Microorganism Billiards

Colin Wahl, Joseph Lukasik, Saverio Spagnolie, Jean-Luc Thiffeault

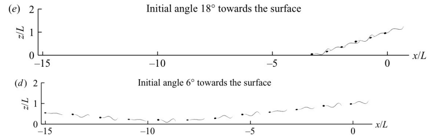
Department of Mathematics
University of Wisconsin – Madison

Fluid Mechanics Seminar, DAMTP, University of Cambridge
23 January 2015

Supported by NSF grant DMS-1109315

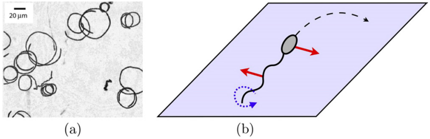
Swimming trajectories near a wall

Accumulation (*E. coli, spermatozoa*)



[Rothschild (1963); Berke *et al.* (2008); Smith *et al.* (2009)]

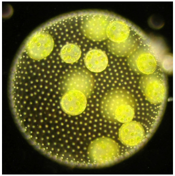
Circular swimming (*E. coli*)

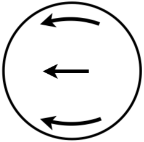


[Lauga *et al.* (2006)]

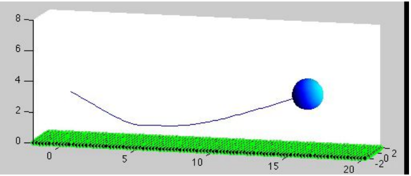
Glancing (potential flow squirmers)

Volvox, paramecia





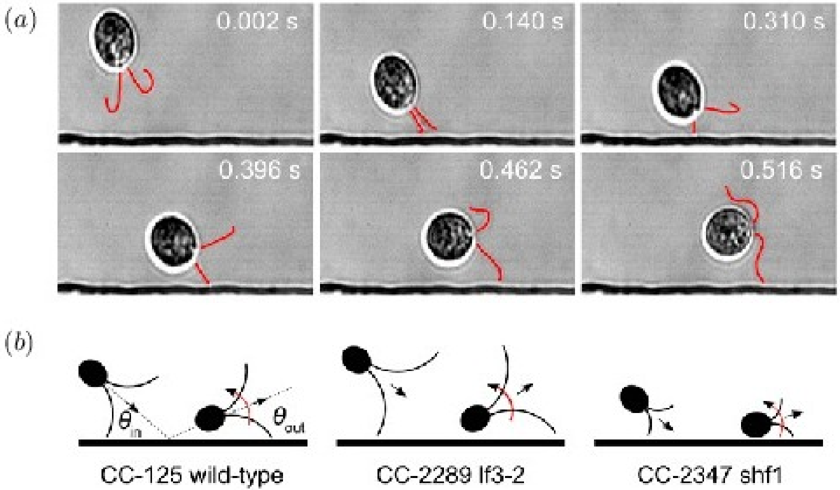
[Goldstein, Jung labs]



[Spagnolie & Lauga (2012); Crowdy & Or (2010); Li & Ardekani (2014)]

