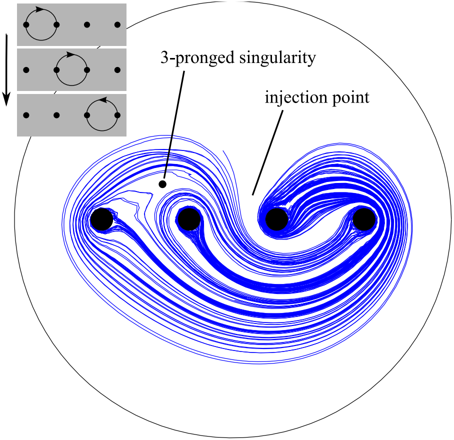


3-pronged singularity



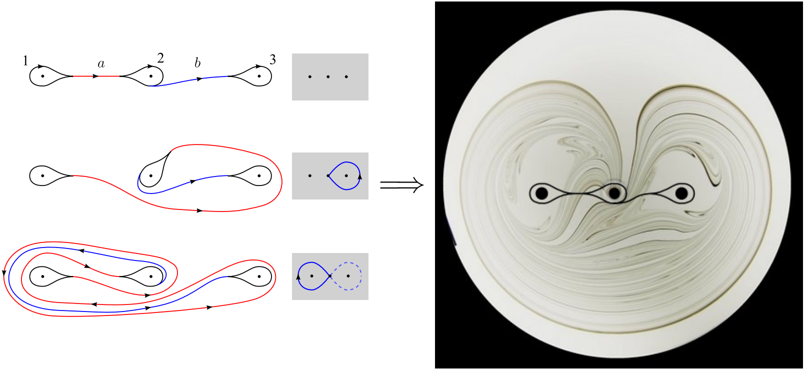
Boundary singularity

Visualising a singular foliation



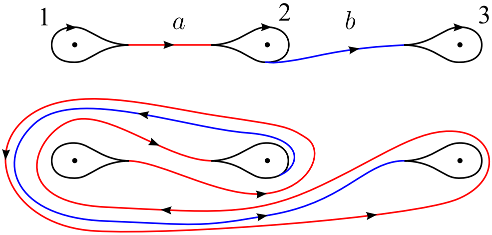
- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Train tracks



Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

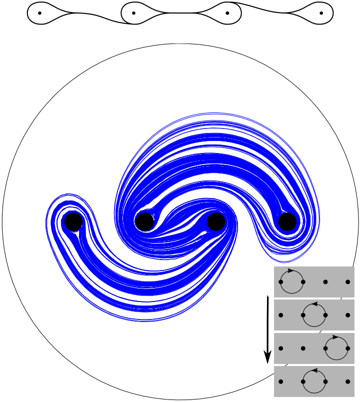
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of a and b, and write as matrix:

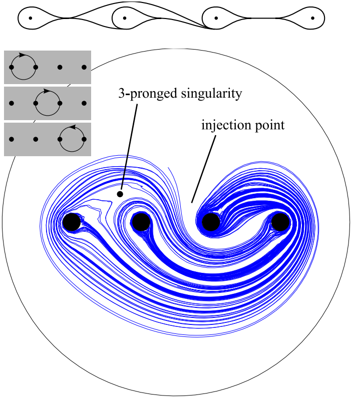
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



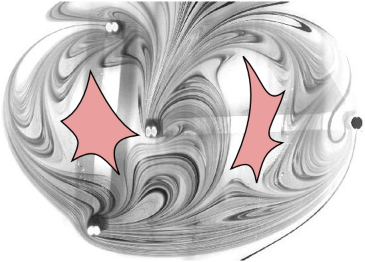
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

Back to the experiment

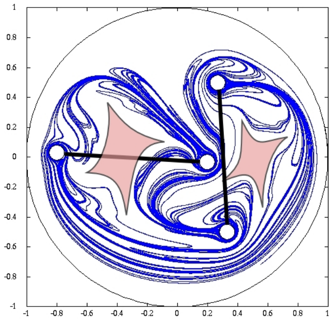


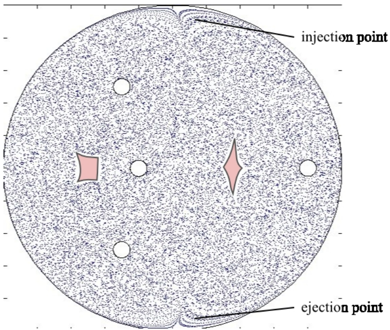
- Two 5-pronged singularities clearly visible;
- Created by the “slicing” of the rods;
- Only one injection point, at the top.

- Each 5-prong rotates unidirectionally;
- They are never interchanged with each other;
- Hence, the experimental picture greatly limits the possible pseudo-Anosovs that can occur.

