
Braiding and Mixing

Jean-Luc Thiffeault

Matthew Finn

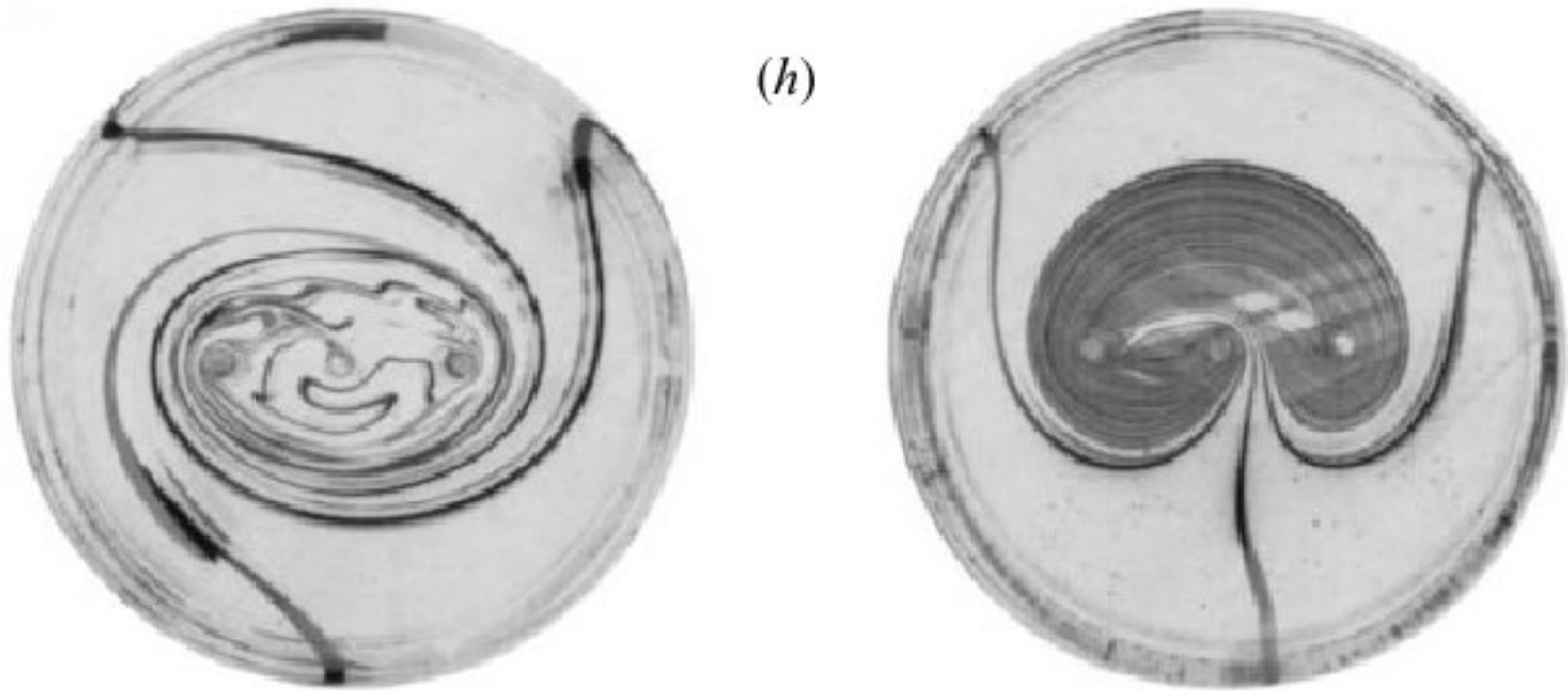
Emmanuelle Guillard

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Department of Mathematics

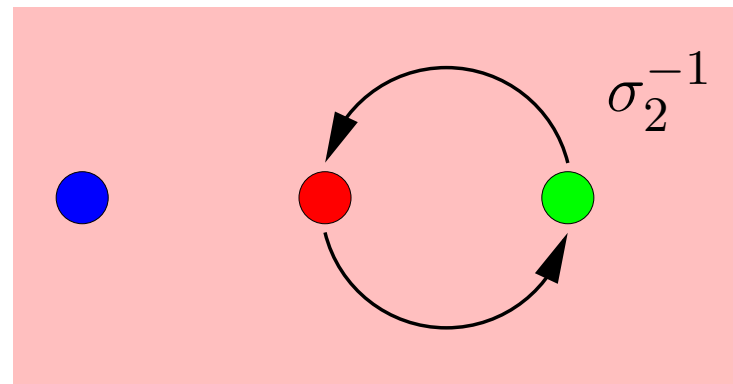
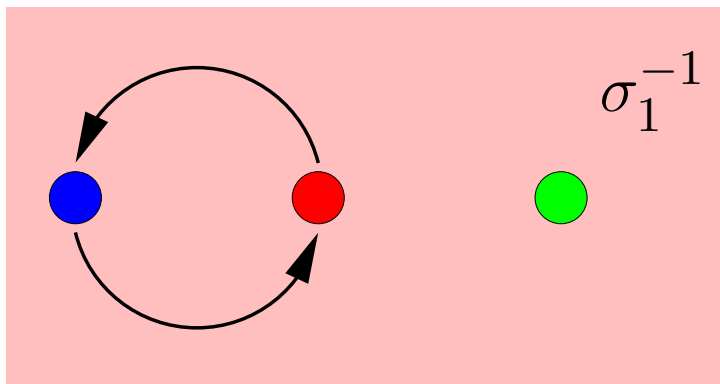
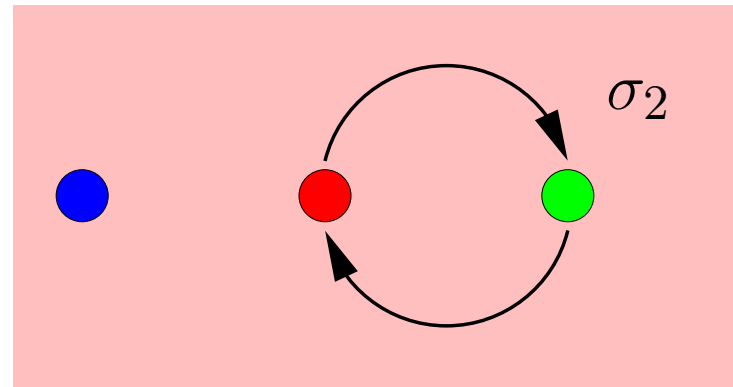
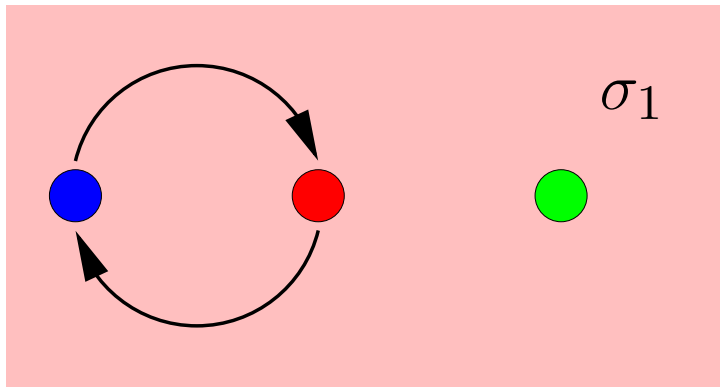
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Experiment of Boyland *et al.*



[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)] (movie by Matthew Finn)

