

Braids of entangled particle trajectories

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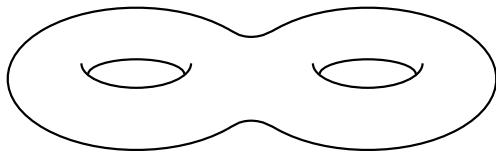
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Surface dynamics

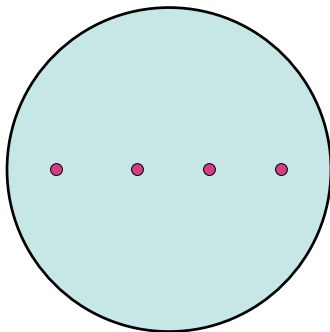
Low-dimensional topologists have long studied **transformations of surfaces** such as the **double-torus**:



The central object of study is the **homeomorphism**: a continuous, invertible transformation whose inverse is also continuous.

Punctured disks

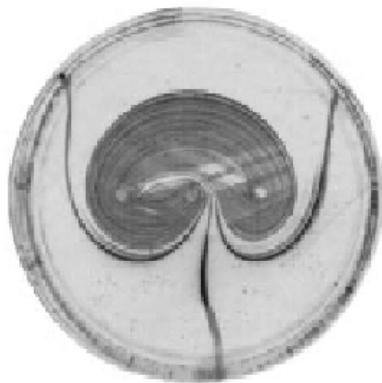
A surface of more practical relevance is the punctured disk:



For instance, it is a model of a two-dimensional vat of viscous fluid with stirring rods.

Punctured disks in experiments

The transformation in this case is given by the solution of a fluid equation over one period of rod motion.

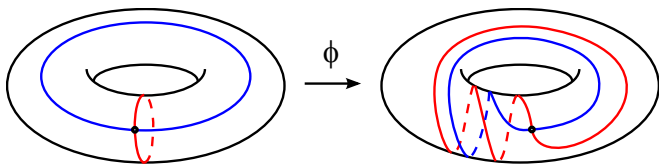


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

[movie 1] [movie 2]

Action on curves

If we don't know anything about a transformation ϕ , we can learn a lot by looking at its action on some representative curves:



This is the action of the famous **cat map** of Arnold. In the language of topology we are looking at its action on the **fundamental group**.

Note that since the curves initially intersect only once, their image only intersects once as well.

Growth of curves for Cat Map

The Cat Map can be written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod{1}$$

where the torus is the bi-periodic domain $[0, 1]^2$.

At each application of the map, curves grow asymptotically by a factor given by the largest eigenvalue of the matrix,

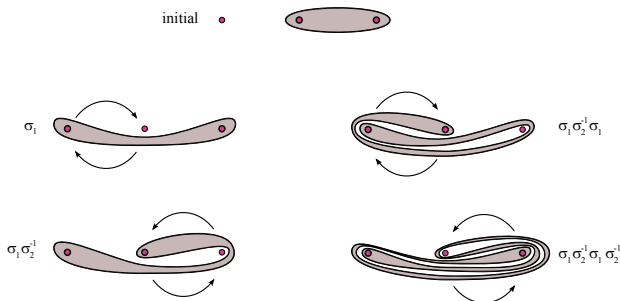
$$\lambda = \frac{1}{2}(3 + \sqrt{5}) = \varphi^2, \quad \varphi := \frac{1}{2}(1 + \sqrt{5})$$

where φ is the Golden Ratio.

The rate of growth $h = \log \lambda$ is called the [topological entropy](#).

Growth of curves on a disk

On a disk with 3 punctures (rods), we can also look at the growth of curves:



We use the **braid generator** notation: σ_i means the clockwise interchange of the i th and $(i + 1)$ th rod. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1 \sigma_2^{-1}$.

Growth of curves on a disk (2)

But how do we find the rate of growth of curves for motions on the disk?

For 3 punctures it's easy: the entropy for $\sigma_1\sigma_2^{-1}$ is $h = \log \phi^2$, just like the Cat Map!

(This is not a coincidence: there is an intimate connection between the two. For the specialist: the key word is [double cover](#).)

For more punctures, this is a hard problem.

Entropy calculation

The problem: given a periodic motion of n punctures on a disk, what is the entropy?

Many approaches available:

- **Interval exchange map** (orientable foliations — not general enough);
- Train tracks and **Bestvina–Handel algorithm** (1995) (computationally very hard — overkill);
- **Burau representation** (Kolev, 1989): super-fast, but only a lower bound;
- **Moussafir iterative technique** (2006): fast and intuitive!

The Moussafir technique allows us to tackle large-scale problems.

