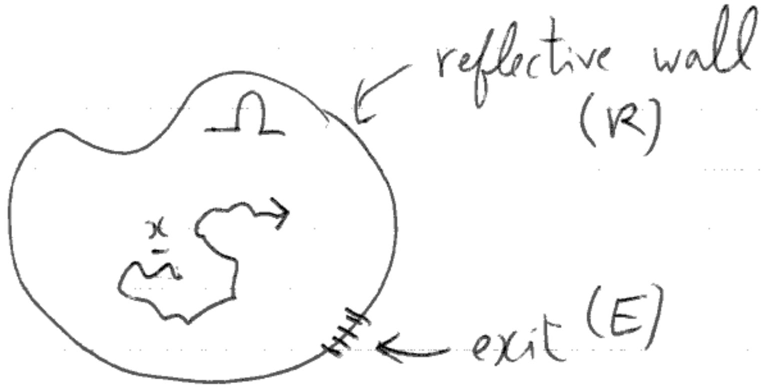
BIRS talk 7/26/18 — 7/23/18

Exit time problems for microswimmers

Basic setup:

Brownian particle (BP)



Q: What is $\tau(\underline{x})$, the mean exit time (MET) for a BP starting at $\underline{x}$?

A: Solve simple elliptic problem:

$$D\Delta\tau \equiv D\nabla^2\tau = -1$$

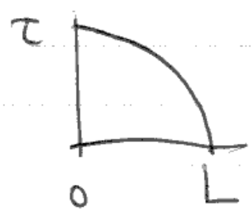
$$\tau\Big|_E = 0, \qquad \frac{\partial\tau}{\partial n}\Big|_R = 0$$

example: interval

$\Omega = [0, L]$

$\tau'(0) = 0, \quad \tau(L) = 0$

$$\tau(x) = \frac{L^2}{2D}\left(1 - \frac{x^2}{L^2}\right)$$

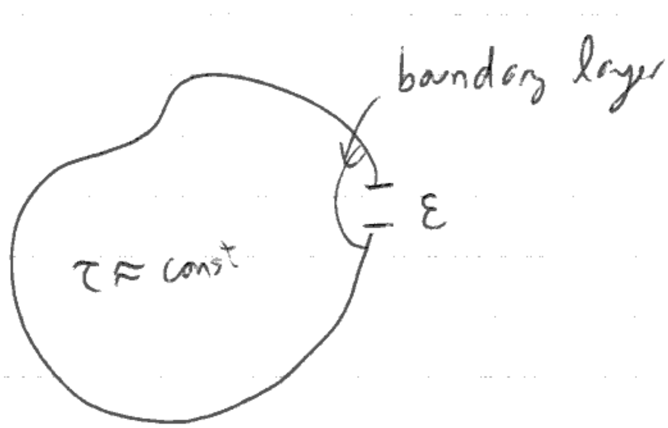


Small exit:

Can exploit small parameter to get asymptotic solution.

(see review by Holcman & Schuss (2014).)

Important observation: as $\varepsilon \to 0$, MET $\tau(\underline{x}) \approx$ const!



A BP typically explores the entire domain before finding the exit. Hence, independent of initial position $\underline{x}$.

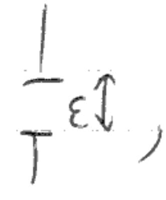
Connected to decaying eigenfunction: $\partial_t c = D\nabla^2 c$

Take $c = \tilde{c}(\underline{x})\, e^{-\sigma t}$: $\quad -\sigma\tilde{c} = D\nabla^2\tilde{c}$

Now take $\tilde{c} = \tau$: $\quad -\sigma\tau = -1 \Rightarrow \sigma = \dfrac{1}{\tau}$

So, for a small exit, the MET^{-1} is the same as the rate of loss through the exit.

For an exit on a ~~smooth~~ smooth wall



find

$$\tau = \frac{|\Omega|}{\pi D}\log\left(\varepsilon d\right)^{-1} + O(\varepsilon^0)$$

in 2D. ($\tau \to \infty$ as $\varepsilon \to 0$)

$|\Omega|$ = area

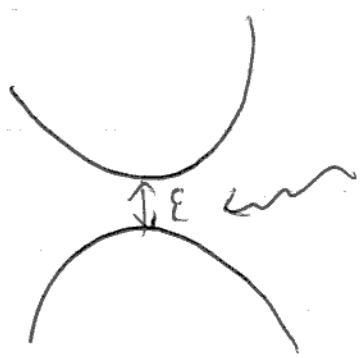
d = "log capacitance" of exit

In 3D, the MET diverges more strongly as $\varepsilon \to 0$:

$$\tau(\underline{x}) = \frac{|\Omega|}{2D\varepsilon} + O(\varepsilon^0) \qquad \text{(circular hole of radius } \varepsilon)$$

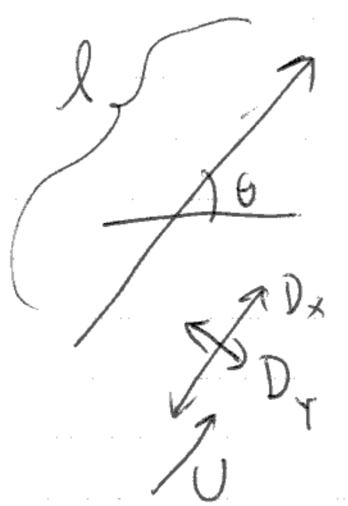
i.e., it is much harder to find the exit in 3D.