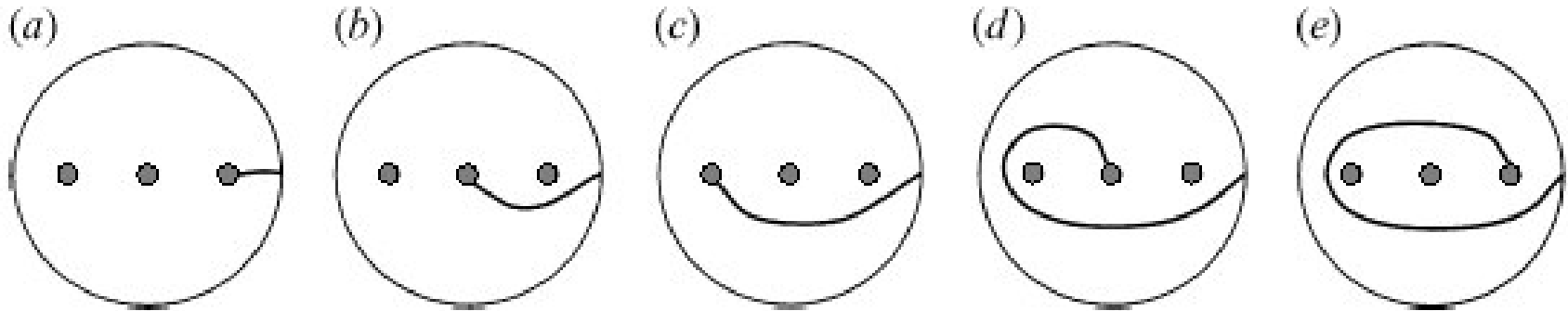
Four Basic Operations



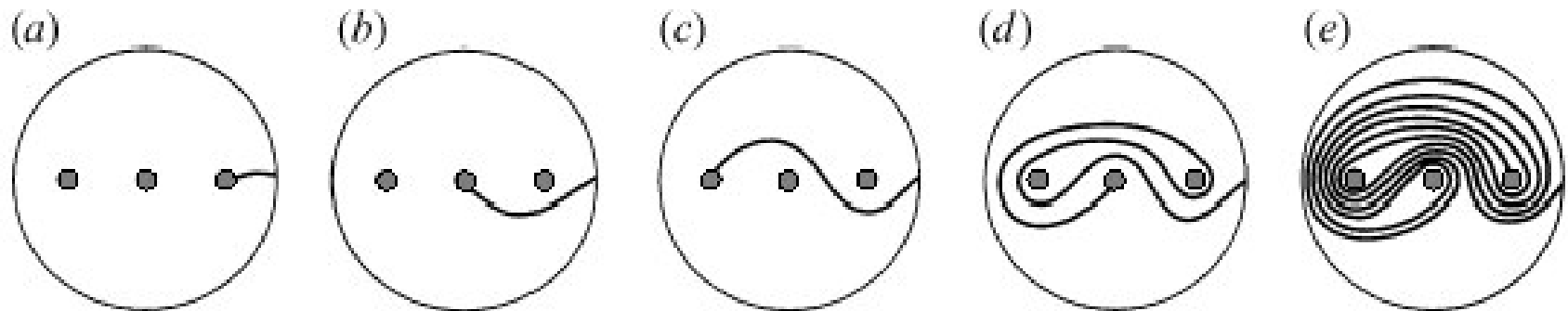
σ_1 and σ_2 are referred to as the **generators of the 3-braid group**.

Two Stirring Protocols

$\sigma_1\sigma_2$ protocol



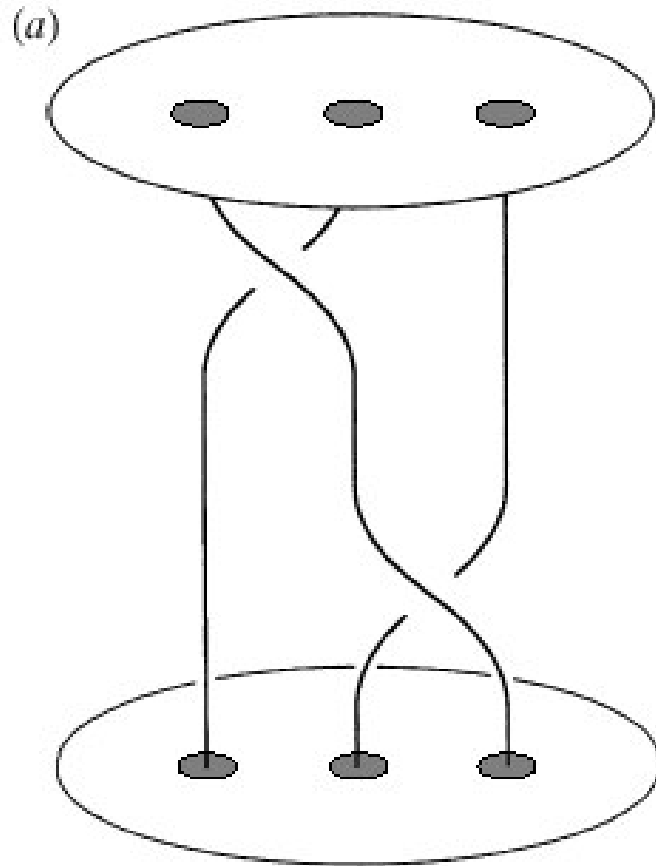
$\sigma_1^{-1}\sigma_2$ protocol



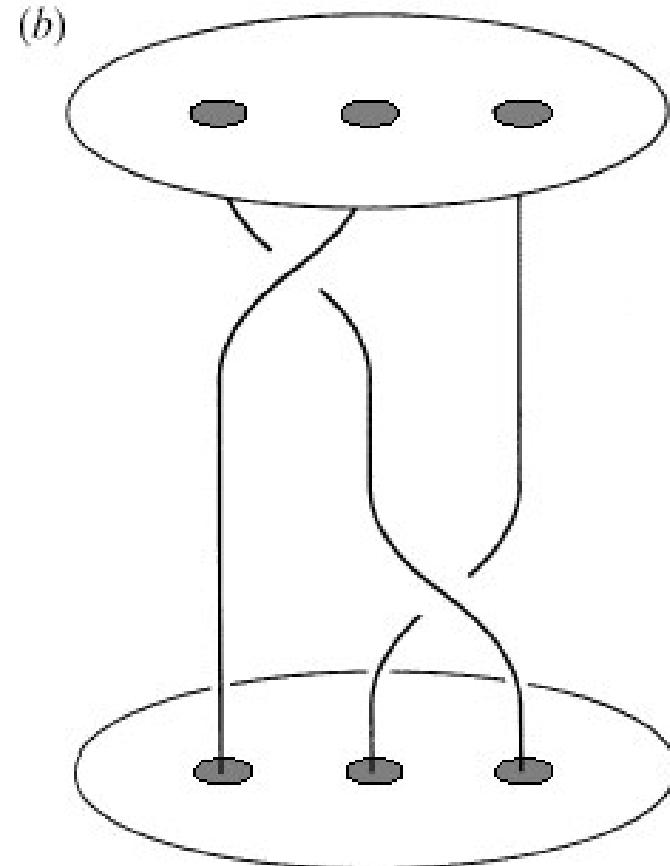
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Braiding

$\sigma_1\sigma_2$ protocol

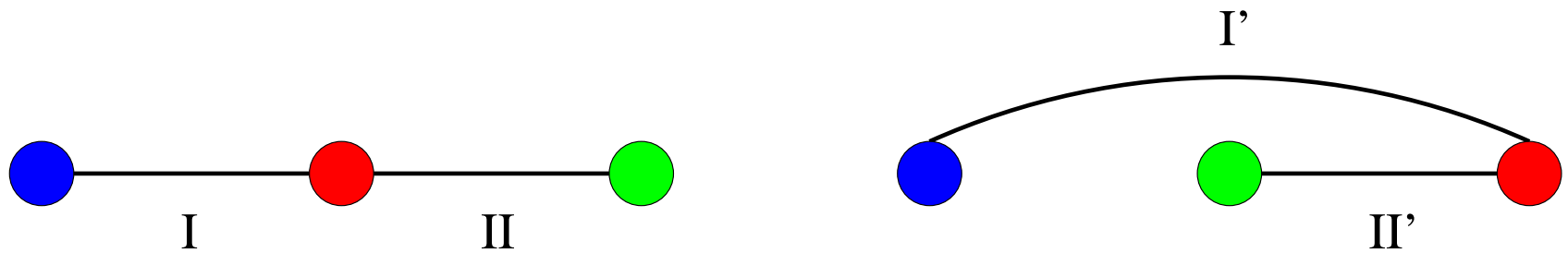


$\sigma_1^{-1}\sigma_2$ protocol



[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)]

Matrix Representation of σ_2



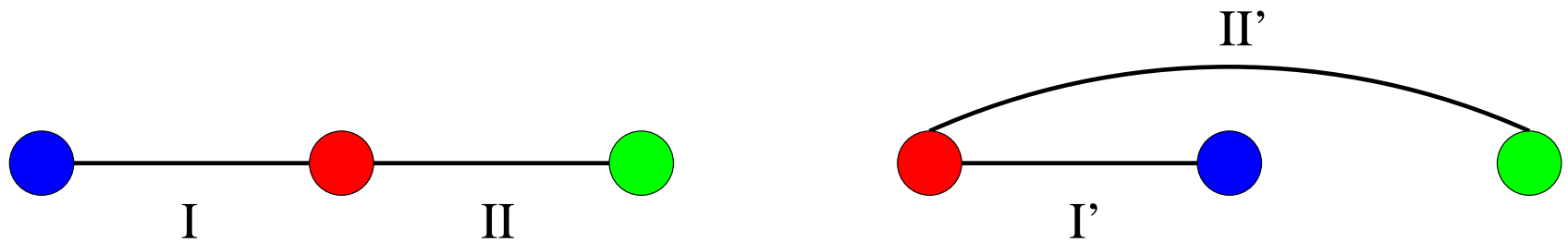
Let I and II denote the lengths of the two segments. After a σ_2 operation, we have

$$\begin{pmatrix} I' \\ II' \end{pmatrix} = \begin{pmatrix} I + II \\ II \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ II \end{pmatrix} = \sigma_2 \begin{pmatrix} I \\ II \end{pmatrix}.$$