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

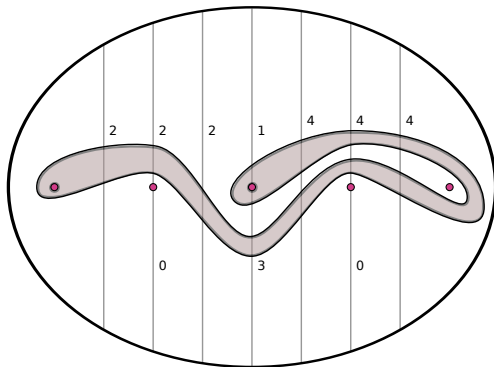
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them **topologically** with very few numbers.

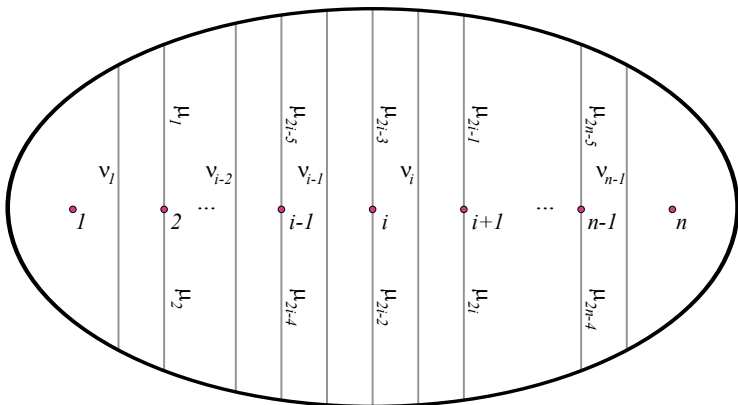
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the [Dynnikov coordinates](#) involve intersections with vertical lines:



Crossing numbers

Label the crossing numbers:



Dynnikov coordinates

Now take the difference of crossing numbers:

$$a_i = \frac{1}{2} (\mu_{2i} - \mu_{2i-1}),$$
$$b_i = \frac{1}{2} (\nu_i - \nu_{i+1})$$

for $i = 1, \dots, n - 2$.

The vector of length $(2n - 4)$,

$$\mathbf{u} = (a_1, \dots, a_{n-2}, b_1, \dots, b_{n-2})$$

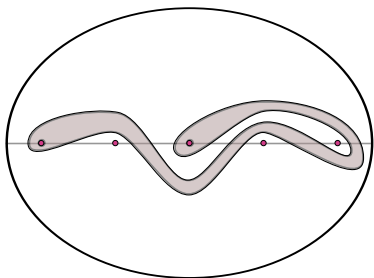
is called the **Dynnikov coordinates** of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can't do it with fewer than $2n - 4$ numbers.

Intersection number

A useful formula gives the **minimum intersection number** with the 'horizontal axis':

$$L(\mathbf{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i|,$$

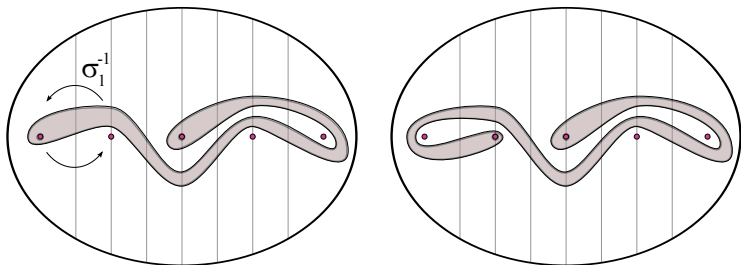


For example, the loop on the left has $L = 12$.

The crossing number grows proportionally to the length.

Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates!

Action on loop coordinates

The **update rules** for σ_i acting on a loop with coordinates (\mathbf{a}, \mathbf{b}) can be written

$$a'_{i-1} = a_{i-1} - b_{i-1}^+ - (b_i^+ + c_{i-1})^+,$$

$$b'_{i-1} = b_i + c_{i-1}^-,$$

$$a'_i = a_i - b_i^- - (b_{i-1}^- - c_{i-1})^-,$$

$$b'_i = b_{i-1} - c_{i-1}^-,$$

where

$$f^+ := \max(f, 0), \quad f^- := \min(f, 0).$$

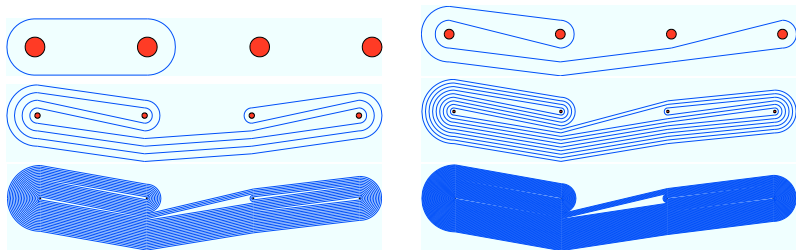
$$c_{i-1} := a_{i-1} - a_i - b_i^+ + b_{i-1}^-.$$

