

# heat exchange and exit times

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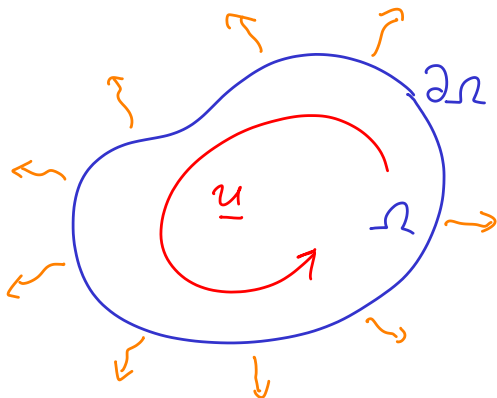
# advection–diffusion equation in a bounded region



Advection and diffusion of heat in a **bounded region**  $\Omega$ , with Dirichlet boundary conditions:

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = D \Delta \theta, \quad \mathbf{u} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0,$$

with  $\nabla \cdot \mathbf{u} = 0$  and  $\theta(\mathbf{x}, t) \geq 0$ .



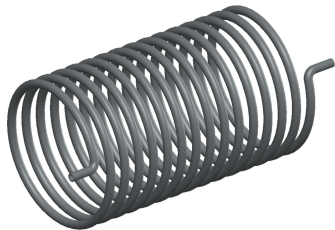
This is the **heat exchanger** configuration: given an initial distribution of heat, it is fluxed away through the cooled boundaries.

This happens through **diffusion** (conduction) alone, but is greatly aided by **stirring**.

# heat exchangers



Our domain will be a 2D cross-section of a traditional coil.



Write  $\langle \cdot \rangle$  for an integral over  $\Omega$ .

$$\langle \cdot \rangle := \int_{\Omega} \cdot \, dV$$

The **rate of heat loss is equal to the flux** through the boundary  $\partial\Omega$ :

$$\partial_t \langle \theta \rangle = D \int_{\partial\Omega} \nabla \theta \cdot \hat{\mathbf{n}} \, dS =: -F[\theta] \leq 0. \quad *$$

**Goal:** find velocity fields  $\mathbf{u}$  that maximize the heat flux.

Note that  $*$  is not so good for this, since velocity does not appear.

The role of  $\mathbf{u}$  is to **increase gradients** near the boundary. What it does internally is not directly relevant. **This is in contrast to the traditional Neumann IVP (chaotic mixing, etc).**



Take **steady velocity**  $\mathbf{u}(\mathbf{x})$ . The **mean exit time**  $\tau(\mathbf{x})$  of a Brownian particle initially at  $\mathbf{x}$  satisfies

$$-\mathbf{u} \cdot \nabla \tau = D \Delta \tau + 1, \quad \tau|_{\partial\Omega} = 0,$$

This is a steady advection–diffusion equation with velocity  $-\mathbf{u}$  and source 1.

Intuitively, a **small integrated mean exit time**  $\langle \tau \rangle = \|\tau\|_1$  implies that the velocity is efficient at taking heat out of the system.

The mean exit time equation is much nicer than the equation for the concentration: it is **steady**, and it applies for any **initial concentration**  $\theta_0(\mathbf{x})$ .

## relationship between exit time and mean temperature

Recall that  $\langle \cdot \rangle$  is an integral over space, and take  $\langle \theta_0 \rangle = 1$ . The quantity

$$\int_0^\infty \langle \theta \rangle dt$$

is a **cooling time**. **Smaller is better** for good heat exchange.

We have the rigorous bounds

$$\int_0^\infty \langle \theta \rangle dt \leq \|\tau\|_\infty \quad \int_0^\infty \langle \theta \rangle dt \leq \|\tau\|_1 \|\theta_0\|_\infty.$$

Thus, decreasing a norm like  $\|\tau\|_1$  or  $\|\tau\|_\infty$  will typically decrease the cooling time, as expected.

# does stirring always help?



[Iyer, G., Novikov, A., Ryzhik, L., & Zlatoš, A. (2010). *SIAM J. Math. Anal.* **42** (6), 2484–2498]

## Theorem (Iyer *et al.* 2010)

$\Omega \in \mathbb{R}^n$  bounded,  $\partial\Omega \in C^1$ . Then

$$\|\tau\|_{L^p(\Omega)} \leq \|\tau_0\|_{L^p(\mathcal{B})}, \quad 1 \leq p \leq \infty,$$

where  $\mathcal{B} \in \mathbb{R}^n$  is a ball of the same volume as  $\Omega$ , and  $\tau_0$  is the 'purely diffusive' solution,  $0 = D\Delta\tau_0 + 1$  on  $\mathcal{B}$ .