Hence, the matrix representation for σ_2 is

$$\sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Matrix Representation of σ_1^{-1}



Similarly, after a σ_1^{-1} operation we have

$$\begin{pmatrix} \text{I}' \\ \text{II}' \end{pmatrix} = \begin{pmatrix} \text{I} \\ \text{I} + \text{II} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \text{I} \\ \text{II} \end{pmatrix} = \sigma_1^{-1} \begin{pmatrix} \text{I} \\ \text{II} \end{pmatrix}.$$

Hence, the matrix representation for σ_1^{-1} is

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Matrix Representation of the Braid Group

We now invoke the faithfulness of the representation to complete the set,

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \sigma_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Our two protocols have representation

$$\sigma_1\sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad \sigma_1^{-1}\sigma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

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- The matrix associated with each generator has unit eigenvalues.

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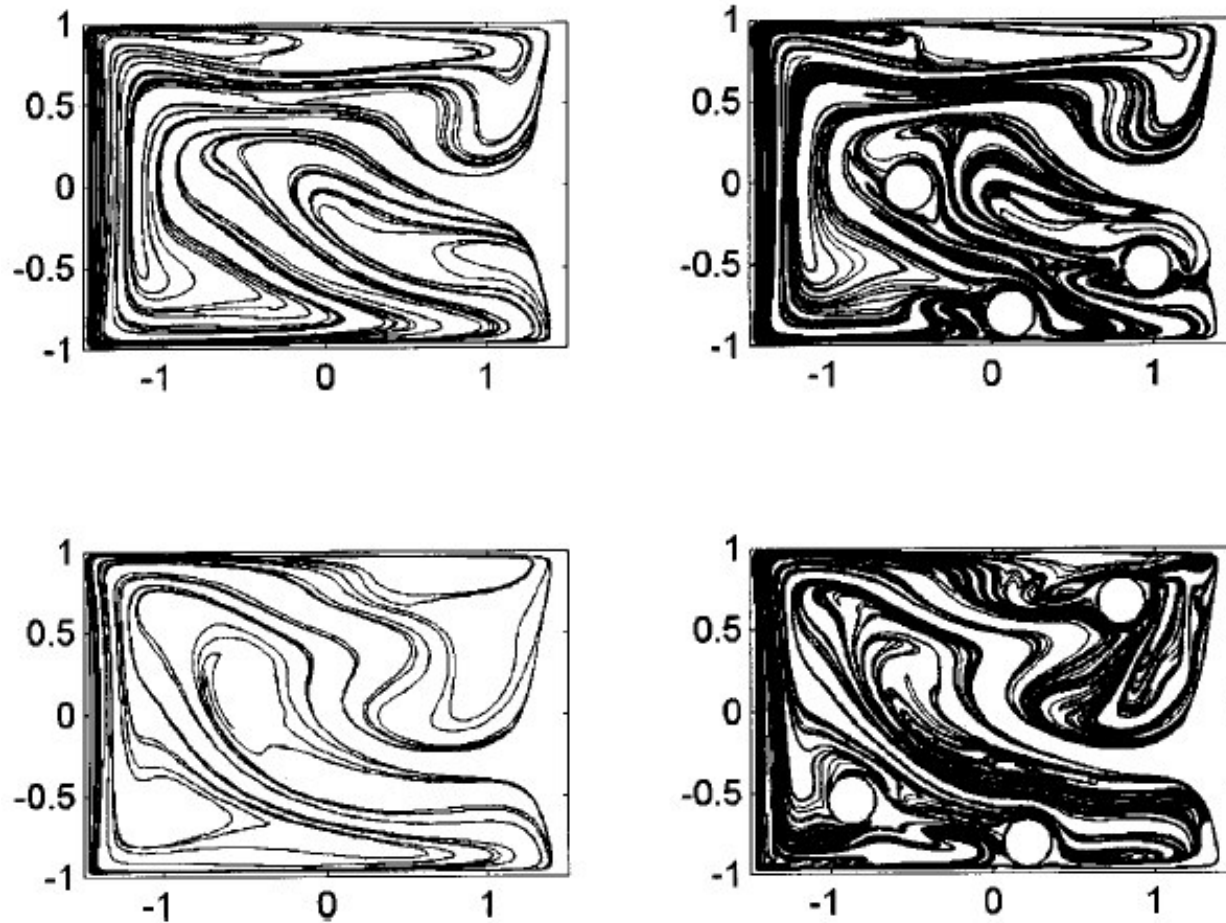
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- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

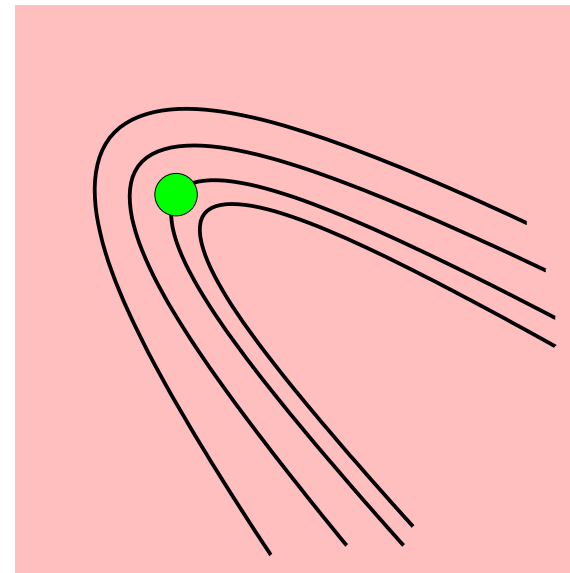
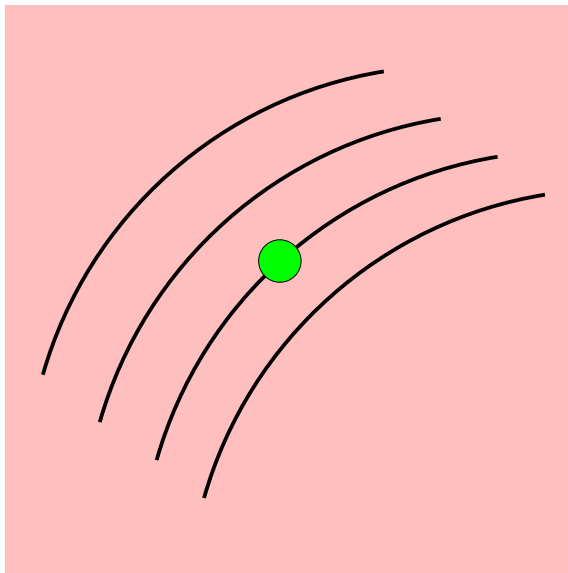
Freely-moving Rods in a Cavity Flow



[A. Vikhansky, *Physics of Fluids* **15**, 1830 (2003)]

Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

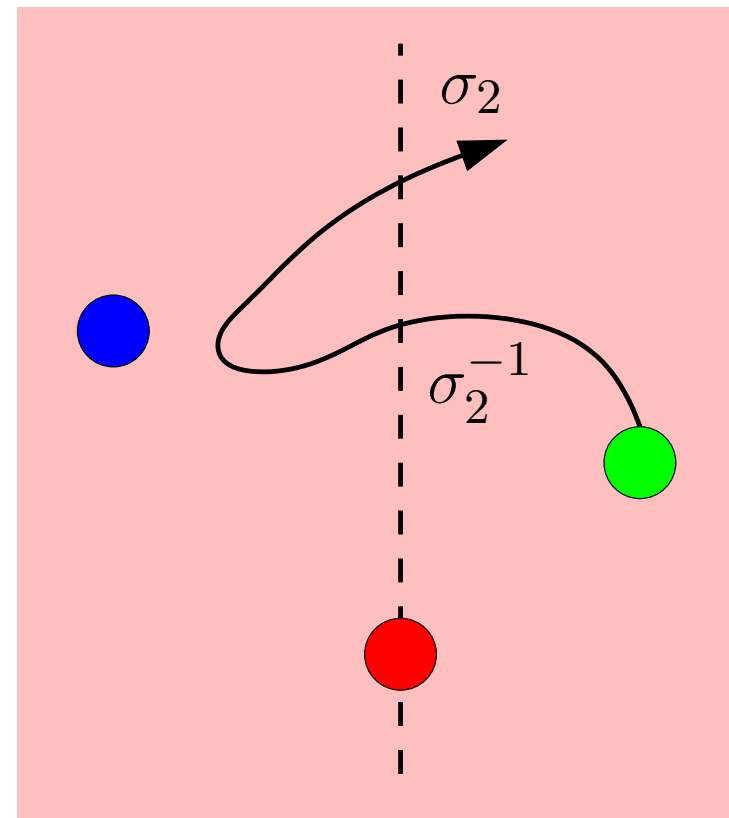
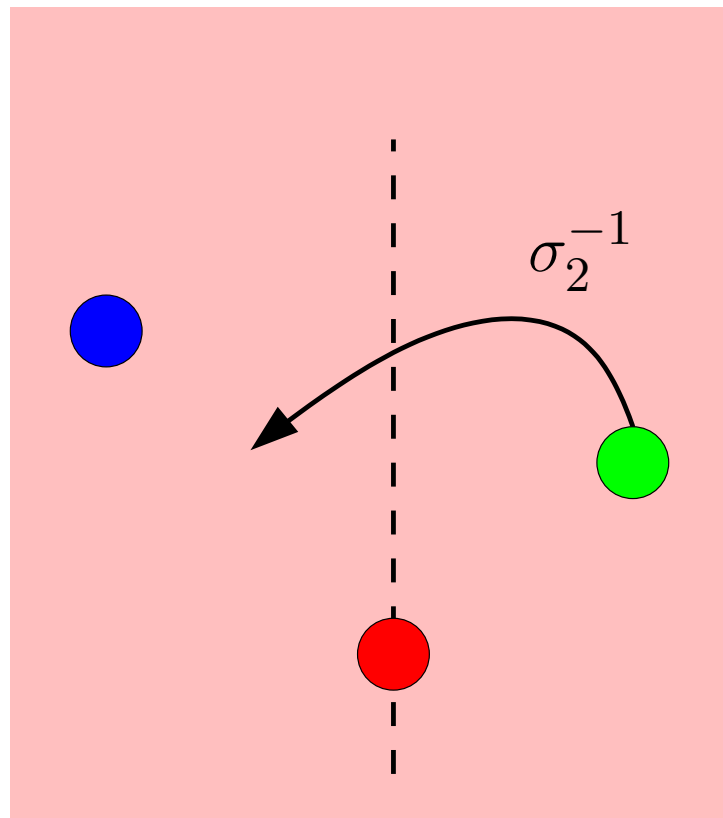


Material lines must bend around the orbit: **it acts just like a “rod”!**

The idea: pick any three fluid particles and follow them.

How do they braid around each other?

Detecting Braiding Events



In the second case there is no net braid: the two elements cancel each other.

Random Sequence of Braids

We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \dots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and N is the number of braiding events detected after a time t .

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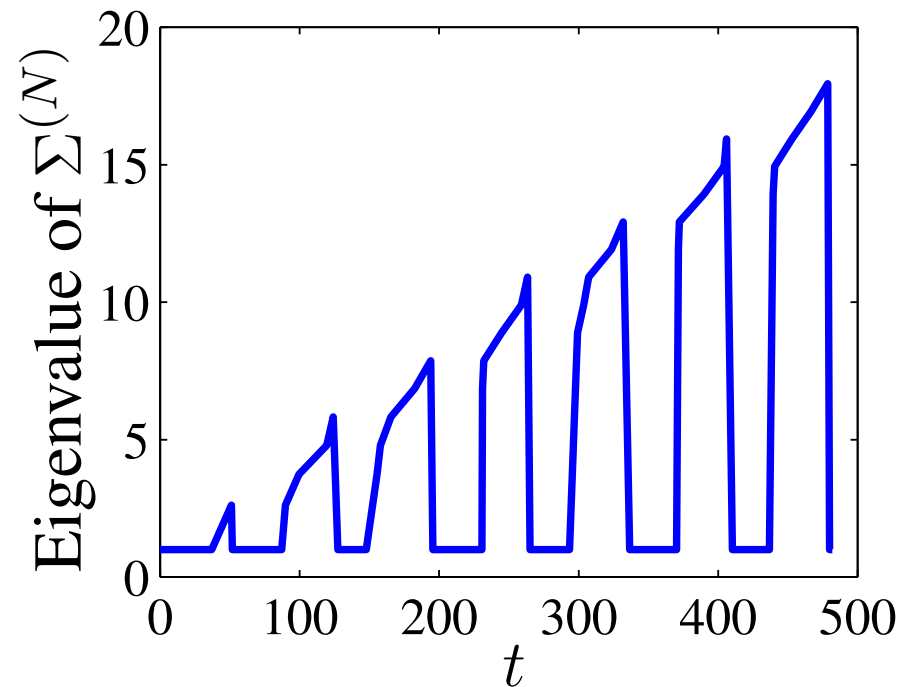
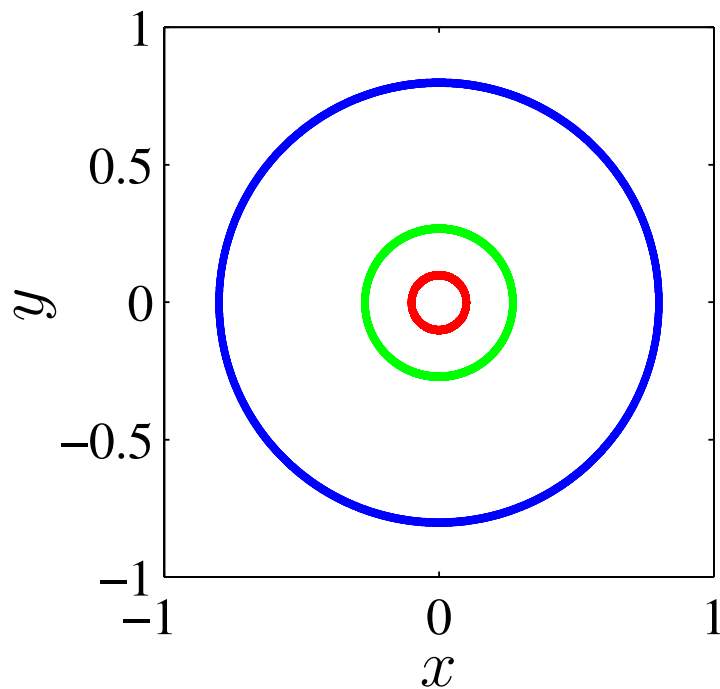
where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and N is the number of braiding events detected after a time t .

The largest eigenvalue of $\Sigma^{(N)}$ is a measure of the **complexity of the braiding motion**, called the **braiding factor**.

Random matrix theory says that the braiding factor can **grow exponentially!** We call the rate of exponential growth the **braiding Lyapunov exponent** or just **braiding exponent**.

Non-braiding Motion

First consider the motion of three points in concentric circles with irrationally-related frequencies.

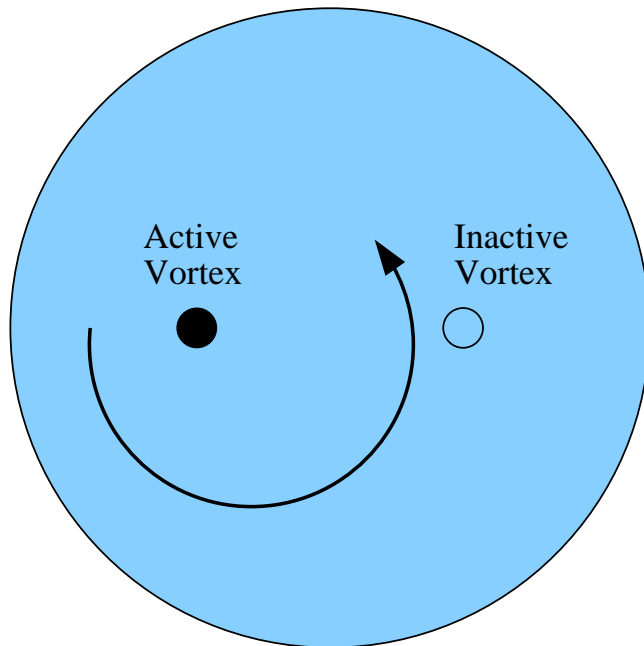


The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.

