

# Collisionless Shocks and the Earth's Bow Shock

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## 1 Introduction

The mean free path for particle collisions in the solar wind at the boundary of the earth's magnetosphere ( $n_e \sim 2\text{--}10 \text{ cm}^{-3}$ ) is about  $10^8 \text{ km}$ , which means that Coulomb collisions are completely negligible. This implies that the collisional viscosity  $\mu$  is effectively zero, so that one would not expect to observe shock waves at all in such a medium because there is no mechanism to prevent wave steepening (see Section 2.1). Nevertheless, satellite observations clearly show the presence of a well-defined shock transition (with a thickness of the order of  $100 \text{ km}$ ), the so-called “bow shock” between the solar wind and the earth's magnetosphere.

I shall give a brief overview of the mathematical feasibility of collisionless shocks, the physical factors that give rise to them in plasmas, and then sum up by describing the important characteristics of the earth's bow shock.

## 2 Collisionless Shocks

### 2.1 The Role of Dissipation

There are some fundamental differences between ordinary, collision-dominated shocks, and collisionless shocks:

- The plasma is in general *not* in thermodynamic equilibrium behind the shock. For example, electron and ion pressure may be quite different from each other, depending on the dissipation mechanism.
- Jump conditions do not completely determine the downstream state.

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- In a collision-dominated shock the shock thickness is only a few collision mean free paths. In a collisionless shock, the thickness over which the shock occurs can be much larger. In fact the concept of shock thickness becomes unclear, because there are many different scale lengths in the system depending on the dissipation mechanism(s) involved (Section 2.3).

For small-amplitude, plane waves propagating along the x-axis, we have

$$\frac{\partial v}{\partial t} + (v \pm c_s) \frac{\partial v}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2}, \quad (1)$$

where  $v$  is the velocity amplitude,  $c_s$  is the velocity of sound ( $v \ll c_s$ ), and  $\mu$  is the kinematic viscosity ( $\mu \sim \lambda_c c_s$ , where  $\lambda_c$  is the mean free path). When can we neglect the viscous term? We compare it with the next larger term:

$$\mu \frac{\partial^2 v}{\partial x^2} \ll v \frac{\partial v}{\partial x}, \quad (2)$$

and we let  $\partial/\partial x \rightarrow 1/L$ , where  $L$  is the scale of the disturbance:

$$\mu \ll Lv, \quad (3)$$

or  $\lambda_c \ll Lv/c_s$  (This just says that the Reynolds number is very large). If we do neglect the viscosity, the solutions to equation (1) are simple waves:

$$v = v_0(x - (v \pm c_s)t), \quad (4)$$

and the initial velocity profile  $v_0(x)$  will steepen since regions of higher  $v$  travel faster. If we take the spatial derivative of equation (4),

$$\frac{\partial v}{\partial x} = \left(1 - t \frac{\partial v}{\partial x}\right) v_0', \quad (5)$$

$$\frac{\partial v}{\partial x} = \frac{v_0'}{1 + tv_0'}, \quad (6)$$

and  $\partial v/\partial x$  will blow up at a point for some time  $t_0$ :

$$t_0 = \frac{1}{\max \left| \frac{\partial v}{\partial x} \right|}. \quad (7)$$

In reality what happens is that the viscous term in equation (1) becomes important in limiting the slope. We see this because  $v/L \sim \partial v/\partial x$ , so that  $L \sim v(\partial v/\partial x)^{-1}$ , and if  $\partial v/\partial x$  blows up at some point then  $L$  goes to zero there, and the condition (3) cannot be satisfied for a nonzero  $\mu$ . So the viscosity becomes important locally, thereby limiting the steepening. This is an essential requirement for the formation of a shock wave: if the steepening is not limited in some way, the wave overturns (onset of multistreaming, see Ref. [1] and [2]... too nasty to be tackled here), in the same way that ocean waves

“break”. So it is all right to let the viscosity go to zero (in which case the shock thickness also goes to zero), but there must still be some dissipation for the shock to occur<sup>1</sup>. For collisionless waves, we must look for other phenomena to limit steepening. We shall look at different methods of generating the necessary dissipation.

## 2.2 Dispersion

For sufficiently small amplitudes a dispersive system can be described by the Korteweg–de Vries (KdV) equation (see, for example, Ref. [3], Ref. [4], or Appendix A):

$$\frac{\partial v}{\partial t} + (v \pm c_s) \frac{\partial v}{\partial x} = \xi \frac{\partial^3 v}{\partial x^3}, \quad (8)$$

which is similar to equation (1), the dissipative term being replaced by a dispersive one. The linear dispersion relation corresponding to equation (8) is  $\omega = \pm k c_s + \xi k^3$ . We shall use equation (8) to show how dispersion can generate a shock wave, though the details may vary for different systems.