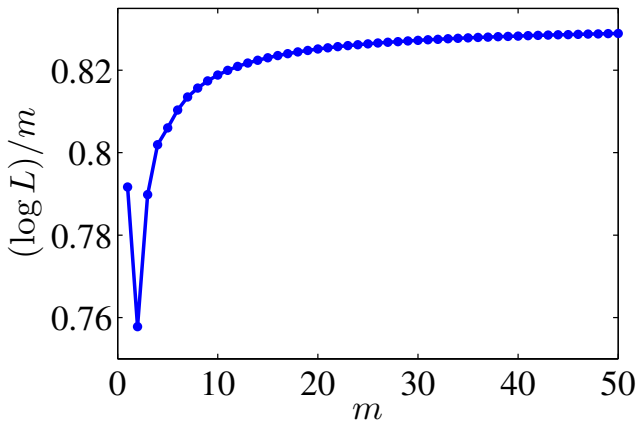
This is called a **piecewise-linear action**.

Easy to code up (see for example Thiffeault (2009)).

Growth of L

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:

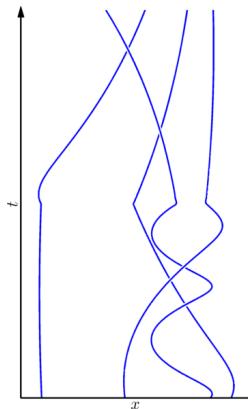
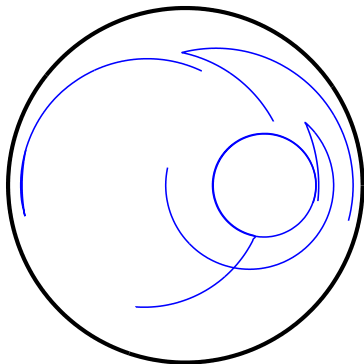


Growth of L (2)

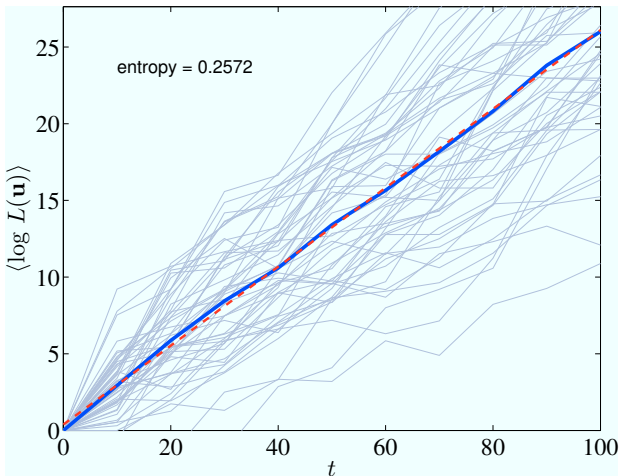
m is the number of times the braid acted on the initial loop.

Random particle trajectories

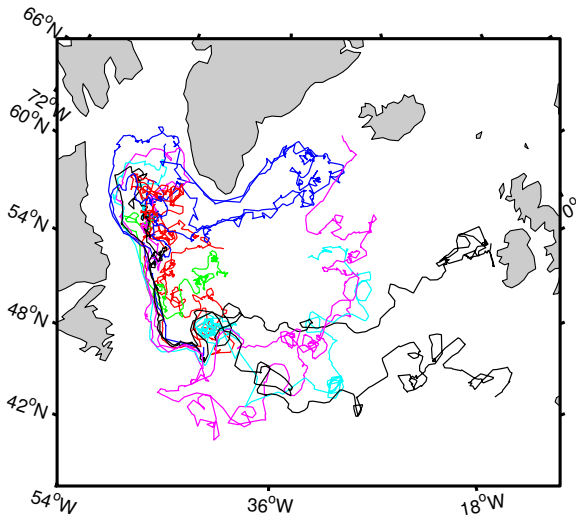
Now consider a set of n particles advected by some flow, such as the blinking vortex flow:



Entropy by averaging over trajectories



Oceanic float trajectories



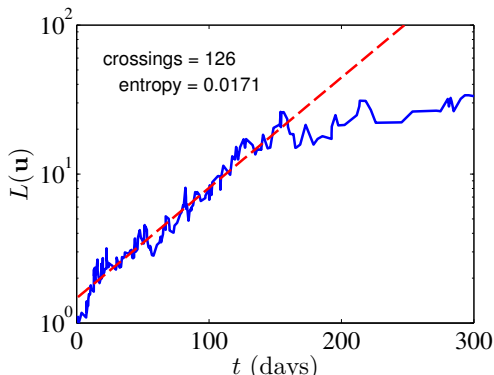
Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Oceanic floats: Entropy