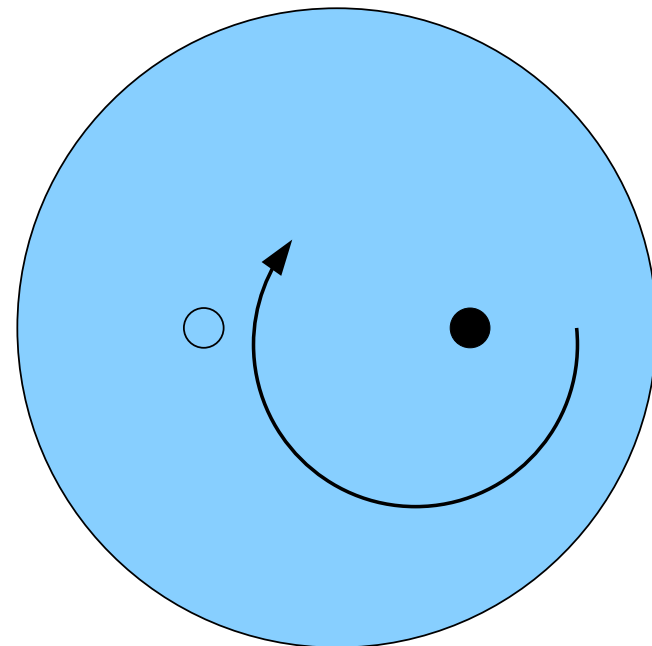
Blinking-vortex Flow

To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won't do).

Aref's **blinking-vortex flow** is ideal.



First half of period

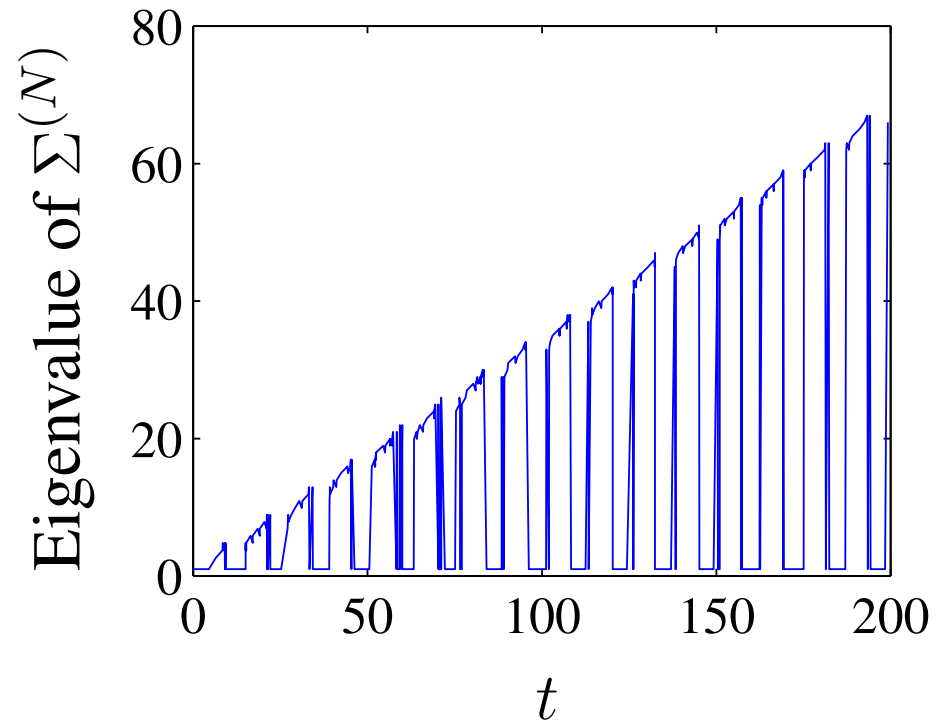
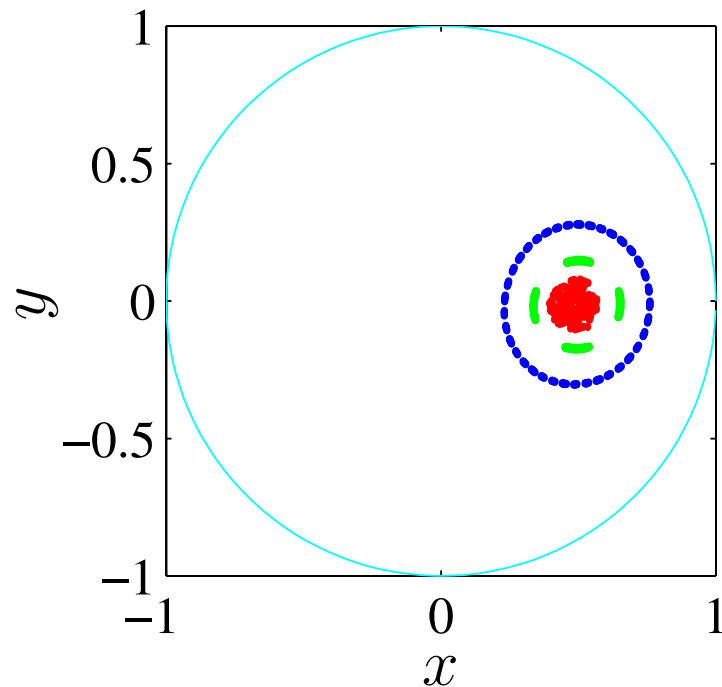


Second half of period

The only parameter is the circulation Γ of the vortices.

Blinking Vortex: Non-braiding Motion

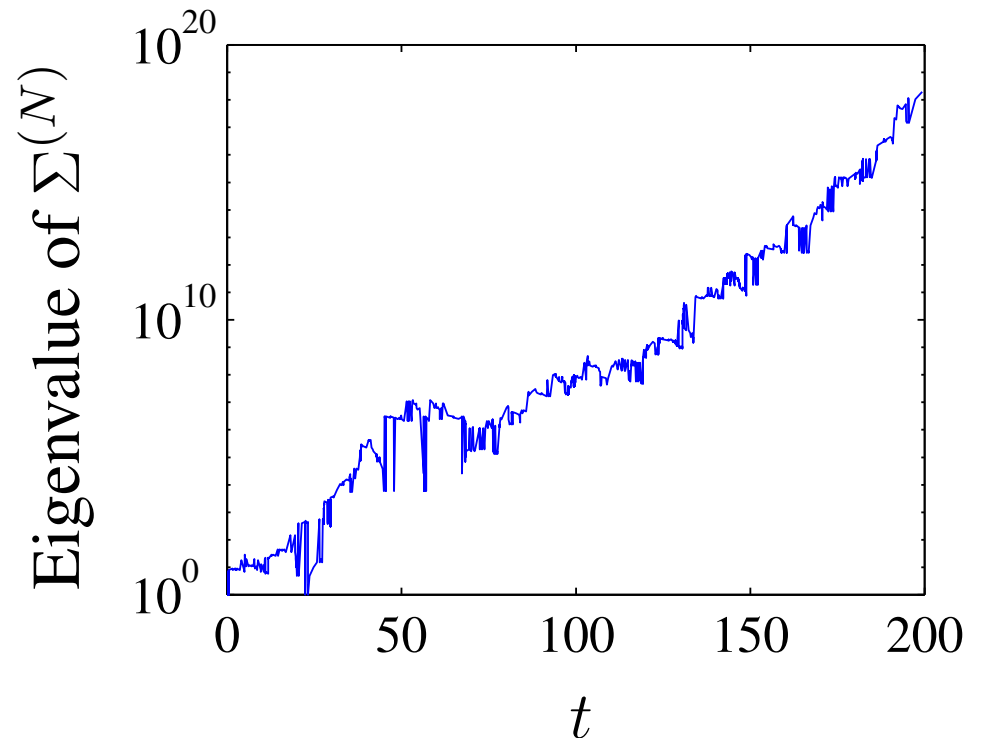
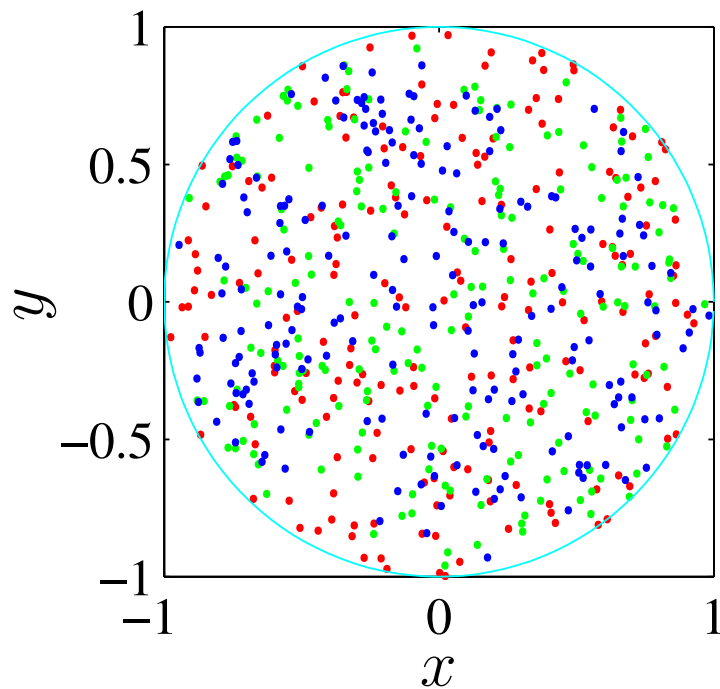
For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.