In Figure 1 we have a graph of the dispersion relation of equation (8). Branch (a) is for  $\xi > 0$ , and branch (b) is for  $\xi < 0$ . If we neglect the dispersion term  $\partial^3 v / \partial x^3$ , then we recover equation (1) with  $\mu = 0$ , and we saw that this leads to steepening of the wavepacket. What is happening is that harmonics of higher and higher  $k$  are being generated, and so eventually the cubic term in the dispersion relation will become important. Then, if  $\xi > 0$ , the large  $k$  (shorter wavelength) harmonics will run ahead of the main pulse. If  $\xi < 0$ , they will fall behind it. This will prevent the pulse from steepening further and create a trailing wave train for  $\xi > 0$ , or a leading wave train (precursor) for  $\xi < 0$ , as seen in Figure 2 (a) and (b).

Well, this isn’t really the whole story. Let’s go through this in more detail. Consider a solution of the type  $v = v(x + M c_s t)$ , i.e. a wave with a constant profile propagating with velocity  $M c_s$  in the negative  $x$  direction (the lower sign in (8)). Here  $M$  is the Mach number. Equation (8) becomes

$$((v + M c_s) - c_s) v' = \xi v''', \quad (9)$$

and if we let  $x \rightarrow x + M c_s t$ ,  $v \rightarrow v + M c_s$ , i.e. transform to the wave frame,

$$(v - c_s) v' = \xi v'''. \quad (10)$$

After one  $x$  integration, we have

$$\xi v'' = \frac{1}{2} v^2 - c_s v + K, \quad (11)$$

$$= \frac{d}{dv} \left( \frac{1}{6} v^3 - \frac{1}{2} c_s v^2 + K v \right). \quad (12)$$

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<sup>1</sup>A similar kind of approximation is done for the ideal gas law: collisions are neglected, but if there were absolutely no collisions then the gas could never reach thermodynamic equilibrium.

We can rewrite this as

$$\frac{d^2v}{dx^2} = -\frac{dU(v)}{dv}, \quad (13)$$

where

$$U(v) \equiv -\frac{1}{6\xi}(v - c_s)^3 + \eta(v - c_s), \quad (14)$$

$$= -\frac{1}{6\xi}(v - c_s) \left( (v - c_s)^2 - 6\xi\eta \right). \quad (15)$$

Equation (13) is like the equation of motion for an oscillator in a potential  $U(v)$ , where  $v$  is like the position and  $x$  is like time. Multiplying by  $v'$  and integrating again, we have the Hamiltonian for this system:

$$H = \frac{1}{2}v'^2 + U(v) = E = \text{constant}. \quad (16)$$

Figure 3 shows a graph of the potential  $U(v)$  as a function of  $v$  for  $\xi > 0$  (branch (a) in Figure 1). There are two types of solutions to equation (13), and their form depend on the dispersion parameter  $\xi$ , the integration constant  $\eta$ , and the oscillator energy  $E$ . If  $v$  starts out in the “well” part of the potential, the solution is a nonlinear periodic wave. As  $E$  is increased, the period increases since the turning point on the right approaches the limit  $dU/dv = 0$ , at  $v = v_1$ . At that point, a solution that starts with  $v = v_1$  at  $x \rightarrow -\infty$  will have an infinite period with minimum value  $v_m$ . That solution is a soliton, in this case a rarefaction soliton (since  $v \propto n^{-1}$ ). For  $\xi < 0$ , we have a compression soliton.

The oscillatory solution obviously cannot represent a shock, and neither can the soliton one since  $v(-\infty) = v(+\infty) = v_1$ . The missing ingredient is, again, dissipation. If there is dissipation the initially flat profile (at  $x = -\infty$ ) of a soliton solution will oscillate as  $x$  increases and decay to the bottom of the well at  $v = v_2$ . Thus this will be a shock-like solution with a jump from  $v_1$  to  $v_2$  with a trailing wavetrain, as seen in Figure 2 (a). For  $\xi < 0$ , the jump will be from  $v_2$  to  $v_1$ , and the shock will have a leading wavetrain, a precursor (Figure 2 (b)).

