The Structure of Lie–Poisson Systems

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Overview

- Many physical systems have a Hamiltonian formulation in terms of Lie–Poisson brackets obtained from Lie algebra extensions.
- For concreteness, we will treat the case of 2D fluid brackets, which give rise to a variety of fluid and plasma systems. (There is an abstract formulation with wider applicability.)
- The simplest extension has a direct sum structure, and leads to multifluid systems.
- We classify low-order brackets, thus showing that there are only a small number of independent normal forms. We make use of Lie algebra cohomology to achieve this.

Hamiltonian Formulation

A system of equations has a Hamiltonian formulation if it can be written in the form

$$\boxed{\dot{\xi}^\lambda(\mathbf{x},t) = \{\xi^\lambda\,, H\}}$$

where H is a Hamiltonian functional, and $\xi(\mathbf{x})$ represents a vector of field variables (vorticity, temperature, ...).

The Poisson bracket $\{\,,\}$ is antisymmetric and satisfies the Jacobi identity,

$$\{F\,, \{G\,, H\}\} + \{G\,, \{H\,, F\}\} + \{H\,, \{F\,, G\}\} = 0.$$

Jacobi tells us that there exist local canonical coordinates.

The Lie–Poisson Bracket

We define the Lie–Poisson bracket for one field variable as

$$\{F\,,G\} := \int_\Omega \omega(\mathbf{x}',t)\left[\frac{\delta F}{\delta\omega(\mathbf{x}',t)}\,,\frac{\delta G}{\delta\omega(\mathbf{x}',t)}\right]\,\mathrm{d}^2x'$$

The spatial coordinates are $\mathbf{x} = (x, y)$, and the inner bracket is the 2-D Jacobian,

$$[\,a\,,b\,] = \frac{\partial a}{\partial x}\,\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\,\frac{\partial a}{\partial y}.$$

The 2-D fluid domain is denoted by Ω.

The 2-D Euler Equation

Consider the Hamiltonian

$$H[\omega] = \tfrac{1}{2} \int_\Omega |\nabla\phi(\mathbf{x},t)|^2 \, \mathrm{d}^2x, \qquad \frac{\delta H}{\delta\omega} = -\phi,$$

where ϕ is the streamfunction and $\omega = \nabla^2\phi$ is the vorticity.

Inserting this into the Lie–Poisson bracket, we have

$$\begin{aligned}
\dot\omega(\mathbf{x},t) = \{\omega\,,H\} &= \int_\Omega \omega(\mathbf{x}',t) \left[\frac{\delta\omega(\mathbf{x},t)}{\delta\omega(\mathbf{x}',t)}\,, \frac{\delta H}{\delta\omega(\mathbf{x}',t)} \right] \mathrm{d}^2x' \\
&= \int_\Omega \omega(\mathbf{x}',t)\,[\,\delta(\mathbf{x}-\mathbf{x}')\,,-\phi(\mathbf{x}',t)\,]\;\mathrm{d}^2x' \\
&= \int_\Omega \delta(\mathbf{x}-\mathbf{x}')\,[\,\omega(\mathbf{x}',t)\,,\phi(\mathbf{x}',t)\,]\;\mathrm{d}^2x' \;\boxed{= [\,\omega(\mathbf{x},t)\,,\phi(\mathbf{x},t)\,]}\,,
\end{aligned}$$

which is Euler's equation for the 2-D ideal fluid.

Lie–Poisson Bracket Extensions

Now, say we wish to describe a physical system consisting of several field variables. The most general linear combination of one-field brackets is

$$\{F\,,G\} = \int_\Omega W_\lambda{}^{\mu\nu}\,\xi^\lambda(\mathbf{x}',t)\left[\frac{\delta F}{\delta\xi^\mu(\mathbf{x}',t)}\,,\frac{\delta G}{\delta\xi^\nu(\mathbf{x}',t)}\right]\mathrm{d}^2x'$$

where repeated indices are summed from 0 to n. The 3-tensor W is constant, and determines the structure of the bracket.