One of the orbits is chaotic, the other two are closed.

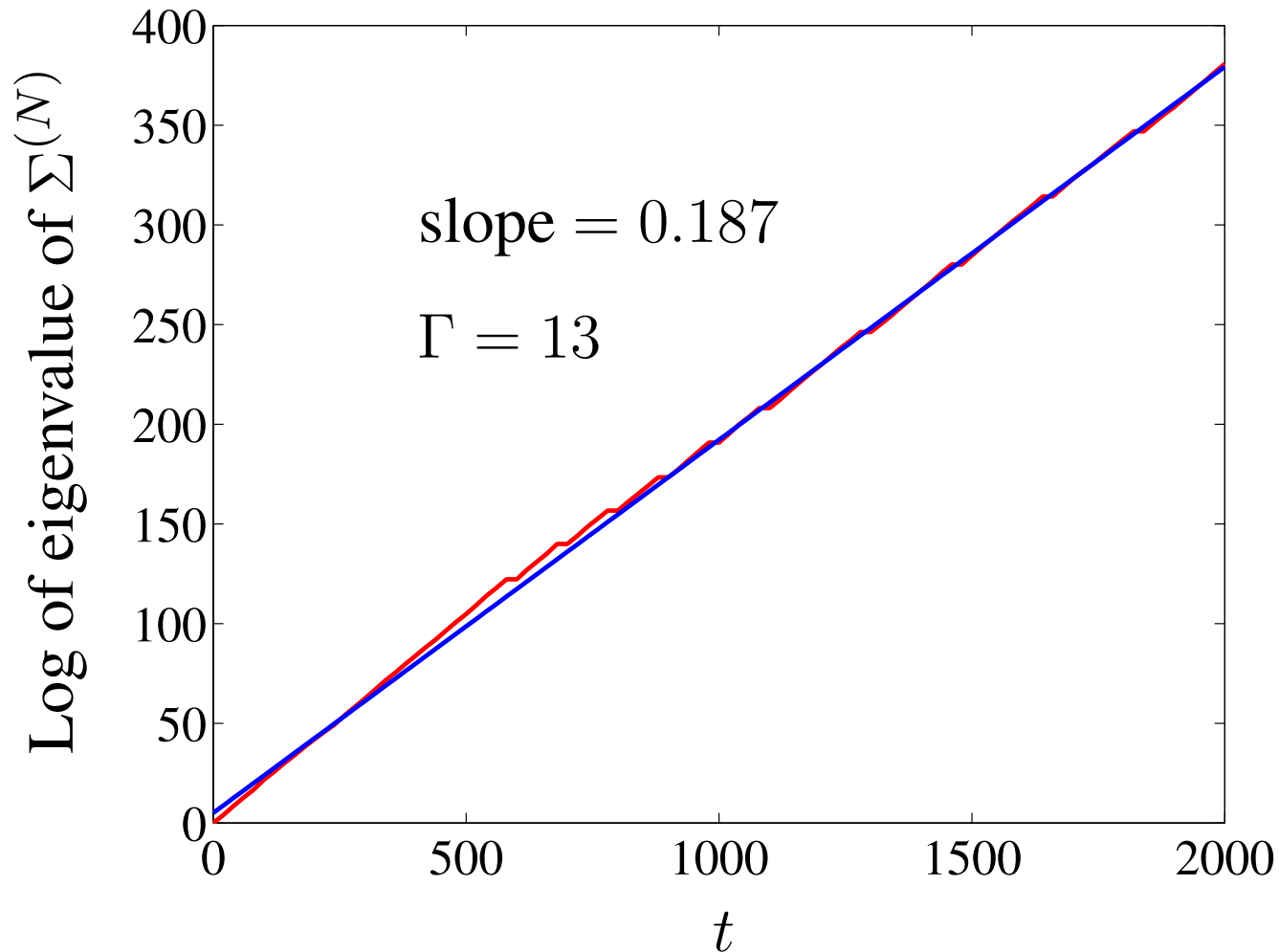
Blinking Vortex: Braiding Motion

For $\Gamma = 13$, the blinking vortex is globally chaotic.



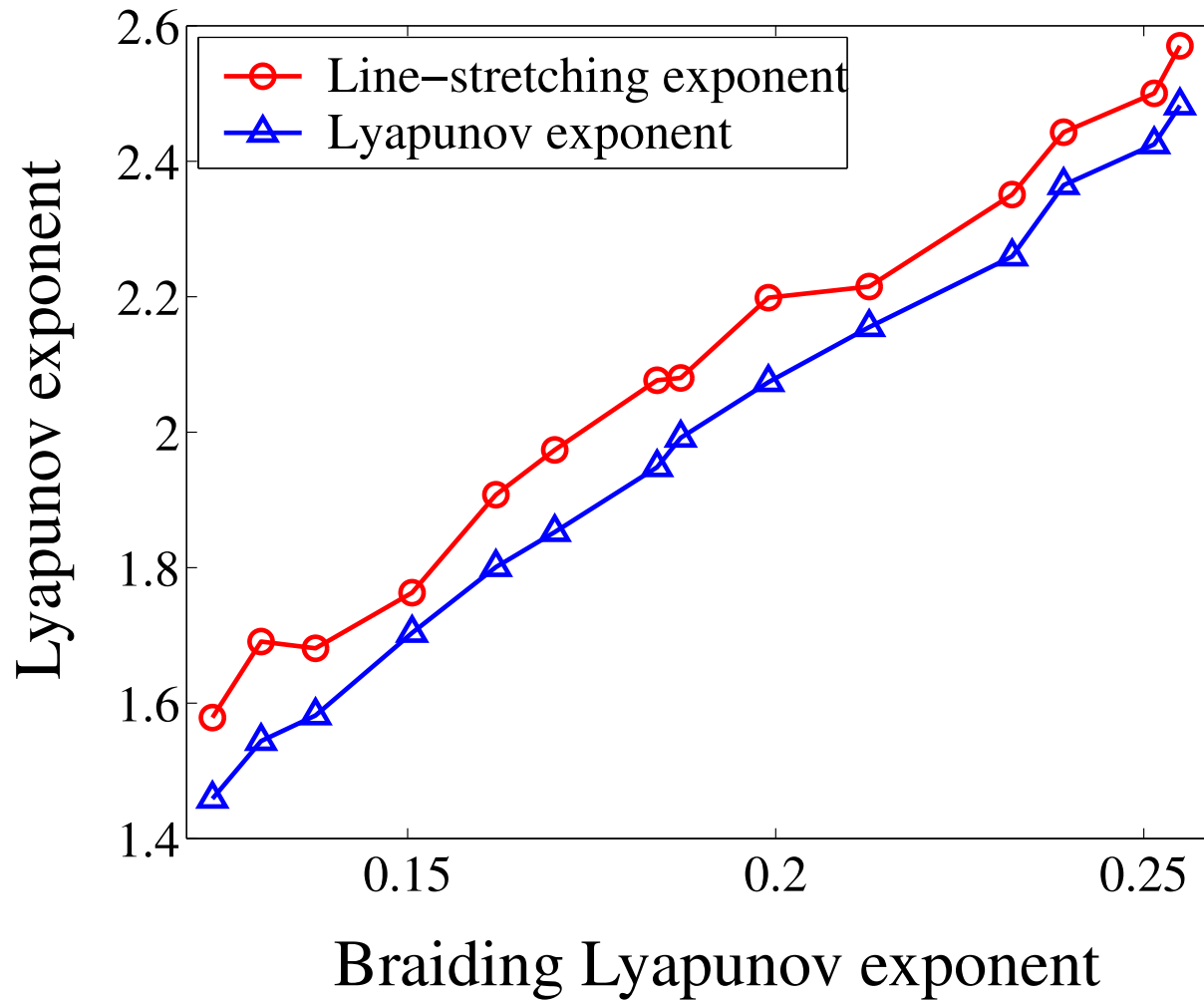
The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order 10^{20} rather than $80!$