play movie

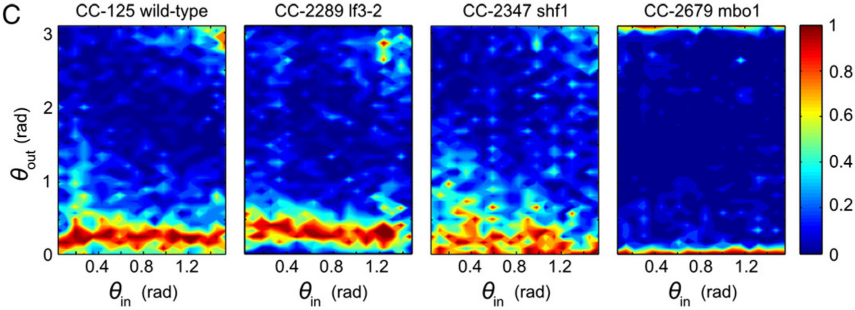
Chlamydomonas reinhardtii near a wall



play movie

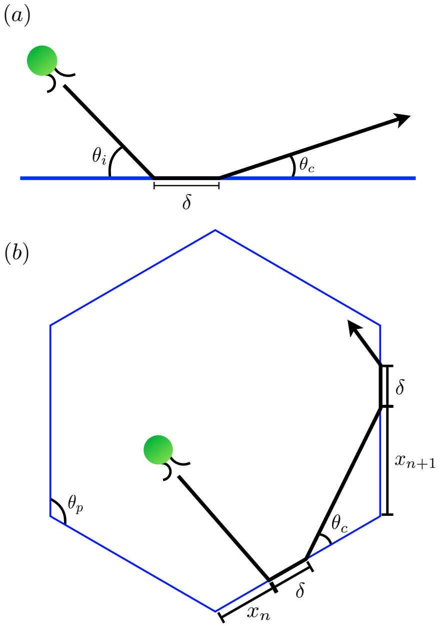
[Kantsler *et al.* (2013)]

Kantsler *et al.* (2013): scattering angles



Typical outgoing angle varies from 12° to 20°, with some spread.

Modeling the swimmer-wall interaction



Simple model of wall interaction:

- swimmers move in straight line between collisions;
- when hitting they ‘hug’ the wall for a distance δ;
- the outgoing angle is θ_c;
- regular polygonal domains, with interior angle $\theta_p = (1 - 2/N)\pi$.
- The position along a side on the nth hit is $x_n \in [0, 1]$.

Regular polygon: hitting the adjacent wall

Interior angle: $\theta_p = (1 - 2/N)\pi$;

First assume reflection angle $\theta_c \in (0, \pi/N)$, so that the swimmer always hits the next adjacent wall.

Take for now $\delta = 0$; we find using simple geometry

$$x_{n+1} = f(x_n) = \beta\,(1 - x_n)$$

where

$$\beta = \sin(\theta_c)/\sin(2\pi/N - \theta_c) \leq 1.$$

A simple linear map tells us where the swimmer hits the sides.

Hitting the adjacent wall (cont'd)

Trivial to solve:

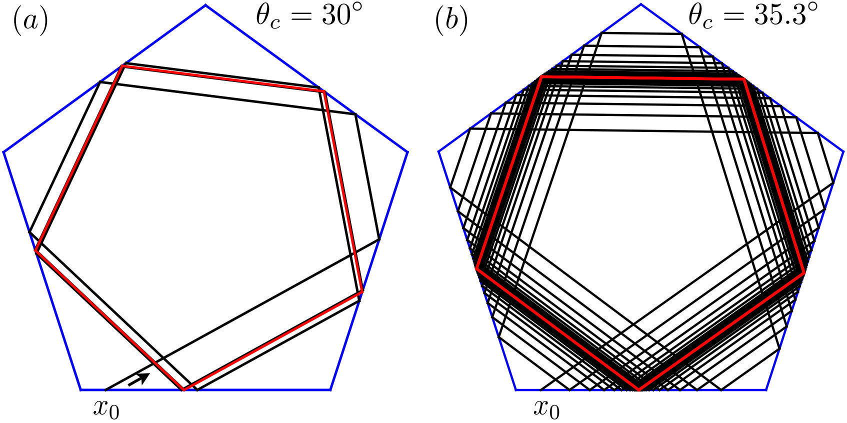
$$\begin{aligned} x_n &= (-\beta)^n x_0 - \sum_{i=1}^{n} (-\beta)^i \\ &= (-\beta)^n x_0 + \beta \, \frac{1-(-\beta)^n}{1+\beta}. \end{aligned}$$

For $\beta < 1$, the dynamics lose memory of the initial position x_0 exponentially fast in the number of impacts, and the swimmer settles in a stable periodic orbit with

$$x^* = \lim_{n\to\infty} x_n = \frac{\beta}{1+\beta} < \tfrac{1}{2}.$$

[$\beta = 1$ leads to an ∞ of neutrally-stable periodic orbits.]

Hitting the adjacent wall (cont'd)



play movie

Converges more slowly as $\beta \uparrow 1$ $(36°)$.