Two 4-pronged singularities

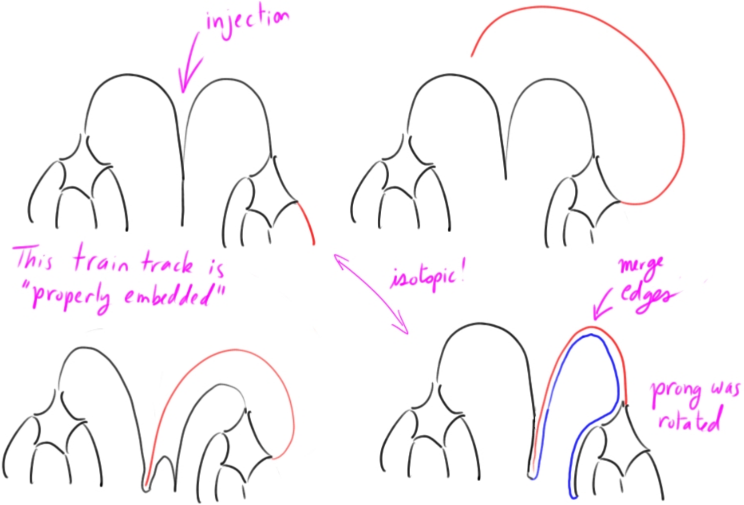
Same protocol, but in a closed container.





Varying the geometry changes the number of prongs: the pronged singularities rotate but lag behind the rods. Smaller rods will increase this lag, and thus the prongness.

A train track folding automaton

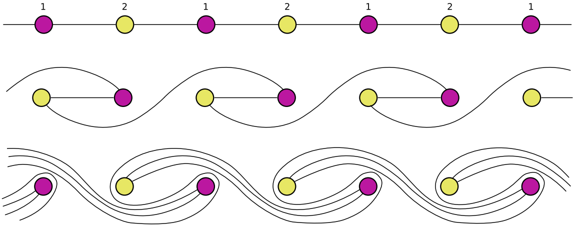


Train track automata (continued)

- Train track automata are a rigorous way of generating all pseudo-Anosovs associated with a train track.
- We know the train track type for our 4-rod experiment, just from watching the movie.
- The tiny automaton we built incorporates the constraints.
- Obtain a train track map by examining how edges are transformed and merged.
- For two k-prongs, the dilatation λ is the largest root of $x^{2k} - x^{2k-1} - 4x^k - x + 1$.
- Decreases with k, which indicates that smaller rods have less effect (shocking!).

Periodic Array of Rods

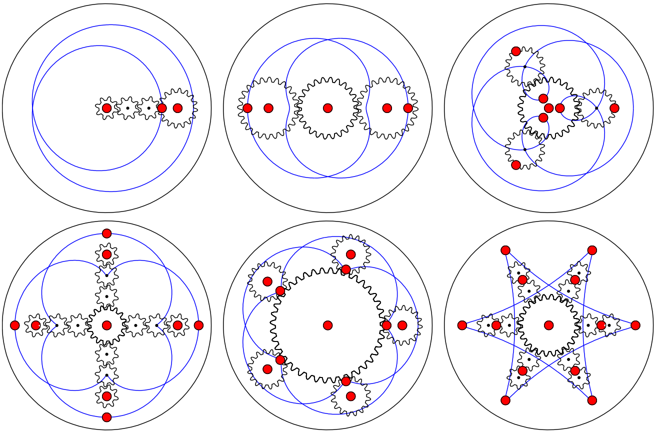
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1\,\sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The entropy per 'switch' is $\log\chi$, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).
- Work with M. D. Finn (Adelaide).

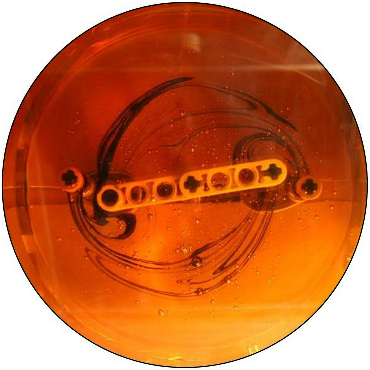
Silver Mixers!

- The designs with entropy given by the silver ratio can be realised with simple gears.
- All the rods move at once: very efficient.



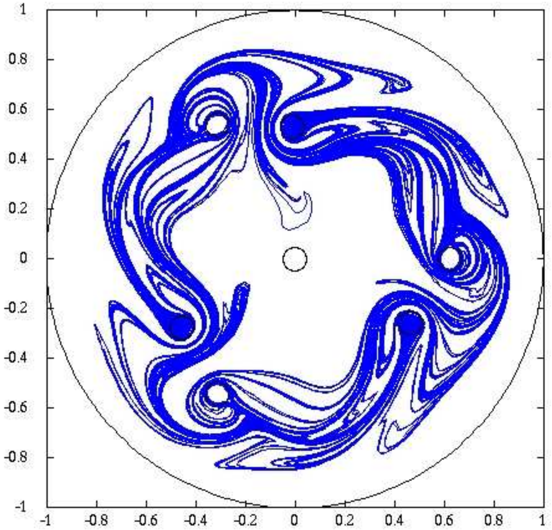
[movie 8]

Four Rods



[movie 9] [movie 10]

Six Rods



[movie 11]

Conclusions

- Walls can have a big impact and slow down mixing.
- It is sometimes possible to shield the walls from the mixing region, for instance by rotating the vessel.
- Having rods undergo 'braiding' motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs, and can be used for rigorous proofs.
- We have an optimal design, the silver mixers.
- Need to also optimise other mixing measures, such as variance decay rate.

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