Averaging over many Triplets



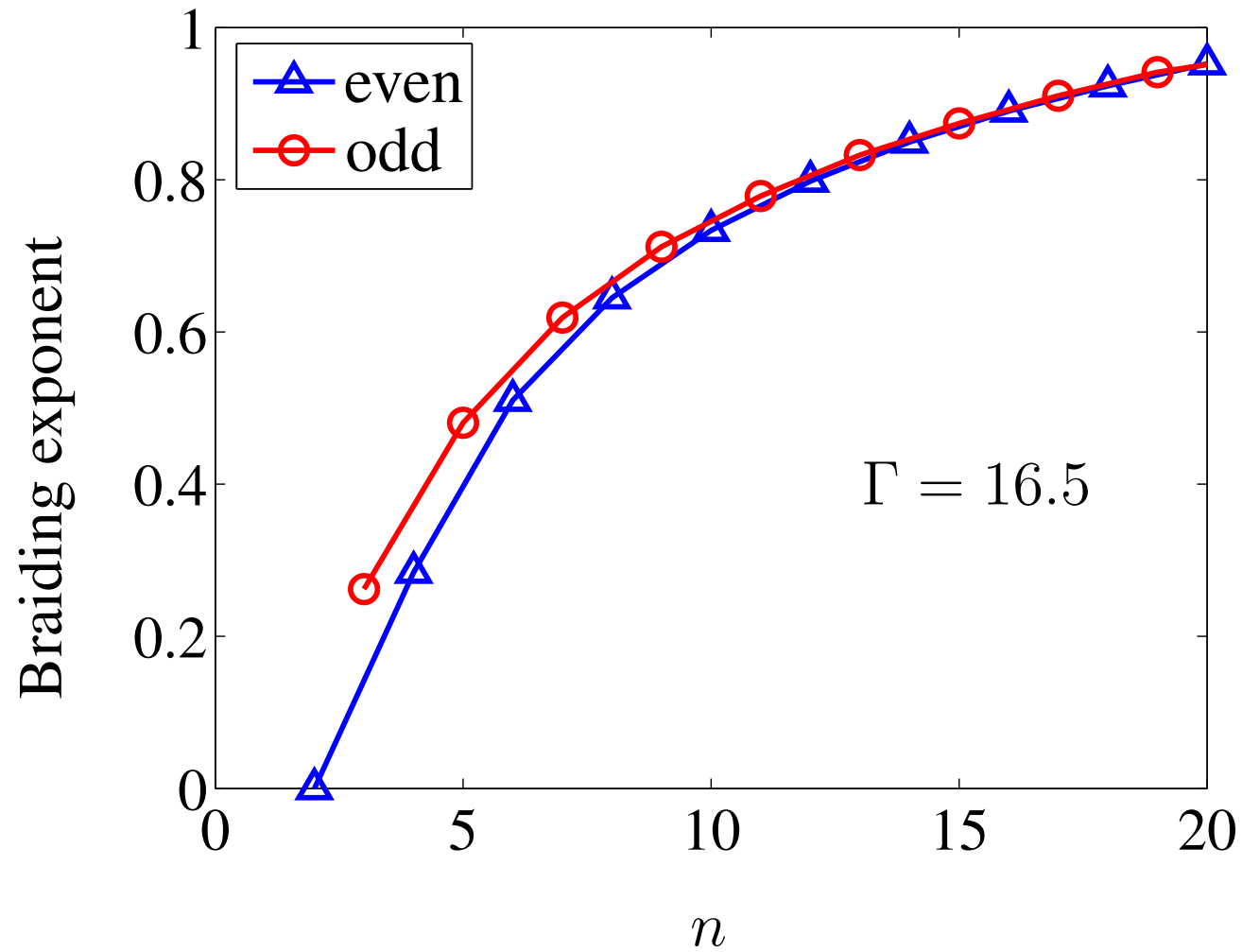
Averaged over 100 random triplets.

Comparison with Lyapunov Exponents

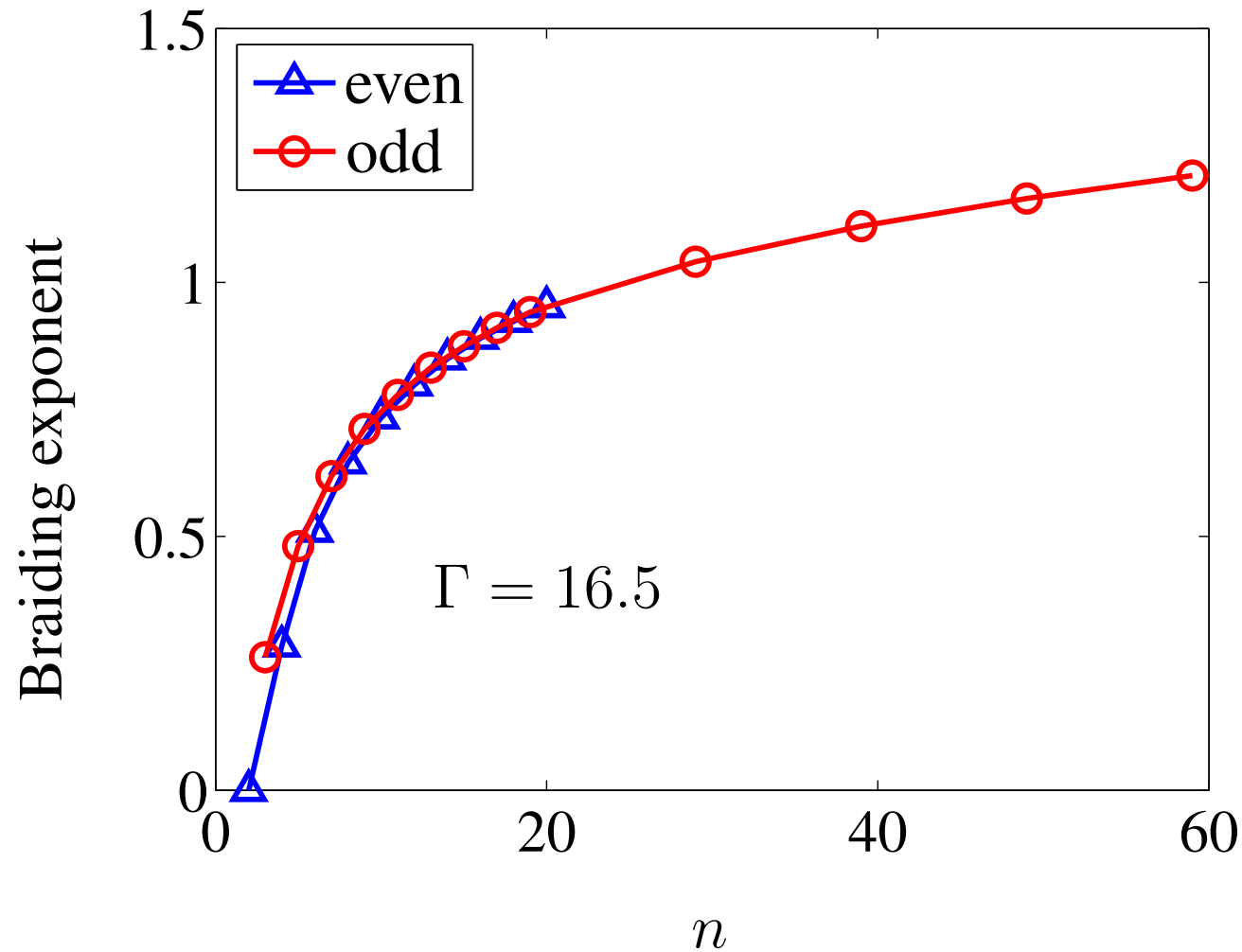


Γ varies from 8 to 20.

Beyond Three Particles



But does it Saturate?



Well, it really should...

One Rod Mixer: The Kenwood Chef



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at CHRISTMAS

and all through the year...

