The Hamiltonian Structure of Fluids and Plasmas:

Reduction and Semidirect Extensions

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Overview

- We explore the analogy between the rigid body and the ideal fluid.
- Define the configuration space for both systems.
- Make things move: we introduce some dynamics (time evolution).
- In both cases, there is a symmetry which permits a reduction of the system to an Euler equation.
- In the reduction process, we have lost information about the configuration of the system. We can recover some information by using semidirect extensions, which reverse reduction.
- We discuss more general extensions.

Configuration Space

We will be concerned with mechanical systems, specifically fluids and rigid bodies.

The configuration of a system is simply a "snapshot." No time evolution is recorded (i.e., we don't know how fast things are going).

The configuration space is some way to encode this information into a set of numbers.

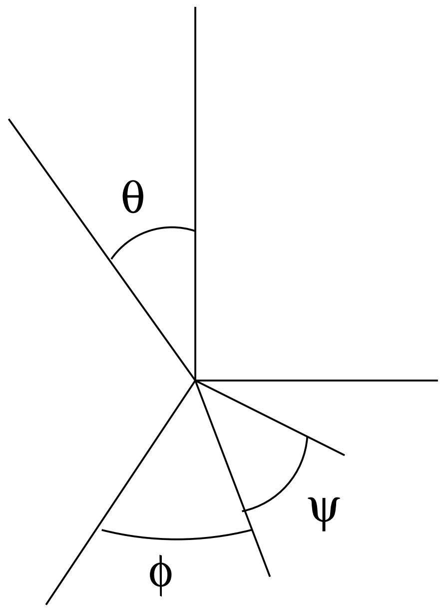
Hopefully, the set of variables chosen is minimal.

The Free Rigid Body

A rigid body is a solid which does not change its shape upon rotation about a fixed point.

"Free" means that there are no external forces acting on the body.

The minimum number of coordinates necessary to characterize the orientation of a rigid body is three: think of two angles to express the position of a vector in spherical coordinates, plus a third angle to express the orientation of the body about that vector.



Configuration Space for the Rigid Body

Thus, the orientation of the rigid body is characterized by a rotation matrix, an element of the (finite-dimensional) Lie group $SO(3)$.

The rotation is defined with respect to some reference orientation.

The "rigidity" of the body is contained in the orthogonality of rotation matrices R. If α_0 is a vector in the reference frame, and $\alpha = R\,\alpha_0$ is this vector in the new frame, then

$$\alpha^T \alpha = (R\,\alpha_0)^T R\,\alpha_0 = \alpha_0^T (R^T R)\,\alpha_0 = \alpha_0^T \alpha_0$$

Vectors do not change length with respect to the fixed rotation point.

The Ideal Fluid

A fluid is ideal if

- It is free of dissipation (inviscid)
  ("perpetual motion")
- It is incompressible
  (Fluid elements retain the same volume)

Water, for example, is incompressible to a large degree, and in some regimes can be thought of as inviscid.

A plasma is usually quite inviscid (collisionless), and in the presence of a large magnetic field can be viewed as a two-dimensional incompressible fluid.

Configuration Space for a Fluid

So how do we describe the configuration of a fluid?

We need to characterize the position of every infinitesimal fluid elements.

First, we label the fluid elements, say by their position in some reference state:
a is the fluid element with position $a = (a_x, a_y, a_z)$ in the reference state.

Then, we define a function $q = q(a)$, where q is a vector that gives the position of fluid element a.

We say that q is a diffeomorphism of the fluid domain to itself.

Incompressibility

There is a restriction on the diffeomorphism q: the fluid is incompressible, which says that fluid elements retain their volume.

This is expressed by the condition that the Jacobian of the transformation be equal to unity:

$$\left| \frac{\partial q}{\partial a} \right| = 1.$$

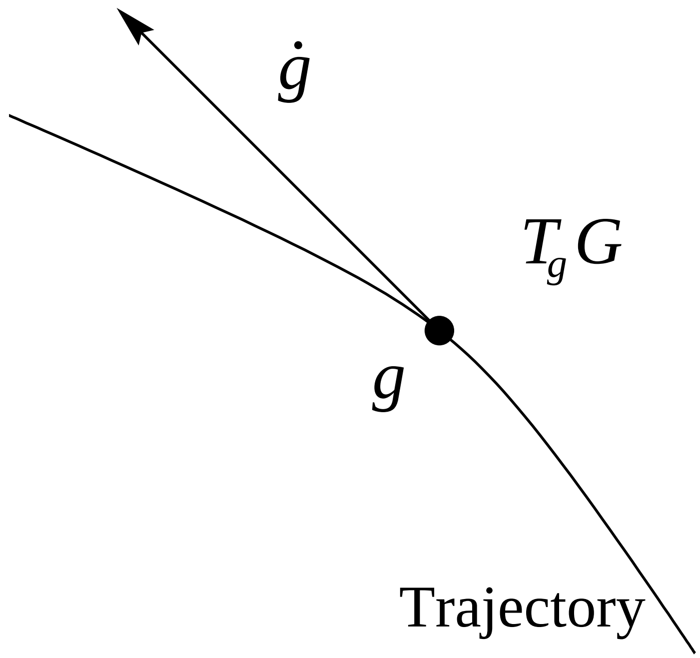
Thus q is a volume-preserving diffeomorphism.