That is, measured in any norm, the exit time is maximized for a disk with no stirring. So **for a disk stirring always helps**, or at least isn't harmful.

They also prove that, surprisingly, if  $\Omega$  is not a disk, then it's **always** possible to make  $\|\tau\|_{L^\infty(\Omega)}$  **increase** by stirring. (Related to unmixing flows? [IMA 2010 gang; see review Thiffeault (2012)])



Let's formulate an optimization problem to find the best incompressible  $\mathbf{u}$ .

Advection–diffusion operator and its **adjoint**:

$$\mathcal{L} := \mathbf{u} \cdot \nabla - D\Delta, \quad \mathcal{L}^\dagger = -\mathbf{u} \cdot \nabla - D\Delta.$$

Minimize  $\langle \tau \rangle$  over steady  $\mathbf{u}(\mathbf{x})$  with fixed total kinetic energy  $E = \frac{1}{2} \|\mathbf{u}\|_2^2$ .

The functional to optimize:

$$\mathcal{F}[\tau, \mathbf{u}, \vartheta, \mu, p] = \langle \tau \rangle - \langle \vartheta (\mathcal{L}^\dagger \tau - 1) \rangle + \frac{1}{2} \mu (\|\mathbf{u}\|_2^2 - 2E) - \langle p \nabla \cdot \mathbf{u} \rangle$$

Here  $\vartheta$ ,  $\mu$ ,  $p$  are **Lagrange multipliers**.



Introduce streamfunction  $\psi$  to satisfy  $\nabla \cdot \mathbf{u} = 0$ :

$$u_x = -\partial_y \psi, \quad u_y = \partial_x \psi.$$

The variational problem gives the [Euler–Lagrange equations](#)

$$\begin{aligned} \mathcal{L}^\dagger \tau &= 1, & \tau|_{\partial\Omega} &= 0; \\ \mathcal{L} \vartheta &= 1, & \vartheta|_{\partial\Omega} &= 0; \\ \mu \Delta \psi &= J(\tau, \vartheta), & \psi|_{\partial\Omega} &= 0; \\ \langle |\nabla \psi|^2 \rangle &= 2E, \end{aligned}$$

with the Jacobian

$$J(\tau, \vartheta) := (\nabla \tau \times \nabla \vartheta) \cdot \hat{\mathbf{z}}.$$



Transform to new functions  $\eta$ ,  $\xi$

$$\tau = \tau_0 + \frac{1}{2}(\eta + \xi), \quad \vartheta = \tau_0 + \frac{1}{2}(\eta - \xi)$$

where recall that  $\tau_0$  is the solution without flow (purely diffusive).

Then by using the Euler–Lagrange equations we can eventually show

$$\langle \tau \rangle = \langle \tau_0 \rangle - \frac{1}{4} \langle |\nabla \xi|^2 \rangle - \frac{1}{4} \langle |\nabla \eta|^2 \rangle.$$

Hence, solutions to E–L equations cannot make  $\langle \tau \rangle$  increase. So stirring is always better than not stirring.

For a disk the purely diffusive solution is  $\tau_0 = \frac{1}{4}(1 - r^2)$ . We then make the *ansatz*

$$\xi = \sqrt{2\mu} B(r) \cos m\theta, \quad \eta = B(r) \sin m\theta, \quad \psi = \xi / \sqrt{2\mu},$$

and look for solutions of that form.