Regular polygon: sliding along the wall

Taking $\delta \neq 0$ (with δ small enough so we don't slide over to the next wall) doesn't modify the map very much:

$$x_n = (-\beta)^n x_0 + (\beta + \delta)\,\frac{1 - (-\beta)^n}{1 + \beta}.$$

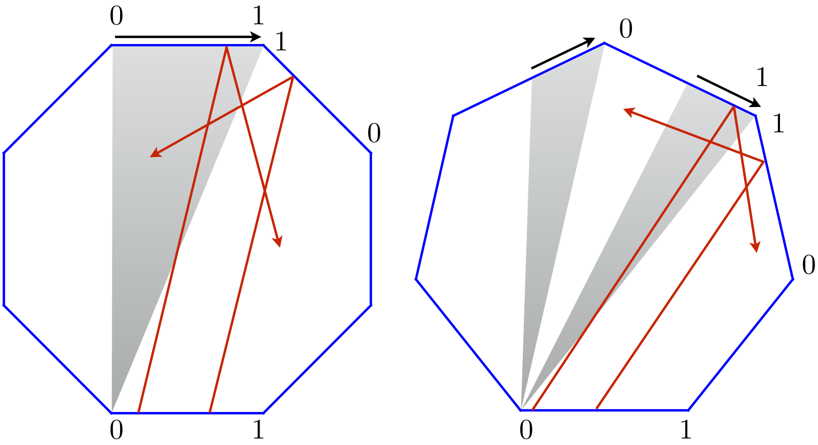
For $\beta < 1$, we get a corrected fixed point

$$x^* = (\delta + \beta)/(1 + \beta)$$

as $n \to \infty$.

Non-adjacent walls

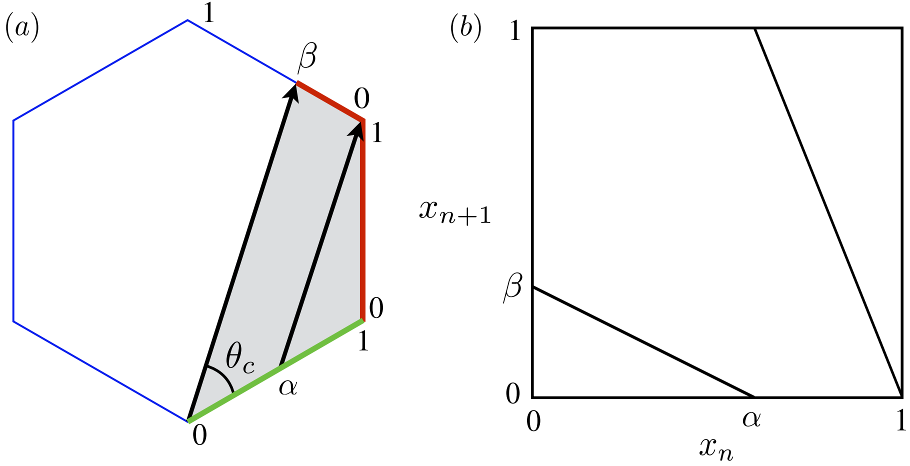
If the angle $\theta_c > \pi/N$ then the swimmer can 'skip' the next adjacent wall:



We need to orient the next side properly with respect to the direction of the bounce. A given side may have different orientations for different hits.

Hitting two walls of a hexagon

Typically swimmers can hit more than one wall depending on x_n:



Get a one-dimensional piecewise-linear discontinuous map!

Hitting two walls of a hexagon: the map

Here is the explicit map:

$$x_{n+1} = f(x_n) = \begin{cases} \beta\alpha^{-1}(\alpha - x_n), & x_n \le \alpha, \\ (1-\alpha)^{-1}(1 - x_n), & x_n > \alpha, \end{cases}$$

where α is the swimmer that hits the corner, and β is the image of 0 (previous slide). These can be worked out from simple geometry.

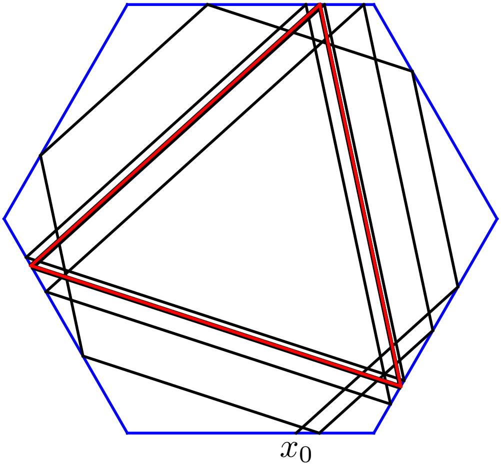
Note that:

$$|f'(x)| < 1 \text{ for } x < \alpha \quad \text{(stable)};$$
$$|f'(x)| > 1 \text{ for } x > \alpha \quad \text{(unstable)}.$$

There is thus a competition between stable and unstable behaviour. . . or is there? Staring at the map, one can see the stable side always wins.