We say that the configuration space of the system is the (Lie) group of volume-preserving diffeomorphisms, $SDiff\ D$, where D is the fluid domain.

Dynamics

Now consider a trajectory, $g(t)$, in the configuration space G.

The velocity (time derivative) $\dot{g}(t)$ is in the tangent space $T_g G$.



For the rigid body, g is a rotation, and $\dot{g}$ is the rotation velocity.

Phase Space

We will find it more convenient to look at the cotangent space T_g^*G. We thus introduce an operator

$$A_g : T_g G \to T_g^* G,$$

which is nondegenerate and symmetric,

$$\left\langle A_g \, \dot{g} \; , \dot{h} \right\rangle_g = \left\langle A_g \, \dot{h} \; , \dot{g} \right\rangle_g .$$

where the angle brackets denote the pairing $\langle \; , \; \rangle_g : T_g^* G \times T_g G \to \mathbb{R}$.

We define the canonical momentum as

$$p := A_g \, \dot{g}$$

The cotangent bundle T^*G is called the phase space.

Hamilton's Equation

A standard way of defining dynamics on phase space is by Hamilton's equations,

$$\dot{g} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial g},$$

where $H : T^*G \to \mathbb{R}$ is the Hamiltonian functional.

(The equations are obtained from a variational principle.)

Hamiltonian flows preserve phase-space area.

Left-invariant Hamiltonian

So far, we have not used the Lie group properties of G at all: the configuration space could be any manifold.

Consider the Hamiltonian

$$H[\,p\,,g\,] = \tfrac{1}{2}\big\langle p\,, A_g^{-1}\,p\big\rangle_g$$

The canonical momentum p can be obtained from some element of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G by the left action $L_{g^{-1}}$:

$$p = L^*_{g^{-1}}\,M,$$

where, for the rigid body, $M \in \mathfrak{g}^*$ is the angular momentum.

Inserting this transformation into H, we get

$$H[\,M\,,g\,] = \tfrac{1}{2}\Big\langle L^*_{g^{-1}}\,M\,,A_g^{-1}\,L^*_{g^{-1}}\,M\Big\rangle_g$$
$$= \tfrac{1}{2}\Big\langle M\,,\,\Big(L_{*g^{-1}}\,A_g^{-1}\,L^*_{g^{-1}}\Big)\,M\Big\rangle_g$$

Thus, if the metric operator is left-invariant

$$L_{*g^{-1}}\,A_g^{-1}\,L^*_{g^{-1}} = A_e^{-1} =: A^{-1},$$

then H becomes a function of M only:

$$\boxed{H[M] = \tfrac{1}{2}\langle M\,,A^{-1}\,M\rangle}$$

where now $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $H : \mathfrak{g}^* \to \mathbb{R}$.

- For the rigid body, the Hamiltonian is rotationally invariant: the absolute orientation of the body (i.e., the choice of a reference coordinate system) does not affect the dynamics.
- For the ideal fluid, the Hamiltonian has a relabeling symmetry: the choice of labels for the fluid elements (i.e., the choice of a reference state for the fluid) does not affect the dynamics.

The rotational symmetry of the rigid body can also be thought of as a “relabeling” of the coordinate axes.

Reduction

The symmetry (left-invariance) of the Hamiltonian is a requirement for a reduction, but is not sufficient.

We will have achieved a reduction if the evolution equation for M,

$$\dot{M} = \frac{d}{dt}\left(L_g^* p\right)$$

depends only on M. This is the case for both the free rigid body and the ideal fluid.

(More usually, one checks that the transformed Poisson bracket depends only on M.)

Euler's Equation

In order to write the equation of motion for M in a concise manner, we define the cobracket $[\ ,\ ]^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ by

$$\boxed{\left\langle [\,\alpha\,,\xi\,]^*\ ,\beta\right\rangle := \langle \xi\,,\,[\,\alpha\,,\beta\,]\rangle}$$

where $\alpha, \beta \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, and $[\ ,\ ]$ is the usual Lie bracket in $\mathfrak{g}$.

Then it is straightforward to show that the equation for M is Euler's equation

$$\boxed{\dot{M} = -\left[\,\frac{\delta H}{\delta M}\,, M\,\right]^*}$$

where

$$H_M := \frac{\delta H}{\delta M} \quad \in \mathfrak{g}$$

Euler's Equation for the Rigid Body

For the rigid body, the cobracket is the vector cross product, and Euler's equation is

$$\boxed{\dot{M} = \left(A^{-1}M\right) \times M}$$

where M is the angular momentum and A is the moment of inertia tensor.