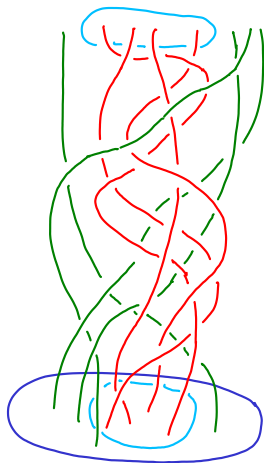
10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Lagrangian Coherent Structures



- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an **isolated region** in the flow that does not interact with the rest, a **Lagrangian coherent structure (LCS)**;
- There is a tool for this: Braid classification algorithms detect **reducing curves**.
- Hence, could identify LCS from particle trajectory data by searching for reducing curves.
- For now: doesn't scale well.

Special issue of *Chaos* on LCS

The New York Times **Science**

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Finding Order in the Apparent Chaos of Currents

By BINA VENKATARAMAN
Published: September 28, 2009

Suppose a blob of dioxin-rich pesticide is spilled into Monterey Bay. It might quickly disperse to the Pacific Ocean. But hours later, a spill of the same size at the same spot could circle near the coastline, posing a greater danger to marine life. The briny surface waters of the bay churn so chaotically that a slight shift in the place or time an oil drop, a buoy — or even a person — falls in can dictate whether it is swept out to the open ocean or swirls near the shore.

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[Enlarge This Image](#) But the results are not unpredictable. A team of scientists studying Monterey Bay since 2000 has found that underlying its complex, seemingly jumbled currents is a structure that guides the dispersal patterns, a structure that changes over time.



New York Times, 29 September 2009.

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (**stretching of material lines**);
- This idea can also be used on fluid particles to estimate entropy;
- Need a way to compute entropy fast: **loop coordinates**;
- There is a lot more information in this braid: extract it! (**Lagrangian coherent structures**);
- Long-term goal: a toolbox of topological methods to analyze and make prediction about general flow properties;
- Holy grail: **Three dimensions!** (though current work applies to many 3D situations. . .)
- To appear in special issue of *Chaos*. Preprint:
<http://arxiv.org/abs/0906.3647>

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References

- Bestvina, M. & Handel, M. 1995 Train-Tracks for Surface Homeomorphisms. *Topology* **34**, 109–140.
- Binder, B. J. & Cox, S. M. 2008 A Mixer Design for the Pigtail Braid. *Fluid Dyn. Res.* **49**, 34–44.
- Bowen, R. 1978 Entropy and the fundamental group. In *Structure of Attractors*, volume 668 of *Lecture Notes in Math.*, pp. 21–29. New York: Springer.
- Boyland, P. L. 1994 Topological methods in surface dynamics. *Topology Appl.* **58**, 223–298.
- Boyland, P. L., Aref, H. & Stremmer, M. A. 2000 Topological fluid mechanics of stirring. *J. Fluid Mech.* **403**, 277–304.
- Boyland, P. L., Stremmer, M. A. & Aref, H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95.
- Dynnikov, I. A. 2002 On a Yang–Baxter map and the Dehornoy ordering. *Russian Math. Surveys* **57**, 592–594.
- Gouillart, E., Finn, M. D. & Thiffeault, J.-L. 2006 Topological Mixing with Ghost Rods. *Phys. Rev. E* **73**, 036311.
- Hall, T. & Yurttaş, S. Ö. 2009 On the Topological Entropy of Families of Braids. *Topology Appl.* **156**, 1554–1564.
- Kolev, B. 1989 Entropie topologique et représentation de Burau. *C. R. Acad. Sci. Sér. I* **309**, 835–838. English translation at arXiv:math.DS/0304105.
- Moussafir, J.-O. 2006 On the Entropy of Braids. *Func. Anal. and Other Math.* **1**, 43–54. arXiv:math.DS/0603355.
- Thiffeault, J.-L. 2005 Measuring Topological Chaos. *Phys. Rev. Lett.* **94**, 084502.
- Thiffeault, J.-L. 2009 Braids of entangled particle trajectories. *Chaos*, in press. arXiv:0906.3647.
- Thiffeault, J.-L. & Finn, M. D. 2006 Topology, Braids, and Mixing in Fluids. *Phil. Trans. R. Soc. Lond. A* **364**, 3251–3266.
- Thurston, W. P. 1988 On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc.* **19**, 417–431.