### 2.3 Dissipation from Collective Interactions

Well obviously the dissipation will have to come from somewhere else than binary collisions. There are a multitude of ways the minute amount of dissipation required can be generated, but they all depend on the *collective* interactions of the particles: the interaction of particles through the electric and magnetic fields they produce, caused by instabilities. One example of such a possible dissipation mechanism is the so-called “decay instability”. It occurs when an excited mode gives away some of its energy (i.e., decays) to some other, much lower amplitude modes (usually two other modes

for three-wave interaction). Sometimes, the modes so excited can propagate out of the shock, thereby effectively radiating wave energy out of the shock.

Most of the processes involved are turbulent, and just how turbulent the process is determines how messy the shock is. If the fields and distributions change in some coherent manner through the shock, the shock is termed *laminar*. If turbulence is strong, then the shocks are appropriately named *turbulent*. There is a third category, *mixed structure* shocks, where the shock has some coherent structure and some chaotic aspects (Ref. [2] is the best source I know on this, though it is slightly outdated. Ref. [5] is also good). The shock fronts we have been looking at (in Figure 2) are of the laminar type, where we just assume the damping is due to some small, well-behaved effective viscosity. How appropriate this is depends on a myriad of factors like the Mach number of the shock, the angle of incidence, and the plasma  $\beta$  (ratio of magnetic pressure to thermal pressure).

## 2.4 Anomalous Resistivity

Another possibility for limiting wave steepening is resistivity. The changing magnetic fields across the shock implies the flow of electric currents. If the field gradient becomes so large that the current exceeds the limit for some current-driven instability (for instance, the ion-sound instability discussed in Ref. [5]) the resulting anomalous resistivity will limit further steepening. This shall not be investigated further as this effect is important only for low Mach-number shocks (the earth's bow shock has an Alfvén-Mach number between 5 and 10, so this type of shock will not generally occur).

## 3 The Earth's Bow Shock

Where does the earth's bow shock fit in? It is a high Mach number shock ( $M_A \sim 5-10$ ), and the majority of shock profiles exhibit a precursor structure consistent with a dispersion having  $\xi > 0$ . This is consistent with the *whistler* mode (see Ref. [6] or your favourite plasma textbook. A whistler wave propagates parallel to the magnetic field, and it has a frequency greater than the ion cyclotron frequency, but less than the electron cyclotron frequency.). The shock also tends to be of the turbulent type, though there are regions where the shock is laminar or mixed, as seen in Figure 4.

At the level of theory given here, we can only claim to qualitatively understand why there *could* be a shock between the solar wind and the magnetosphere. For more details, see [5] and [7].

## A Derivation of the Korteweg–de Vries Equation

In this appendix we shall derive a version of the KdV equation applicable to nonlinear ion sound waves. We shall follow closely the derivation given by Davidson [4]. The ions are assumed cold and nondrifting with respect to the electrons ( $T_i \ll T_e$ ), and we use a one-dimensional description. Moreover, electron inertia effects are neglected ( $m_e \rightarrow 0$ ) and the isothermal equation of state,  $P_e = n_e k_B T_e$  ( $T_e = \text{constant}$ ), is adopted for the electrons. We write the momentum balance equation for the electrons:

$$m_e n_e \frac{dv_e}{dt} = 0 = e \frac{\partial \phi}{\partial x} - \frac{k_B T_e}{n_e} \frac{\partial n_e}{\partial x}, \quad (17)$$

where  $-e$  is the charge of the electron,  $n_e(x, t)$  is the electron number density, and  $\phi(x, t)$  is the electrostatic potential. Equation (17) can be integrated to give

$$n_e = n_0 \exp(e\phi/k_B T_e), \quad (18)$$

where  $n_0$  is the uniform background electron density. Poisson's equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - n_i) = 4\pi e(n_0 \exp(e\phi/k_B T_e) - n_i). \quad (19)$$

For the ions, the equation of motion and the equation of continuity are

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{e}{m_i} \frac{\partial \phi}{\partial x}, \quad (20)$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial n_i v_i}{\partial x} = 0, \quad (21)$$

where  $n_i(x, t)$  is the ion number density,  $v_i(x, t)$  is the ion mean velocity, and  $e$  and  $m_i$  the ion charge and mass, respectively. At this point we introduce the dimensionless variables  $n = n_i/n_0$ ,  $u = v_i/c_{si}$ ,  $z = x/\lambda_D$ ,  $\tau = c_{si}t/\lambda_D$ , and  $\varphi = e\phi/k_B T_e$ , where  $c_{si} = (k_B T_e/m_i)^{1/2}$  is the ion acoustic speed and  $\lambda_D = (k_B T_e/4\pi n_0 e^2)^{1/2}$  is the Debye length. Equations (20), (21), and (19) become

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial z} = 0, \quad (22)$$

$$\frac{\partial n}{\partial \tau} + \frac{\partial nu}{\partial z} = 0, \quad (23)$$

$$\frac{\partial^2 \varphi}{\partial z^2} = e^\varphi - n. \quad (24)$$

The approximation usually made is charge neutrality, in which case  $\partial^2 \phi/\partial x^2 = 0$  and so  $n_i = n_e$ . But it can be shown that this leads to unlimited wave steepening, which is what we're trying to avoid. So this approximation will not be used here.