Note that the scaling with ε depends strongly on the shape of the domain near the exit:



$$\tau \sim \frac{1}{\sqrt{\varepsilon}} \qquad (\text{rather than } \log \varepsilon^{-1})$$

Of course, this is for point particles. Swimmers are better modeled as rods:



SDE:

$$dx = \left(U\,dt + \sqrt{2D_x}\,dW_1\right)\cos\theta - \sin\theta\,\sqrt{2D_Y}\,dW_2$$

$$dy = \left(U\,dt + \sqrt{2D_x}\,dW_1\right)\sin\theta + \cos\theta\,\sqrt{2D_Y}\,dW_2$$

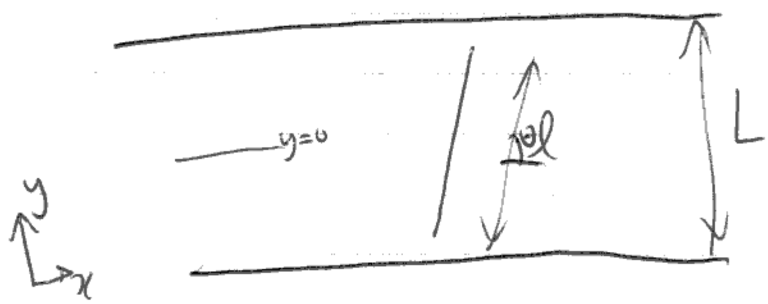
$$d\theta = \sqrt{2D_\theta}\,dW_3$$

$W_i(t)$ are independent Brownian motions

The exit time equation for the needle is ($D_x = D_y = 0$):

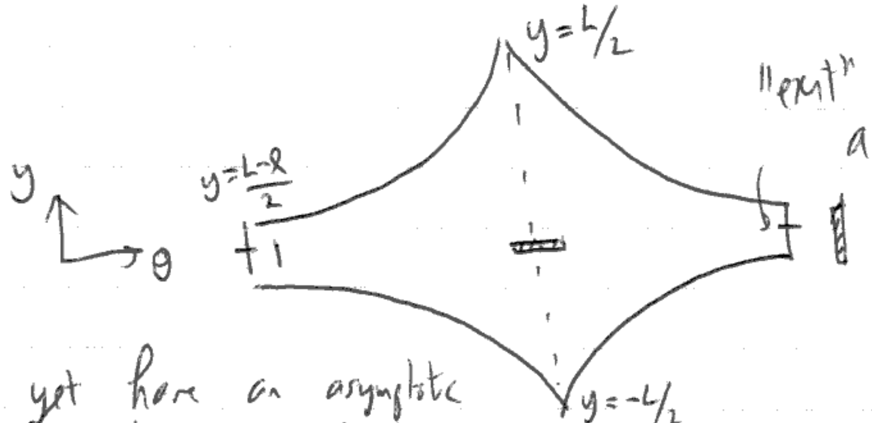
$$U\cos\theta\,\partial_x \tau + U\sin\theta\,\partial_y \tau + D_\theta\,\partial^2_{\theta\theta}\tau = -1.$$

But what's the domain? Take a channel of width L:



Take $L-\ell \ll 1$, i.e., the channel is tight

x is irrelevant.



"exit" corresponds to a reversal.

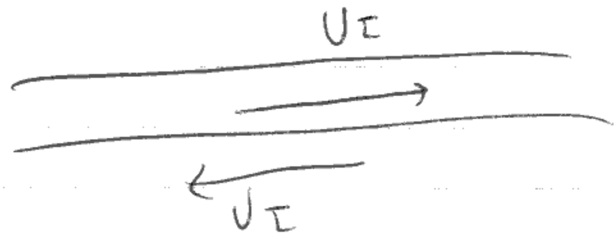
$\varepsilon = L - \ell$

We don't get here an asymptotic formula for the reversal time. (Just numerics.)

H & S ($D_x, D_y \neq 0$, $U = 0$) find $\tau \sim \dfrac{\pi(\pi-2)}{D_\theta\,(L(L-\ell))^{1/4}}\sqrt{\dfrac{D_x}{D_\theta}} + o(\sqrt{\varepsilon})$

$$\sim \frac{1}{\sqrt{\varepsilon}}$$

This reversal time can, presumably, be tied to an effective diffusivity for a confined swimmer:



$$D_{eff} \sim \frac{(U\tau)^2}{\tau} = U^2\tau$$

(Could also try to solve for the effective diff directly — in progress)

Q: In 3D, model organism in a pipette?

Are there interesting geometrical problems that are mathematically tractable, but biologically relevant?

Consumption problems?

5 words: Small exits mean easier math

Haiku: τ Neumann at walls / Delta τ is minus one / Dirichlet exits

Limerick: There once was a microbe from Limerick
Who wanted to visit the arctic
She wandered about
But soon found out
It would take a time logarithmic