We call this type of bracket an extension of the one-field bracket.

Properties of W

In order for the extension to be a good Poisson bracket, it must satisfy

1. Antisymmetry: Since the inner bracket [,] is already antisymmetric, W must be symmetric in its upper indices:

$$\boxed{W_\lambda{}^{\mu\nu} = W_\lambda{}^{\nu\mu}}\ .$$

2. Jacobi identity: assuming the inner bracket [,] satisfies Jacobi, it is easy to show that W must satisfy

$$\boxed{W_\lambda{}^{\sigma\mu}\, W_\sigma{}^{\tau\nu} = W_\lambda{}^{\sigma\nu}\, W_\sigma{}^{\tau\mu}}\ .$$

If we look at W as a collection of matrices $W^{(\mu)}$, then this means that these matrices commute.

Example: Compressible Reduced MHD

The four-field model derived by Hazeltine et al. (1987) for 2-D compressible reduced MHD (CRMHD) has a Lie–Poisson structure. The model includes compressibility and finite ion Larmor radius effects. The field variables are

ω	vorticity
v	parallel velocity
p	pressure
ψ	magnetic flux

and are functions of (x, y, t).

There is also a constant parameter β_{e} that measures compressibility.

The equations of motion for CRMHD are

$$\begin{aligned}\dot{\omega} &= [\,\omega\,,\phi\,] + [\,\psi\,,J\,] + 2\,[\,p\,,x\,]\\ \dot{v} &= [\,v\,,\phi\,] + [\,\psi\,,p\,] + 2\beta_{\rm e}\,[\,x\,,\psi\,]\\ \dot{p} &= [\,p\,,\phi\,] + \beta_{\rm e}\,[\,\psi\,,v\,]\\ \dot{\psi} &= [\,\psi\,,\phi\,]\,,\end{aligned}$$

where $\omega = \nabla^2\phi$, ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current.

The Hamiltonian functional is just the total energy,

$$H[\omega, v, p, \psi] = \frac{1}{2}\int_\Omega \left(|\nabla\phi|^2 + v^2 + \frac{(p - 2\beta_{\rm e}\,x)^2}{\beta_{\rm e}} + |\nabla\psi|^2\right)\,{\rm d}^2x.$$

The equations for CRMHD can be obtained by inserting this Hamiltonian into the Lie–Poisson bracket

$$\{F\,,G\} = \int_\Omega \Bigg(\omega \left[\frac{\delta F}{\delta\omega}\,, \frac{\delta G}{\delta\omega} \right] + v \left(\left[\frac{\delta F}{\delta\omega}\,, \frac{\delta G}{\delta v} \right] + \left[\frac{\delta F}{\delta v}\,, \frac{\delta G}{\delta\omega} \right] \right) + p \left(\left[\frac{\delta F}{\delta\omega}\,, \frac{\delta G}{\delta p} \right] + \left[\frac{\delta F}{\delta p}\,, \frac{\delta G}{\delta\omega} \right] \right) + \psi \left(\left[\frac{\delta F}{\delta\omega}\,, \frac{\delta G}{\delta\psi} \right] + \left[\frac{\delta F}{\delta\psi}\,, \frac{\delta G}{\delta\omega} \right] \right) - \beta_{\mathrm{e}}\, \psi \left(\left[\frac{\delta F}{\delta p}\,, \frac{\delta G}{\delta v} \right] + \left[\frac{\delta F}{\delta v}\,, \frac{\delta G}{\delta p} \right] \right) \Bigg)\, \mathrm{d}^2x.$$

Comparing this to our definition of the Lie–Poisson bracket, with the identification $\boxed{(\xi^0, \xi^1, \xi^2, \xi^3) = (\omega, v, p, \psi)}$, we can read off the tensor W...