Hitting two walls of a hexagon: stable

For some values we recover stable orbits, but for a smaller polygon:

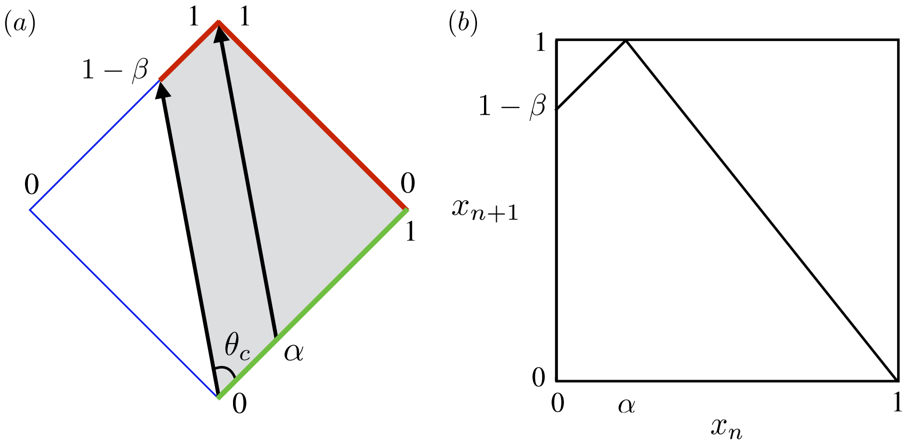


play movie

Here we get an inscribed triangle.

A square domain: simpler?

Let's try something else, a simple square:



This time the map is continuous, owing to the reversal of one interval.

A square domain: the map

Here's the map for this case:

$$x_{n+1} = f(x_n) = \begin{cases} \beta\alpha^{-1}(\alpha - x_n) & x_n \le \alpha, \\ (1-\alpha)^{-1}(1 - x_n) & x_n > \alpha, \end{cases}$$

This is a classic tent map.

For $x < \alpha$, we have $|f'(x)| = 1$ (neutrally stable);

For $x > \alpha$, we have $|f'(x)| > 1$ (unstable).

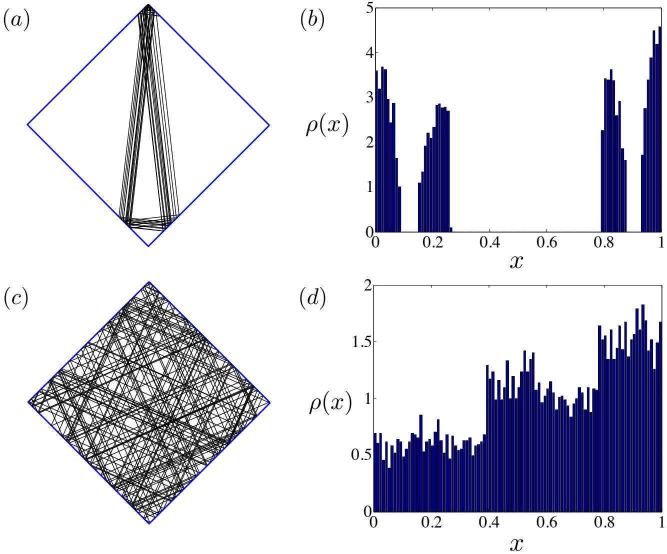
There is no stable region, and the single fixed point is unstable. Likely to get chaotic dynamics!

play movie

A square domain: invariant measures

$\theta_c = 52^\circ$

$\theta_c = 72^\circ$



Lyapunov exponents

For a one-dimensional map, the Lyapunov exponent is defined

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|$$

(There's only one exponent.) It describes the exponential rate of separation of neighbouring trajectories. It is a measure of chaos (> 0).