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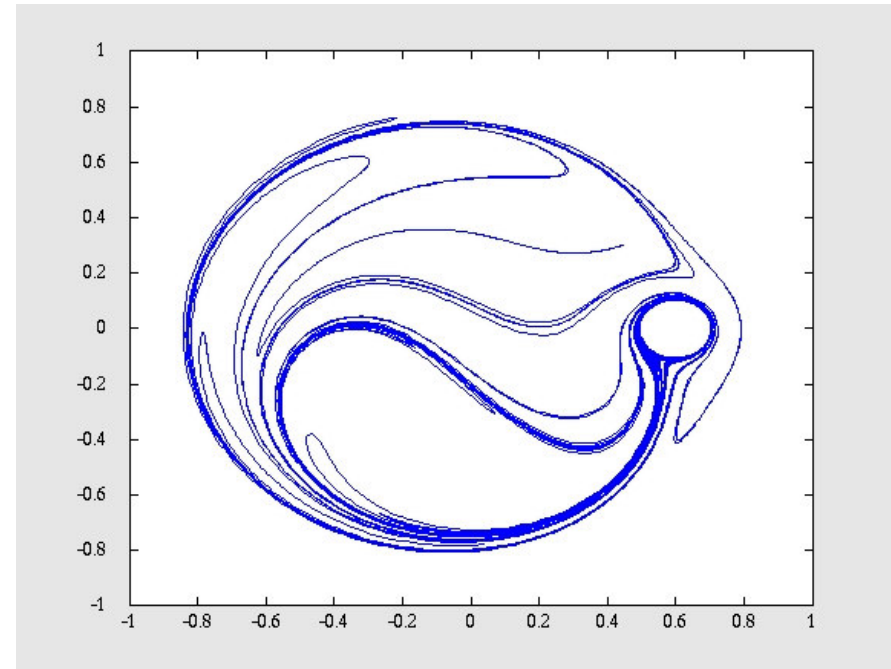
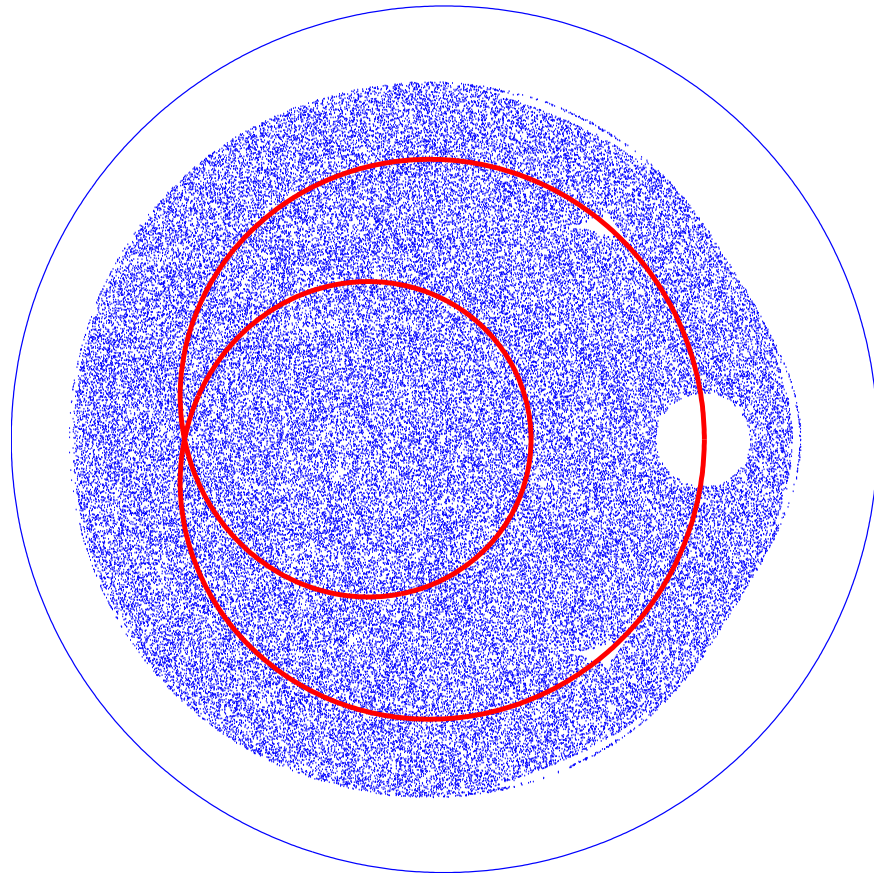
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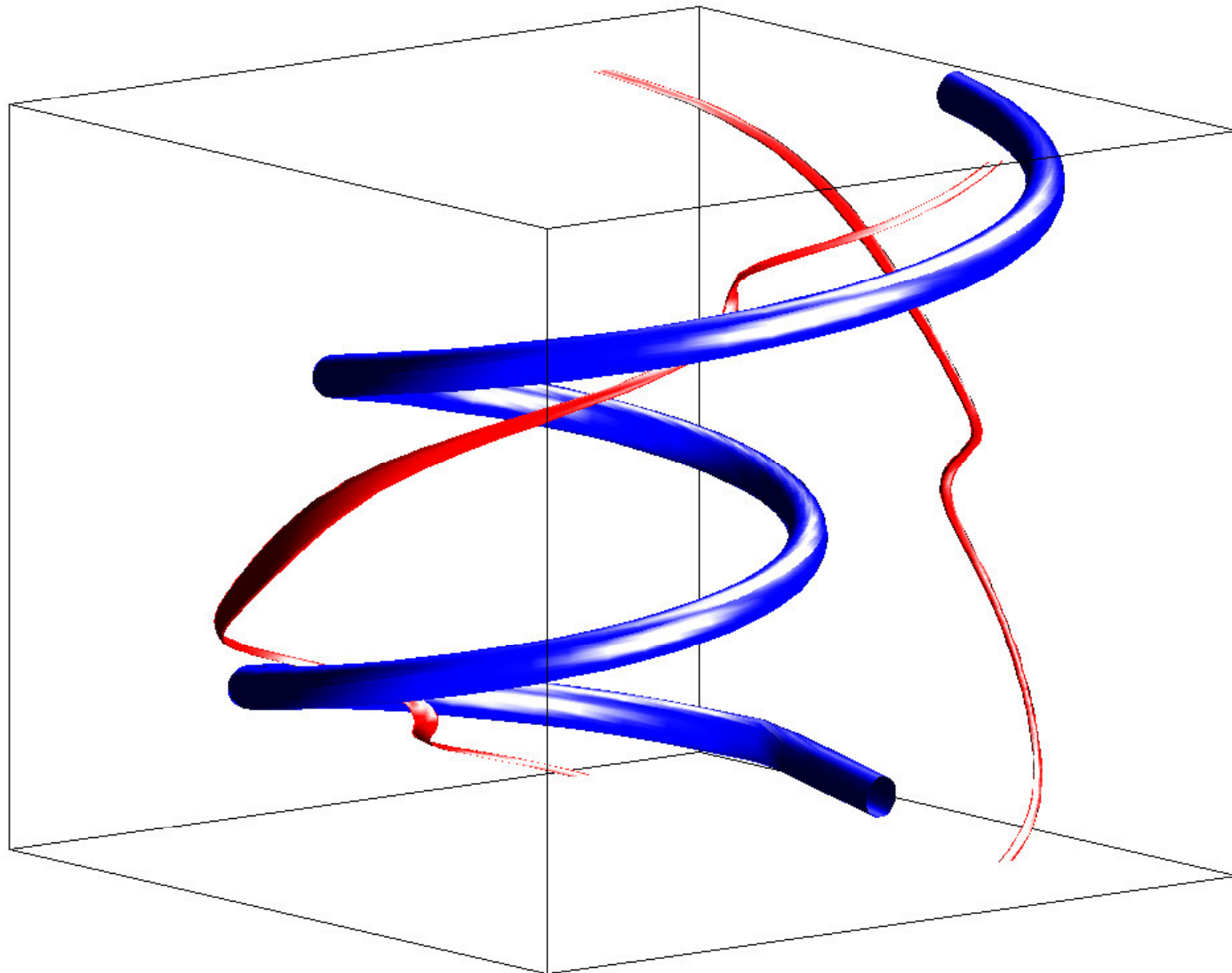
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Epicyclic Motion of the Rod



Topologically, the KC really shouldn't mix well...

But the Good People at Kenwood Knew Better



Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding factor.
- Any collection of n particles can potentially braid.
- The complexity of the braid is a good measure of chaos.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field.
- Many issues to investigate: faithfulness of representation, lower-bound for topological entropy...