Moving walls accelerate mixing

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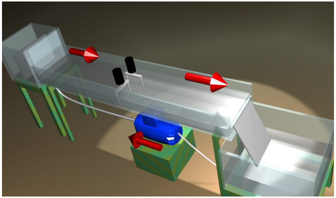
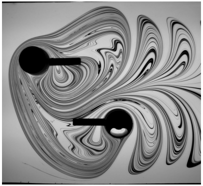
Stirring and Mixing of Viscous Fluids


- Viscous flows ⇒ no turbulence! (laminar)
- Open and closed systems
- Active (rods) and passive

Understand the mechanisms involved.
Characterise and optimise the efficiency of mixing.

Stirring and Mixing: What's the Difference?

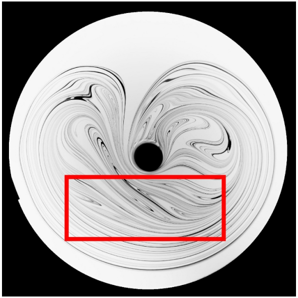
- Stirring is the mechanical motion of the fluid (cause);
- Mixing is the homogenisation of a substance (effect, or goal);
- Two extreme limits: Turbulent and laminar mixing, both relevant in applications;
- For very viscous flows, use simple time-dependent flows to create chaotic mixing.
- Here we look at the impact of the vessel walls on mixing rates.

A Simple Example: Planetary Mixers

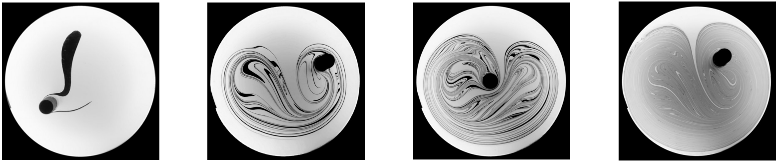
In food processing, rods are often used for stirring.

[movie 1] ©BLT Inc.

The Figure-Eight Stirring Protocol



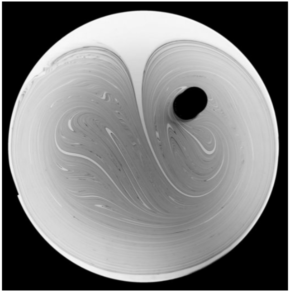
- Circular container of viscous fluid (sugar syrup);
- A rod is moved slowly in a 'figure-eight' pattern;
- Gradients are created by stretching and folding, the signature of chaos.

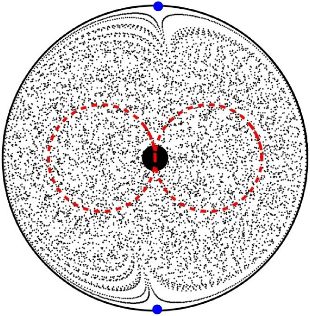


[movie 2] Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

The Mixing Pattern

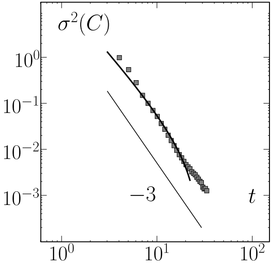
- Kidney-shaped mixed region extends to wall;
- Two distinguished parabolic points on the wall, one associated with injection of material into the central mixing region;
- Asymptotically self-similar, so expect an exponential decay of the concentration ('strange eigenmode' regime).
  (Pierrehumbert, 1994; Rothstein et al., 1999; Voth et al., 2003)



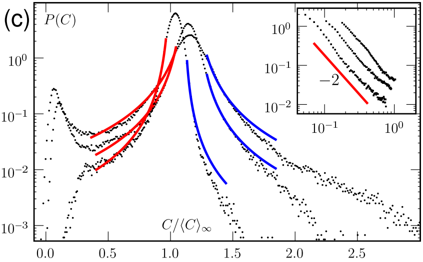


Mixing is Slower Than Expected

Concentration field in a well-mixed central region



Variance $= \int |\theta|^2 dV$

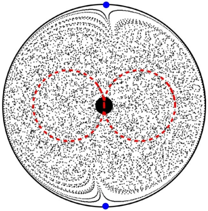


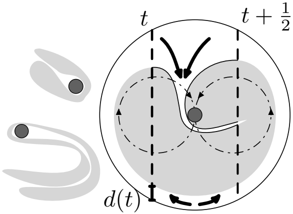
Concentration PDFs

$\Rightarrow$ Algebraic decay of variance $\neq$ Exponential

The 'stretching and folding' action induced by the rod is an exponentially rapid process (chaos!), so why aren't we seeing exponential decay?

Walls Slow Down Mixing





- Trajectories are (almost) everywhere chaotic
  $\Rightarrow$ but there is always poorly-mixed fluid near the walls;
- Re-inject unmixed (white) material along the unstable manifold of a parabolic point on the wall;
- No-slip at walls $\Rightarrow$ width of "white stripes" $\sim t^{-2}$ (algebraic);
- Re-injected white strips contaminate the mixing pattern, in spite of the fact that stretching is exponential in the centre.