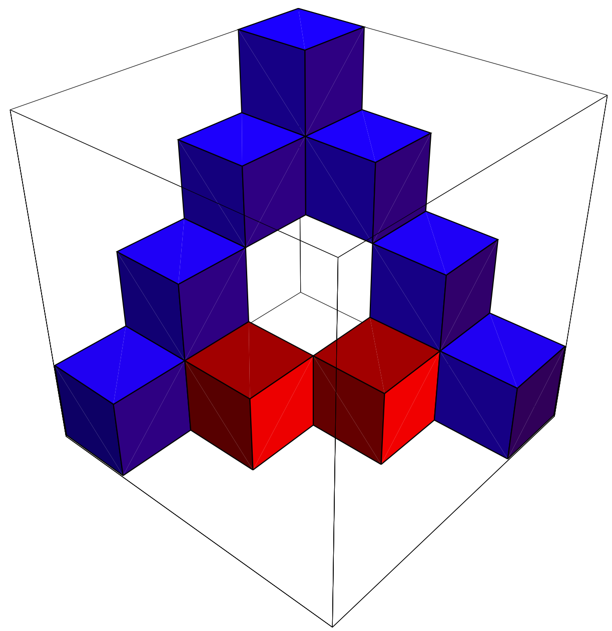
The W tensor for CRMHD

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{\mathrm{e}} & 0 \end{pmatrix},$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\beta_{\mathrm{e}} & 0 & 0 \end{pmatrix}, \qquad W^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that these commute, so that the Jacobi identity holds. (Note the lower-triangular structure.)

Since W is a 3-tensor, we can represent it as a cube:



The vertical axis is the lower index of $W_\lambda{}^{\mu\nu}$, with the origin at the top rear. The two horizontal axes are the symmetric upper indices.

Classification of Brackets

How many independent extensions are there?

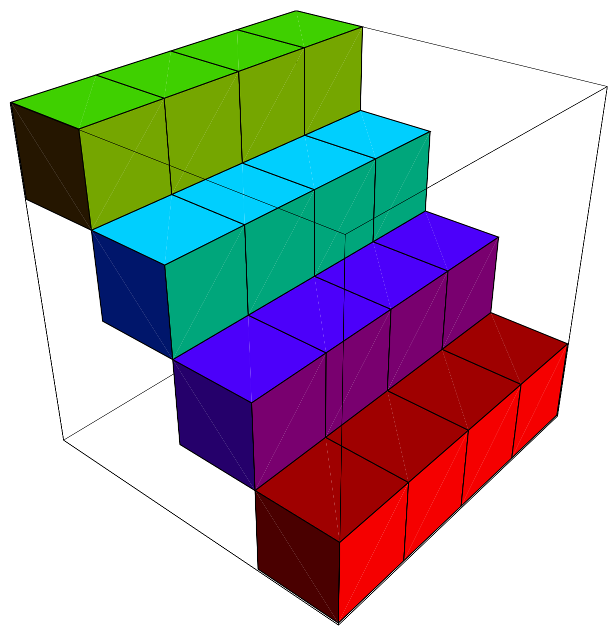
The answer amounts to finding normal forms for W, independent under coordinate transformations.

Threefold process:

1. Decomposition into a direct sum.
2. Transforming the matrices $W^{(\mu)}$ to lower-triangular form.
3. Finally, the hard part is to use Lie algebra cohomology to (almost) achieve the classification.