Inserting this into the full system gives solutions provided the radial functions  $B(r)$  satisfy the **nonlinear eigenvalue problem**

$$r^2 B'' + rB' + (r^2 \lambda - m^2)B = \frac{1}{2} m^2 B^3, \quad \lambda = m / \sqrt{2\mu}.$$

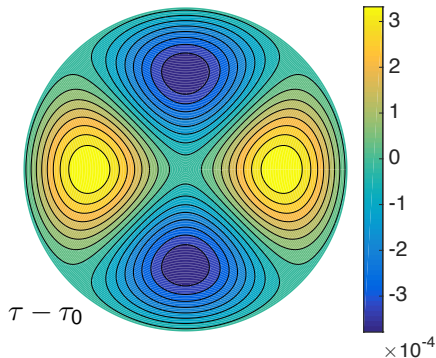
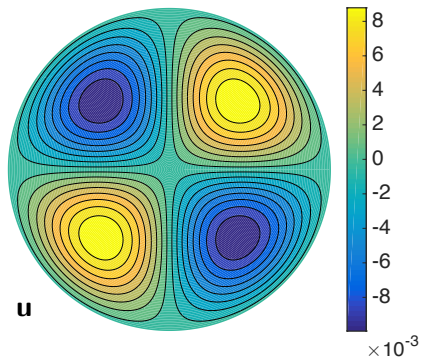
The left-hand side is Bessel's equation.

Note that it is rather unusual for such a linear-type ansatz to give nonlinear solutions. We also have no guarantee that this is the **true optimal solution**.

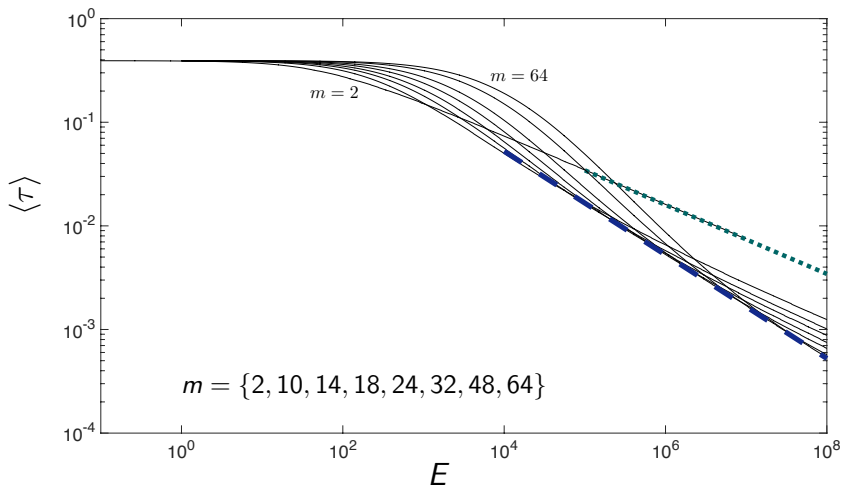
For **small energy**  $E$ , exact solution in terms of Bessel functions  $J_m(\rho_{mn}r)$ , where  $\rho_{mn}$  are zeros:

$$\langle \tau \rangle / \langle \tau_0 \rangle = 1 - (4m^2 / \pi \rho_{mn}^4) E + O(E^2).$$

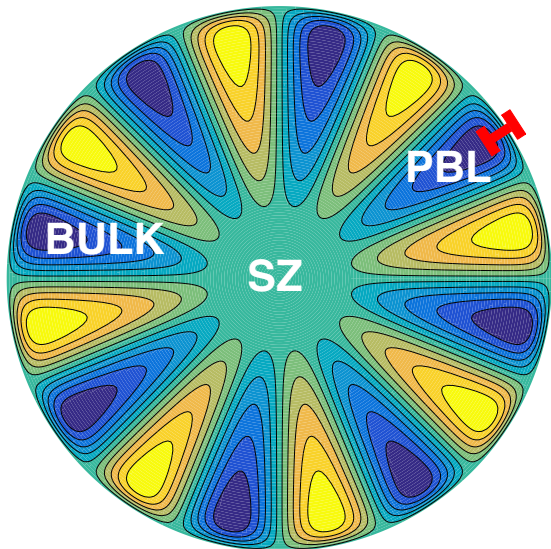
Pick the solution with the smallest  $\langle \tau \rangle$ :  $m = 2, n = 1$  for all  $E \ll 1$ :



Numerical solution with Matlab's **bvp5c**, using a continuation method:



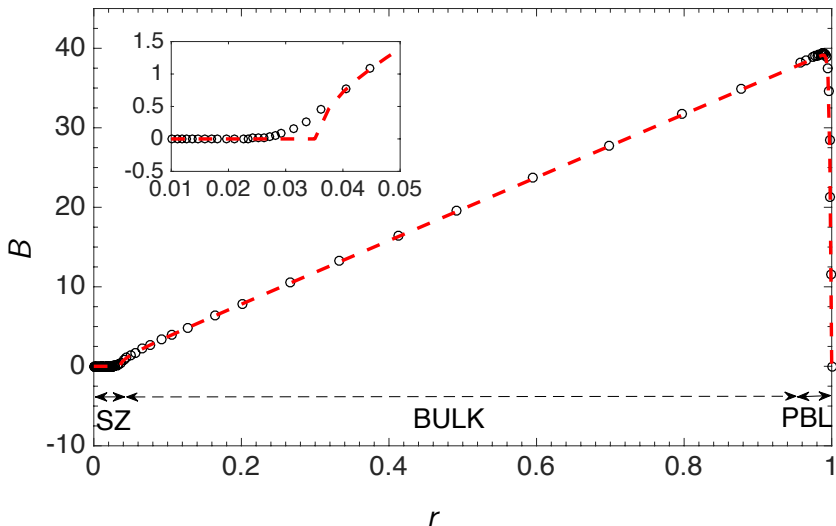
Larger  $m$  worse at small  $E$ , then better, then maybe worse again?



Three regions:

- Stagnation zone (SZ)
- Bulk
- Peripheral boundary layer (PBL)

# structure of the radial solution $B(r)$ for large $E$





Rescaled variables  $B = E^\alpha \tilde{B}$  and  $\lambda = E^\beta \tilde{\lambda}$ :

$$r^2 \tilde{B}'' E^\alpha + r \tilde{B}' E^\alpha + r^2 \tilde{\lambda} \tilde{B} E^{\alpha+\beta} - m^2 \tilde{B} E^\alpha = \frac{1}{2} m^2 \tilde{B}^3 E^{3\alpha}.$$

Outside the boundary layer, the large- $E$  balance must occur between the terms  $r^2 \tilde{\lambda} \tilde{B} E^{\alpha+\beta}$  and  $\frac{1}{2} m^2 \tilde{B}^3 E^{3\alpha}$ , so  $\beta = 2\alpha$ .

This gives the outer solution

$$B_{\text{outer}} = E^\alpha \tilde{B} = \sqrt{2/m^3 \tilde{\lambda}} E^\alpha r.$$

(This does not include the stagnation zone in the center. Neglect for now.)

Cannot satisfy  $B_{\text{outer}}(1) = 0$ : need **boundary layer**.





Inner variable  $r = 1 - \epsilon\rho$ :

$$\begin{aligned} \frac{(1 - \epsilon\rho)^2}{\epsilon^2} \bar{B}'' E^\alpha + \frac{(1 - \epsilon\rho)}{\epsilon} \bar{B}' E^\alpha + (1 - \epsilon\rho)^2 \tilde{\lambda} \bar{B} E^{3\alpha} - m^2 \bar{B} E^\alpha \\ = \frac{1}{2} m^2 \bar{B}^3 E^{3\alpha}. \end{aligned}$$

**Dominant balance:** highest derivative with  $E^\alpha = \epsilon^{-1}$ :

$$\bar{B}'' + \tilde{\lambda} \bar{B} = \frac{1}{2} m^2 \bar{B}^3.$$

This has an exact **tanh** solution, which after matching with the outer solution as  $\rho \rightarrow \infty$  gives

$$B_{\text{inner}} = \sqrt{2\tilde{\lambda}/m^2} E^\alpha \tanh\left(\sqrt{\lambda/2} \rho\right)$$



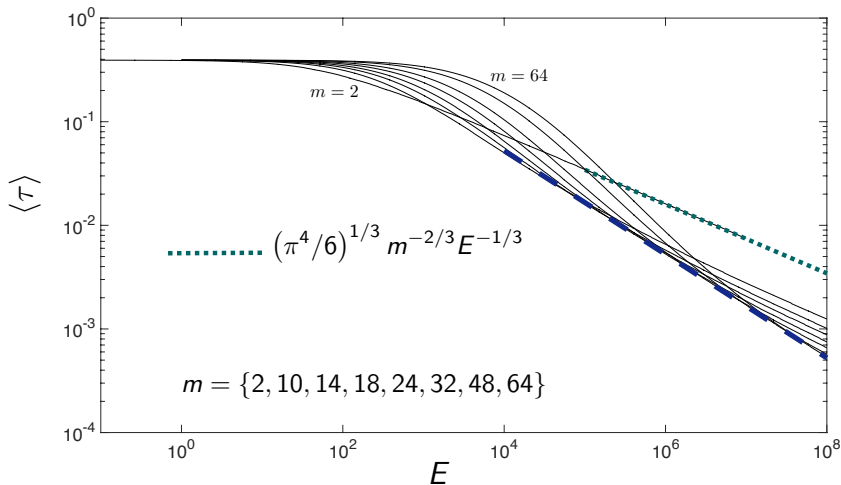
Finally we apply the energy constraint, which reads