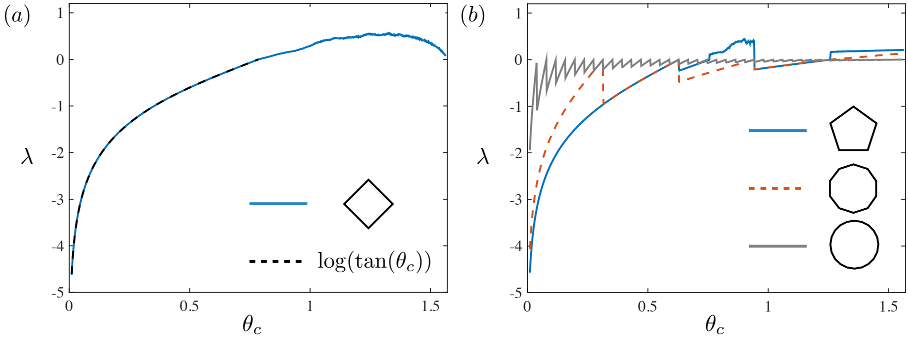
Note that λ is dimensionless: we can convert it to an inverse time to get the 'physical' exponent Λ by dividing by the mean time between hits, $\overline{T}$:

$$\Lambda = \lambda \,/\, \overline{T}, \qquad |\Lambda| \geq |\lambda|/\tau.$$

Here τ is the maximum time between hits. The important thing is that Λ and λ have the same sign (both chaotic, or not).

Lyapunov exponents as a function of angle

Fix shape and vary angle, for $N = 4, 5, 10, 20$ sides:



Note that when the exponent is negative we can in principle get an analytic formula, since it simply corresponds to a stable periodic orbit.

The effect of noise

Of course, since these are biological systems we expect lots of noise and uncertainty. For instance, maybe the swimmer doesn't travel in a straight line, so we add a Gaussian noise term to the simplest map:

$$x_{n+1} = \beta\,(1 - x_n) + \sigma Z_{n+1}$$

This can be solved exactly:

$$x_n = (-\beta)^n x_0 + \beta\,\frac{1-(-\beta)^n}{1+\beta} + \sigma\,\sqrt{\frac{1-\beta^{2n}}{1-\beta^2}}\,\tilde{Z}_n$$

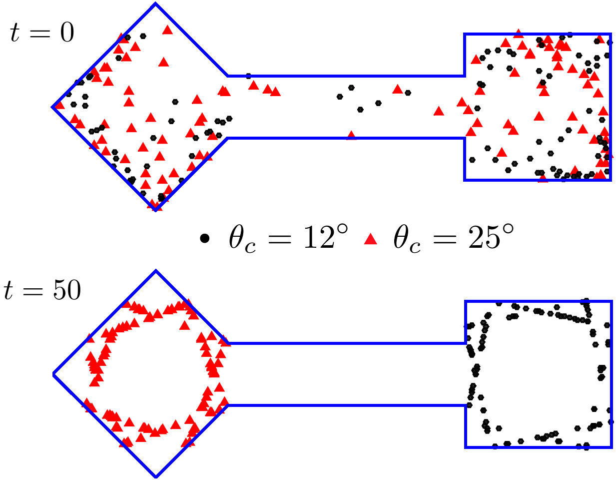
Asymptotes to (for $\beta < 1$)

$$x_n \sim \frac{\beta}{1+\beta} + \frac{\sigma \tilde{Z}_n}{\sqrt{1-\beta^2}} \qquad n \to \infty.$$

Thus the fixed point survives as long as β is not too close to 1.

Sorting with small noise

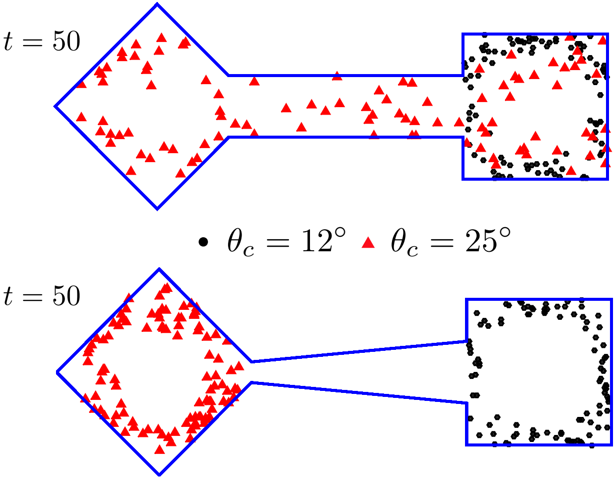
Two different swimmers:



The orientations are chosen such that the fixed points of the two types of swimmers are at the exits.