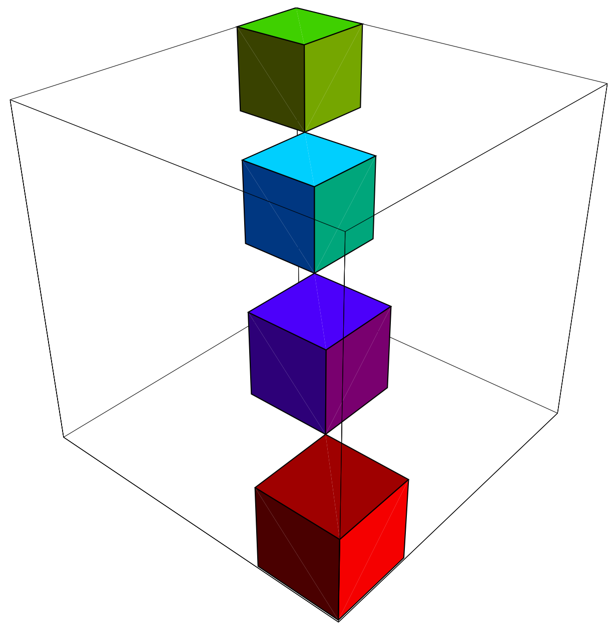
Classification 1: Direct Sum Structure

A set of commuting matrices, by a coordinate transformation, can always be put in block-diagonal form. The 3-tensor W then looks like:



Each “step” corresponds to a degenerate eigenvalue of the $W^{(\mu)}$.

Then, the symmetry of the upper indices of W implies the following structure:



We can focus on each block independently.

Direct Sum Structure: Example

Consider the 2D model for reduced MHD with electron inertia (d_{e}) and compressibility effects (ϱ_{s}) (Cafaro *et al.*, 1998):

$$\frac{\partial F}{\partial t} + [\,\phi\,, F\,] = \varrho_{\mathrm{s}}^2\,[\,\omega\,, \psi\,]$$
$$\frac{\partial \omega}{\partial t} + [\,\phi\,, \omega\,] = [\,\psi\,, J\,]\,,$$

where $F := \psi - d_{\mathrm{e}}^2\,J$.

The conserved Hamiltonian (energy) is

$$H = \int_\Omega (|\nabla\psi|^2 + d_{\mathrm{e}}^2\,J^2 + \varrho_{\mathrm{s}}^2\,\omega^2 + |\nabla\phi|^2)\,\mathrm{d}^2x$$

The equations are generated by a bracket given by

$$W^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad W^{(2)} = \begin{pmatrix} 0 & d_{\mathrm{e}}^2\,\varrho_{\mathrm{s}}^2 \\ 1 & 0 \end{pmatrix}.$$

These matrices commute. $W^{(1)}$ has eigenvalues equal to unity, but $W^{(2)}$ has distinct eigenvalues

$$\lambda_{\pm} = \pm\, d_{\mathrm{e}}\,\varrho_{\mathrm{s}}$$

if $d_{\mathrm{e}}\,\varrho_{\mathrm{s}} \neq 0$. We can thus diagonalize $W^{(2)}$ by using

$$\boxed{G_{\pm} := F \pm d_{\mathrm{e}}\,\varrho_{\mathrm{s}}\,\omega}$$

as new coordinates.

In these coordinates, the bracket (W) becomes

$$\overline{W}^{(1)} = \begin{pmatrix} 2d_{\rm e}\,\varrho_{\rm s} & 0 \\ 0 & 0 \end{pmatrix} \qquad \overline{W}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & -2d_{\rm e}\,\varrho_{\rm s} \end{pmatrix},$$

which are manifestly diagonal matrices. The equations of motion are now

$$\frac{\partial G_\pm}{\partial t} + [\,\phi_\pm\,, G_\pm\,] = 0,$$

where $\phi_\pm := \phi \pm (\varrho_{\rm s}/d_{\rm e})\psi$.

These equations are in conservative form, with advecting velocities,

$$\mathbf{v}_\pm = \hat{\mathbf{z}} \times \nabla\phi_\pm .$$

- The conservative form has important implications: even though the topology of the magnetic potential ψ can change (reconnection), the topology of $G_{\pm}$ is invariant.
- In the singular limit $d_{\rm e}\,\varrho_{\rm s} \to 0$, the direct sum structure disappears (eigenvalues degenerate), and the topology of ψ is conserved.
- In fact, the bracket formulation gives us a way of numerically solving these equations while preserving the Hamiltonian structure (Zeitlin truncations). No spurious reconnection due to numerical dissipation!