Euler's Equation for the Ideal Fluid

For simplicity, we treat the two-dimensional ideal fluid. In that case, we can write the Hamiltonian as (here, $\omega \equiv M$)

$$H[\omega] = \tfrac{1}{2}\left\langle \omega\,, (-\nabla^2)^{-1}\omega \right\rangle = \int_D \omega(x,y)\,(-\nabla^2)^{-1}\omega(x,y)\,\mathrm{d}^2x,$$

where the velocity field is given in terms of the streamfunction $\phi(x,y)$ by $\mathbf{v}(x,y) = \hat{\mathbf{z}} \times \nabla\phi(x,y)$, and the scalar vorticity is $\omega(x,y) = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}} = \nabla^2\phi(x,y)$.

From the Hamiltonian, we see that the metric operator is the negative of the Laplacian,

$$A = -\nabla^2,$$

so that A^{-1} is a Green's function.

The Lie bracket in the algebra corresponding to the group of volume preserving diffeomorphisms *SDiff* D in two dimensions is

$$[\,\alpha\,,\beta\,] = \frac{\partial\alpha}{\partial x}\,\frac{\partial\beta}{\partial y} - \frac{\partial\beta}{\partial x}\,\frac{\partial\alpha}{\partial y},$$

which is the two-dimensional Jacobian. In this case, the cobracket is just the negative of the bracket, so Euler's equation is

$$\boxed{\dot{\omega} = -[\,\phi\,,\omega\,]}$$

where we have used $\delta H/\delta\omega = -\phi$.

Advection

Just like the Euler "description" of the rigid body rigid body contains no information about the orientation, for the fluid there is no information about the location of fluid elements. What the reduction tells us is that this information is not relevant for the dynamics.

(In fluid dynamical terms, we have gone from a Lagrangian description to an Eulerian one.)

But what if we want that information? What if I am an oceanographer tracking the salt concentration in the Atlantic? What equation does the salt concentration satisfy?

The salt is said to be advected by the flow.

Semidirect Structure

The symmetry group underlying advection is called a semidirect product of groups. The equations of motion for such a system are

$$\dot{\omega} = -\left[\frac{\delta H}{\delta \omega}, \omega\right]^* - \left[\frac{\delta H}{\delta c}, \omega\right]^*$$
$$\dot{c} = -\left[\frac{\delta H}{\delta \omega}, c\right]^*.$$

Here c is the concentration of some quantity tied to the fluid elements, such as salt or temperature. If the Hamiltonian depends on c then it will affect the ω equation, such as for Rayleigh-Bénard convection or reduced magnetohydrodynamics (in which the magnetic flux is advected).

In finite dimensions such a structure describes the heavy top in a constant gravitational field.

General Extensions

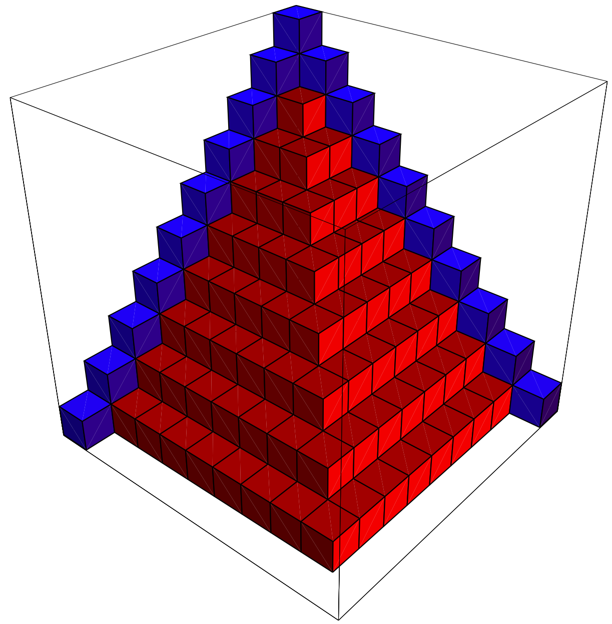
The semidirect structure is the simplest example of a bracket extension, where the equations of motion for n field variables ξ^μ are given by

$$\dot{\xi}^\mu = -W_\lambda{}^{\mu\nu} \left[\frac{\delta H}{\delta \xi^\nu}, \xi^\lambda \right]^*$$

(Repeated greek indices are summed.)

We have classified all possible extensions up to order $n = 5$ [Thiffeault and Morrison, 1999]. The structure of the W tensors is constrained by the requirement that the system be Hamiltonian.

Pictorially, we can represent the 3-tensors ${W_\lambda}^{\mu\nu}$ as cubes, to get a feel for their structure.



The part in red is called a nontrivial cocycle. If it vanishes, the extension is semidirect. The classification was achieved using Lie algebra cohomology.

(There are physical systems with cocycles: CRMHD, twisted top.)

Conclusions

- There is a close analogy between the rigid body and the ideal fluid: both have Lie groups as configuration spaces, and since the Hamiltonian is left-invariant under the group action we can effect a reduction.
- The information about the configuration of the system was not relevant to the dynamics.
- We recovered some information by using semidirect extensions, which reversed the reduction process (by symmetry breaking).
- There are many more types of extensions, which can be understood in terms of constraints in the system.