$$\begin{aligned} \frac{2E}{\pi} &= \int_0^1 \left\{ rB'^2 + \frac{m^2}{r} B^2 \right\} dr \\ &= \int_0^{1-\delta} \left\{ rB_{\text{outer}}'^2 + \frac{m^2}{r} B_{\text{outer}}^2 \right\} dr + \int_{1-\delta}^1 \left\{ B_{\text{inner}}'^2 + m^2 B_{\text{inner}}^2 \right\} dr. \end{aligned}$$

We skip the details, but dominant balance requires  $\alpha = 1/3$ , and so  $\beta = 2\alpha = 2/3$ .

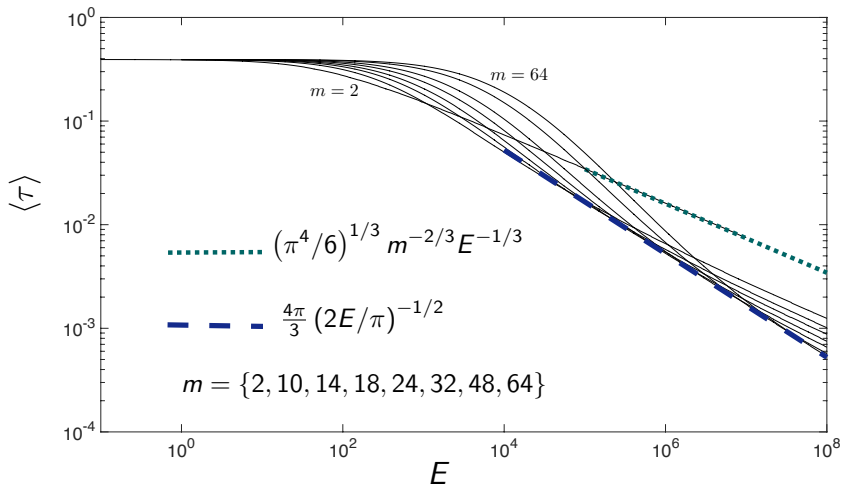
The optimal integrated exit time thus scales as  $m^{-2/3} E^{-1/3}$ .

# large- $E$ case: asymptotics at fixed $m$



Fixed- $E$  asymptotic optimal  $\langle \tau \rangle$  seems to decrease to zero as  $m^{-2/3}$ . This implies no optimal flow, since arbitrarily efficient at large  $m$ . Not so!

# large- $E$ , large- $m$ case



To truly capture the optimal solution, have to let  $m \sim E^{1/4}$ .  
This is the **dashed line** (envelope).

- Transport in heat exchangers has a very different character than ‘freely-decaying’ problem.
- Using the probabilistic **mean exit time** formulation simplifies the problem. (Idea came from Iyer et al. 2010.)
- Optimal solutions for  $\mathbf{u}$  are reminiscent of **Dean flow**.
- At small energy optimal solution has  $m = 2$ ,  $n = 1$ .
- At larger energy there is a boundary layer, which enhances the heat transfer or decreases exit time:  $\langle \tau \rangle \sim m^{-2/3} E^{-1/3}$ .
- This asymptotic solution breaks down when  $m$  gets too large. The **stagnation zone** becomes larger and penalizes large  $m$ .
- A distinguished limit in  $m$  gives  $\langle \tau \rangle \sim E^{-1/2}$ .
- Generalizations: use different norms, spatial weight. . .



- Iyer, G., Novikov, A., Ryzhik, L., & Zlatoš, A. (2010). *SIAM J. Math. Anal.* **42** (6), 2484–2498.
- Thiffeault, J.-L. (2012). *Nonlinearity*, **25** (2), R1–R44.