Sorting with larger noise

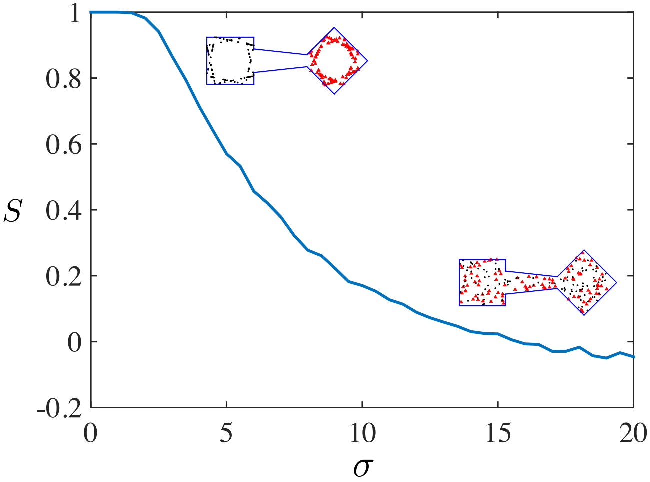
Now with more noise: the old design doesn't work so well.



Need to compensate for larger spread of the triangle swimmers (larger β).

Sorting with smaller angles

The small angles in Kantsler *et al.* (2013) make it more difficult to sort:

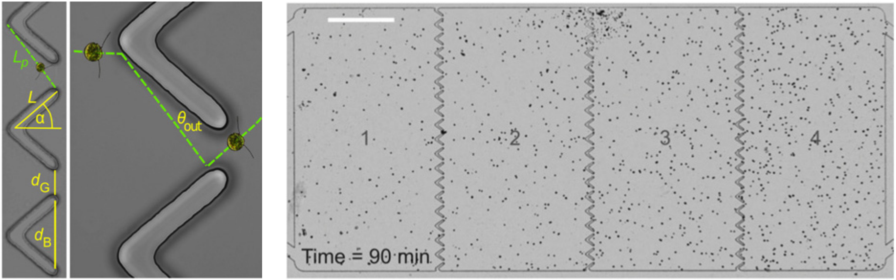


Too much noise (σ) makes it impossible to sort.

play movie

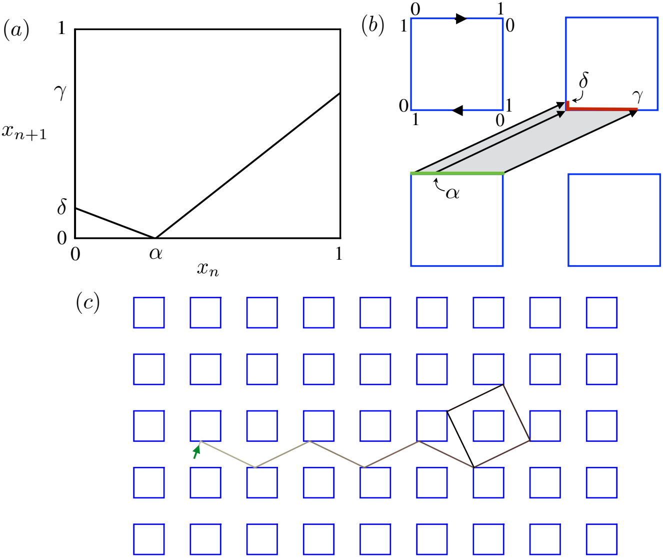
Open sorting

Note that Kantsler *et al.* (2013) also proposed (and built) a sorting mechanism using scattering and rectification.