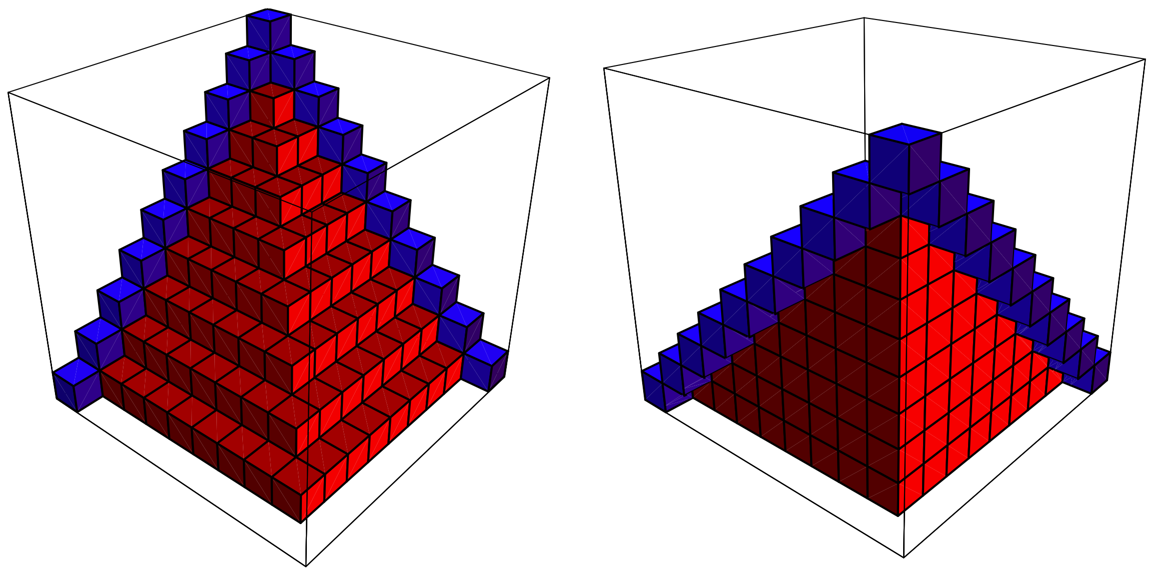
Classification 2: Lower-triangular Form

We focus on a single block, and thus assume that the $W^{(\mu)}$ have $(n+1)$-fold degenerate eigenvalues.

A set of commuting matrices can always be put into lower-triangular form by a coordinate transformation.

Once we do this, by the symmetry of the upper indices of W it is easy to show that only the eigenvalue of $W^{(0)}$ can be nonzero. Furthermore, if it is nonzero it can be rescaled to unity. We assume this is the case.

The most general form of W for an extension is thus



The red cubes form a solvable subalgebra, and are constrained by the commutation requirement. The blue cubes represent unit elements.

Classification 3: Cohomology

The problem of classifying extensions is reduced to classifying the solvable (red) part of the extension. This is achieved by the techniques of Lie algebra cohomology.

Cohomology gives us a class of linear transformations that preserve the lower-triangular structure of the extensions.

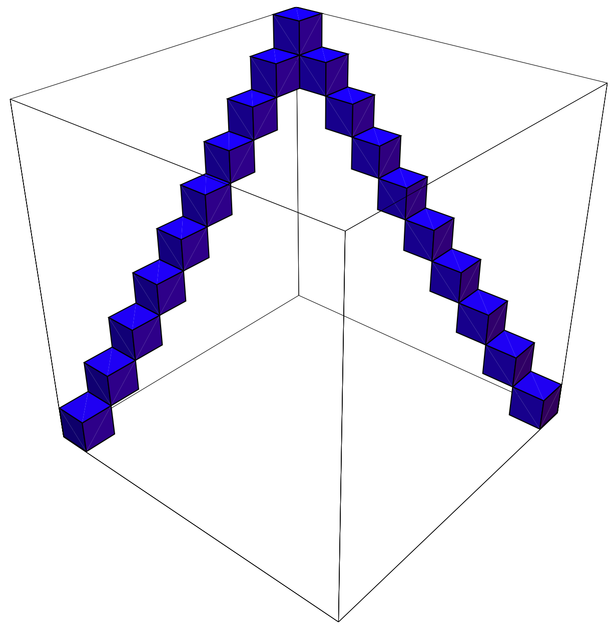
The parts of the extension that can be removed (i.e., made to vanish) by such transformations are called coboundaries.

