A Topological Theory of Rod Stirring

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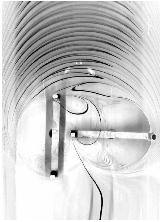
Channel flow: Injection into mixing region

Injection against flow



Injection with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 1] [movie 2] [movie 3]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

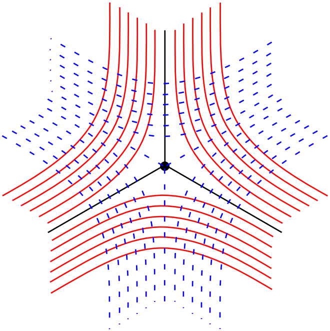
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

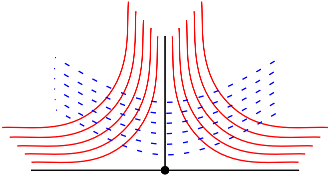
Number 3 is the one we want for good mixing

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

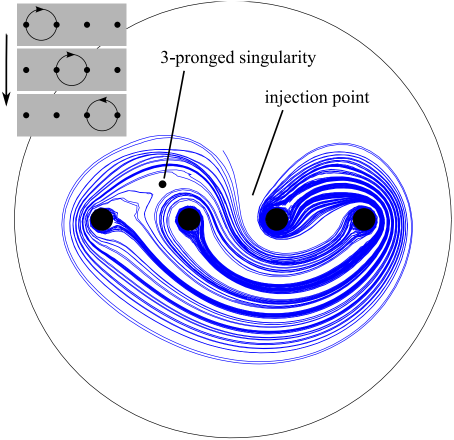


3-pronged singularity



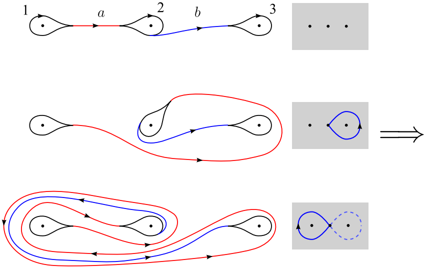
Boundary singularity

Visualising a singular foliation



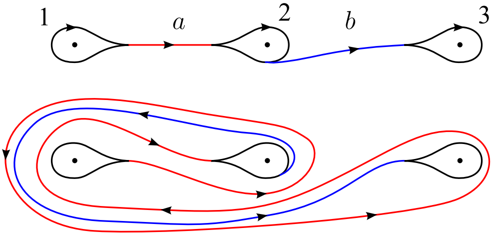
- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Train tracks



Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

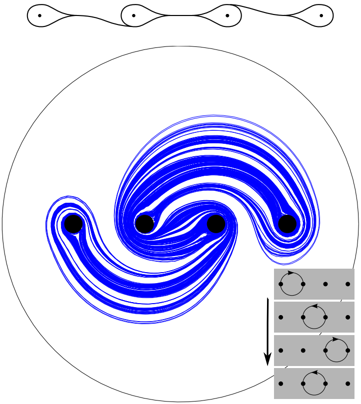
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of a and b, and write as matrix:

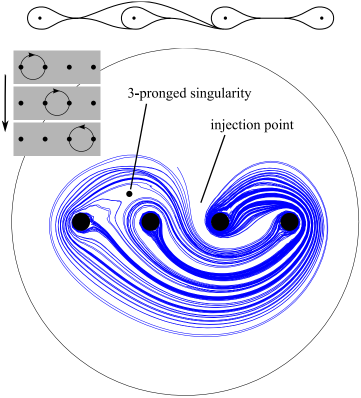
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the ‘blob’ is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



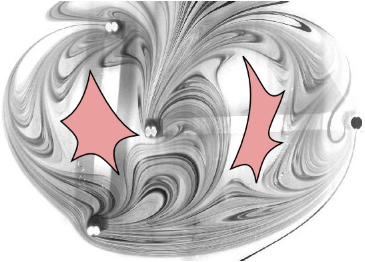
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

Back to the experiment

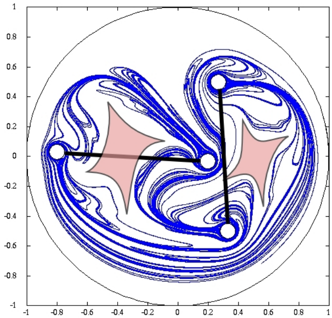


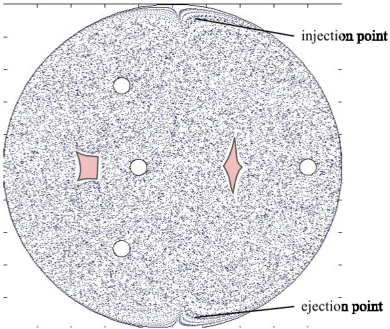
- Two 5-pronged singularities clearly visible;
- Created by the "slicing" of the rods;
- Only one injection point, at the top.

- Each 5-prong rotates unidirectionally;
- They are never interchanged with each other;
- Hence, the experimental picture greatly limits the possible pseudo-Anosovs that can occur.