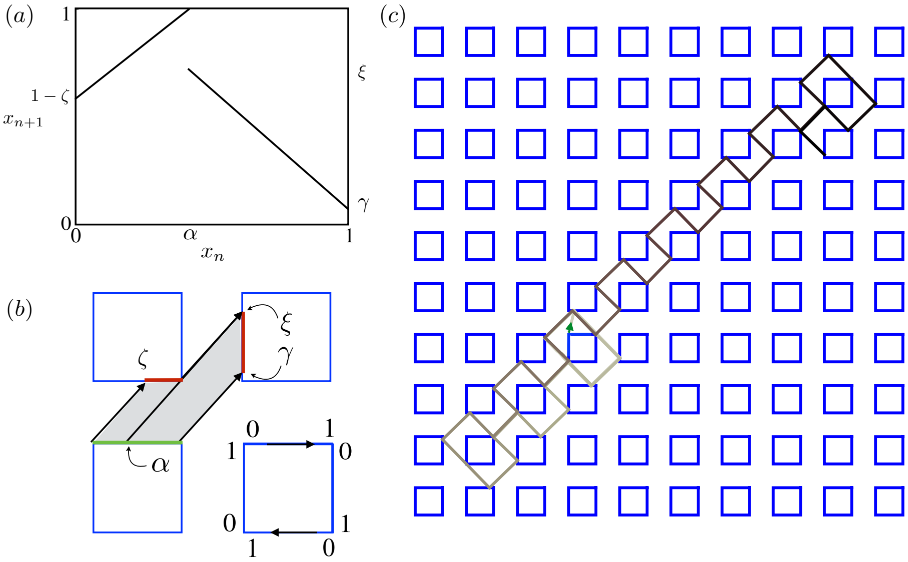
Theirs works by a cascading process. (Ours doesn't use the rectification mechanism.)

Lattice of obstacles: the exterior problem



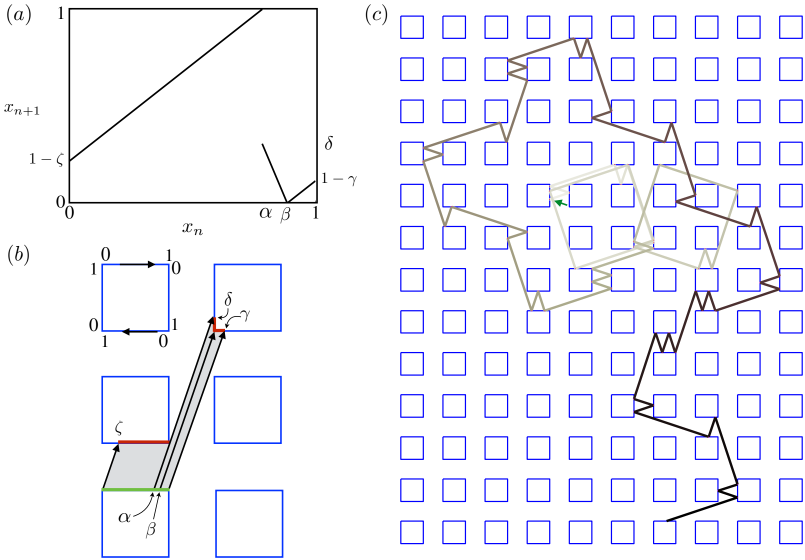
play movie play movie

The exterior problem: larger angle



play movie

The exterior problem: even larger angle



play movie

Some research directions

Some things that remain to be done:

- A full classification, connecting to the theory of discontinuous piecewise-linear maps. (Mostly for mathematicians. . .)
- Irregular shapes? Channels?
- Large angles not observed (chaos), but small angles good for sorting.
- Three-dimensional swimming? (More input from experiments.)
- Exterior problem: when do swimmers escape the lattice?

References

Berke, A. P., Turner, L., Berg, H. C., & Lauga, E. (2008). *Phys. Rev. E,* **101**, 038102.

Crowdy, D. G. & Or, Y. (2010). *Phys. Rev. E,* **81**, 036313.

Kantsler, V., Dunkel, J., Polin, M., & Goldstein, R. E. (2013). *Proc. Natl. Acad. Sci. USA,* **110** (4), 1187–1192.

Lauga, E., DiLuzio, W. R., Whitesides, G. M., & Stone, H. A. (2006). *Biophys. J.* **90**, 400–412.

Li, G.-J. & Ardekani, A. M. (2014). *Phys. Rev. E,* **90**, 013010.

Rothschild, A. J. (1963). *Nature,* **198**, 1221–1222.

Smith, D. J., Gaffney, E. A., Blake, J. R., & Kirkman-Brown, J. C. (2009). *J. Fluid Mech.* **621**, 289–320.

Spagnolie, S. E. & Lauga, E. (2012). *J. Fluid Mech.* **700**, 1–43.

Wahl, C., Lukasik, J., Spagnolie, S. E., & Thiffeault, J.-L. (2015). Preprint.