Equations (22), (23), and (24) have a solitary wave (soliton) solution given by

$$\varphi = 3(M - 1)\text{sech}^2 \left[ \left( \frac{1}{2}(M - 1) \right)^{1/2} (z - M\tau) \right]. \quad (25)$$

Where  $M$  is the Mach number. If we let  $\epsilon = M - 1 \ll 1$  (a measure of the amplitude of the wave), the argument of (25) may be expressed as

$$\epsilon^{1/2}(z - \tau) - \epsilon^{3/2}\tau. \quad (26)$$

This gives the appropriate scaling (in a frame moving with  $M = 1$ ) of the space and time variables that is required to obtain the KdV equation from (22)–(24). The present goal is to construct a weakly nonlinear theory of ion sound waves which describes the evolution of small but finite amplitude disturbances which are traveling near  $M = 1$ . In order to recover the solitary wave solution as a special case, which is the minimum demand we make of the theory, the “stretched” variables

$$\chi = \epsilon^{1/2}(z - \tau), \quad (27)$$

$$\zeta = \epsilon^{3/2}\tau, \quad (28)$$

are introduced, and so

$$\frac{\partial}{\partial z} = \epsilon^{1/2} \frac{\partial}{\partial \chi}, \quad (29)$$

$$\frac{\partial}{\partial \tau} = \epsilon^{3/2} \frac{\partial}{\partial \zeta} - \epsilon^{1/2} \frac{\partial}{\partial \chi}. \quad (30)$$

We now assume that  $n$ ,  $\varphi$ , and  $u$  have power series expansions in  $\epsilon$  about a homogenous field-free equilibrium, i.e.,

$$n = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \dots, \quad (31)$$

$$\varphi = \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + \dots, \quad (32)$$

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \dots. \quad (33)$$

To lowest order in  $\epsilon$ , equations (22), (23), and (24) become (using (29)–(33))

$$\frac{\partial u^{(1)}}{\partial \chi} = \frac{\partial \varphi^{(1)}}{\partial \chi}, \quad (34)$$

$$\frac{\partial u^{(1)}}{\partial \chi} = \frac{\partial n^{(1)}}{\partial \chi}, \quad (35)$$

$$\varphi^{(1)} = n^{(1)}, \quad (36)$$

which gives  $n^{(1)} = \varphi^{(1)} = u^{(1)}$ , since all three variables have to vanish for  $|\chi| \rightarrow \infty$ .

To next order in  $\epsilon$ , we have

$$-\frac{\partial u^{(2)}}{\partial \chi} + \frac{\partial u^{(1)}}{\partial \zeta} + u^{(1)} \frac{\partial u^{(1)}}{\partial \chi} + \frac{\partial \varphi^{(2)}}{\partial \chi} = 0, \quad (37)$$

$$-\frac{\partial n^{(2)}}{\partial \chi} + \frac{\partial n^{(1)}}{\partial \zeta} + \frac{\partial(n^{(1)}u^{(1)})}{\partial \chi} + \frac{\partial v^{(2)}}{\partial \chi} = 0, \quad (38)$$

$$\frac{\partial^2 \varphi^{(1)}}{\partial \chi^2} - \varphi^{(2)} - \frac{(\varphi^{(1)})^2}{2} - n^{(2)} = 0. \quad (39)$$

Adding (37) and (38), using (39) to eliminate  $\varphi^{(2)}$ , and using the first-order result to eliminate  $\varphi^{(1)}$  and  $n^{(1)}$ , we obtain

$$\frac{\partial u^{(1)}}{\partial \zeta} + u^{(1)} \frac{\partial u^{(1)}}{\partial \chi} + \frac{1}{2} \frac{\partial^3 u^{(1)}}{\partial \chi^3} = 0. \quad (40)$$

This is the Korteweg–de Vries equation, with dispersion parameter  $\xi = -\frac{1}{2}$ .

## References

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