Two 4-pronged singularities

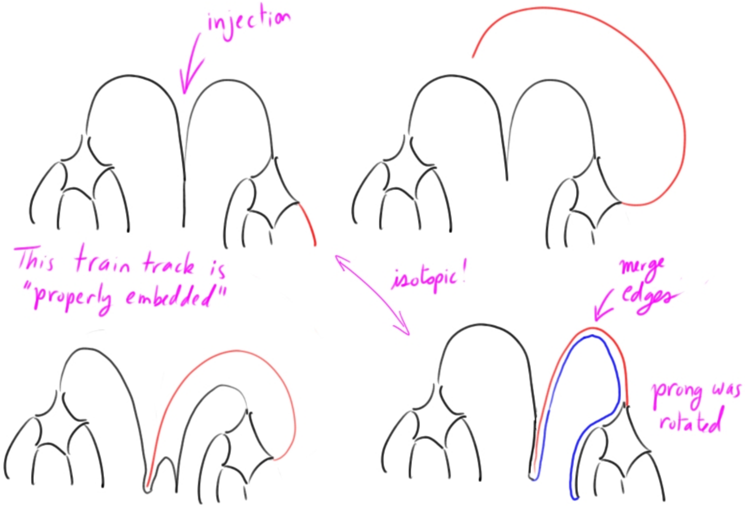
Same protocol, but in a closed container.





Varying the geometry changes the number of prongs: the pronged singularities rotate but lag behind the rods. Smaller rods will increase this lag, and thus the prongness.

A train track folding automaton

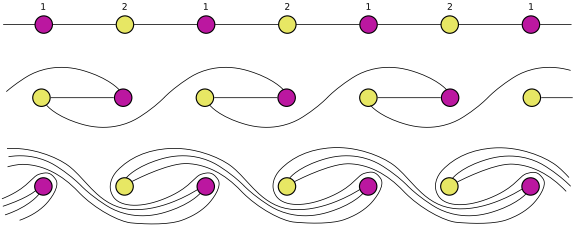


Train track automata (continued)

- Train track automata are a rigorous way of generating all pseudo-Anosovs associated with a train track.
- We know the train track type for our 4-rod experiment, just from watching the movie.
- The tiny automaton we built uniquely incorporates the constraints.
- Obtain a train track map by examining how edges are transformed and merged.
- For two k-prongs, the dilatation λ is the largest root of $x^{2k} - x^{2k-1} - 4x^k - x + 1$.
- Decreases with k, which indicates that smaller rods have less effect (shocking!).

Periodic Array of Rods

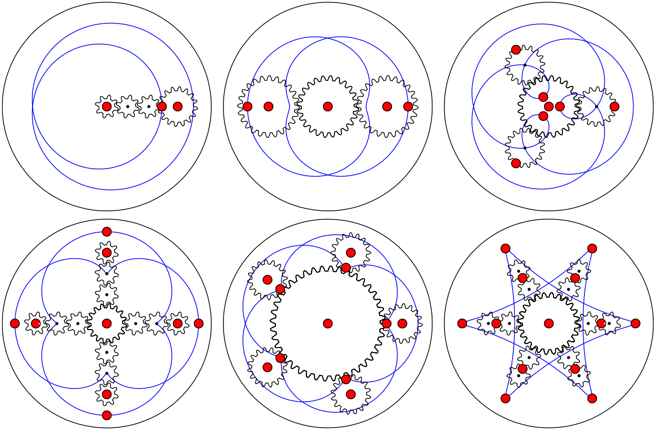
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The entropy per 'switch' is $\log \chi$, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).
- Work with postdoc M. D. Finn (now in Adelaide).

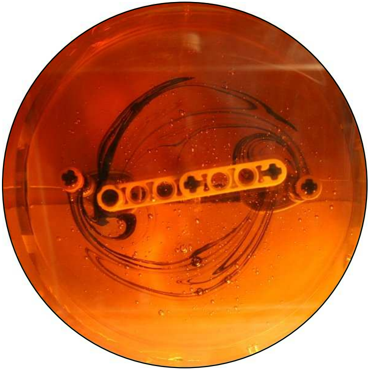
Silver Mixers!

- The designs with entropy given by the silver ratio can be realised with simple gears.
- All the rods move at once: very efficient.



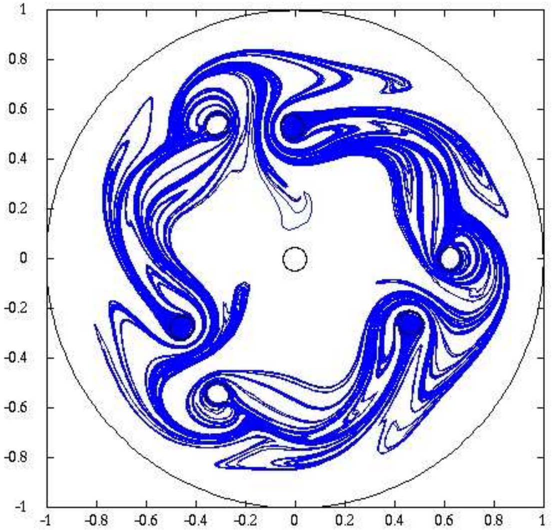
[movie 4]

Four Rods



[movie 5] [movie 6]

Six Rods



[movie 7]

Conclusions

- Having rods undergo 'braiding' motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs, and can be used for rigorous proofs.
- We have an optimal design, the silver mixers.
- Need to also optimise other mixing measures, such as variance decay rate.
- Holy grail: Three dimensions! (though current work applies to many 3D situations. . .)

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