What is left are nontrivial cocycles.

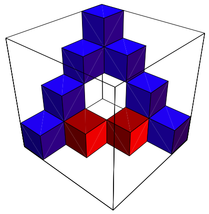
(Cohomology does not quite get it all...)

Pure Semidirect Sum

A common form for the bracket is the semidirect sum (SDS), for which the solvable part of W vanishes:

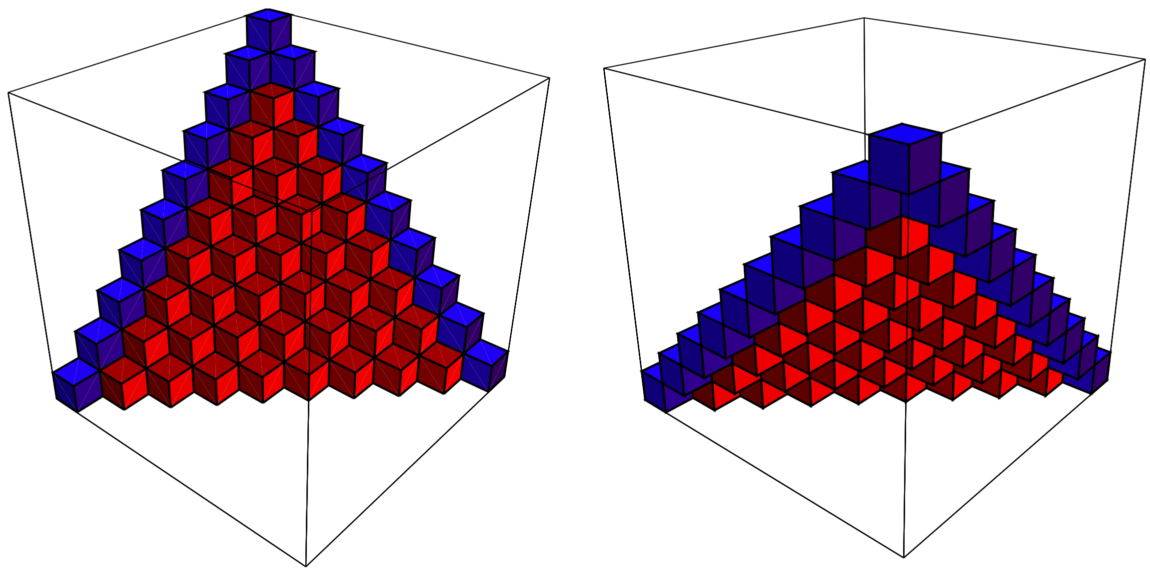


Note that CRMHD does not have a semidirect sum structure because of its extra nonzero blocks, proportional to β_e (a cocycle).



Leibniz Extension

The opposite extreme to the pure semidirect sum is the case for which none of the $W^{(\mu)}$ vanish. Then W must have the structure



This is called the Leibniz extension. All the cubes, red and blue, are equal to unity.

Alternate name: Q*Bert extension. . .

In between these two extreme cases, there are other possible extensions, including the CRMHD bracket.

Order	Number of extensions
1	1
2	1
3	2
4	4
5	9

None of these normal forms contains any free parameter!

(Do not expect this to be true at order 6 and beyond.)

Conclusions

- We gave examples of Lie–Poisson systems in fluid dynamics and plasma physics.
- An extension with a direct sum structure leads to a multifluid system.
- We classified Lie–Poisson bracket extensions, and found that for low orders there are very few independent brackets, with no free parameters.
- In other work, we have developed techniques for finding Casimir invariants of Lie–Poisson brackets (coextension) (Thiffeault and Morrison, submitted).
- Can use brackets or Casimirs to obtain general criteria for stability of Lie–Poisson systems (Ph.D. Thesis).