Hydrodynamics Near the Wall

We can characterize white strips in terms of hydrodynamics near the no-slip wall. $x_\parallel$ and $x_\perp$ denote respectively the distance along and $\perp$ to the wall. No-slip boundary conditions impose

$$v_\parallel \sim x_\perp, \qquad \text{near the wall:} \quad x_\perp \ll 1.$$

Incompressibility

$$\frac{\partial v_\parallel}{\partial x_\parallel} + \frac{\partial v_\perp}{\partial x_\perp} = 0,$$

implies

$$v_\perp \simeq -a\, x_\perp^2\,.$$

Solve $\dot{x}_\perp = v_\perp$:

$$x_\perp \simeq \frac{x_0}{1 + at\, x_0}.$$

Hydrodynamics Near the Wall (continued)

Hence, the distance between the wall and a particle in the lower part of the domain (where $v_\perp < 0$) shrinks as

$$d(t) \simeq 1/at, \qquad t \gg 1.$$

This scaling was derived in Chertkov & Lebedev (2003), and we verified it experimentally. [See also Lebedev & Turitsyn, 2004; Salman & Haynes, 2007; Chernykh & Lebedev, 2008.]

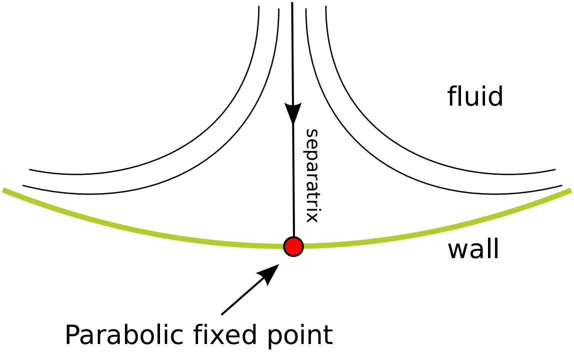
The amount of white that is 'shaved off' at each period is thus

$$\dot{d} \sim T/at^2, \qquad t \gg 1,$$

where T is the period. This is the origin of the power-law decay. Corrections due to the stretch/fold action are described in [Gouillart et al., *Phys. Rev. Lett.* **99**, 114501 (2007)].

The Problem: Separatrix at the Wall

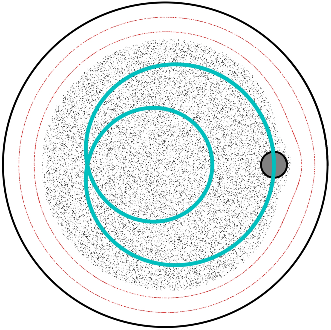
The decay is algebraic near a reattachment point at the wall:



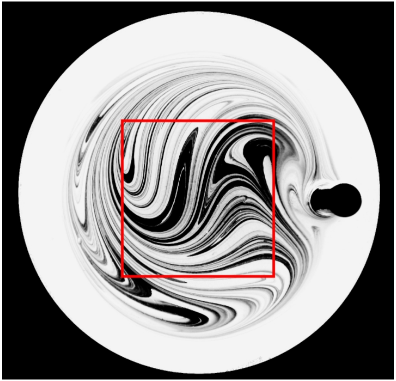
A fluid particle following the separatrix approaches the wall as $1/t$.
[Chertkov & Lebedev (2003); Wang et al. (2003); Lebedev & Turitsyn (2004); Salman & Haynes (2007); Gouillart et al. (2007, 2008, 2009); Chernykh & Lebedev (2008)]

A Second Scenario

How do we mimic a slip boundary condition?



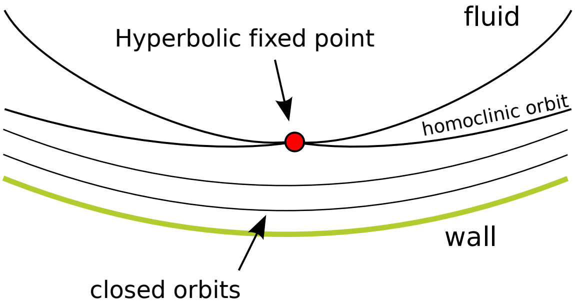
"Epitrochoid" protocol



Central chaotic region + regular region near the walls.

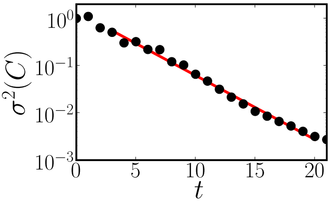
Mimic a Slip Boundary Condition

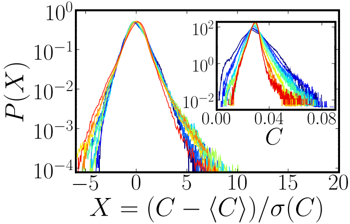
Create closed orbits near the wall:



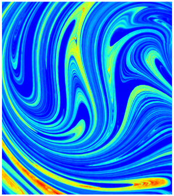
There will be a 'last closed orbit' followed by one or more fixed or periodic points and a separatrix, for example a hyperbolic orbit. Particles approach the hyperbolic fixed point exponentially fast.
[Thiffeault et al. (2011)]

Recover Exponential Decay

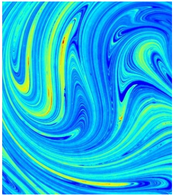




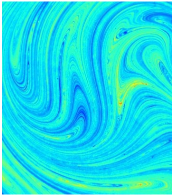
$t = 8$



$t = 12$



$t = 17$



. . . as well as ‘true’ self-similarity (eigenmode).

Another Approach: Rotate the Bowl!

The Edge Map

Can use a simplified 'edge map' to model the near-wall region: [see also Wang et al. (2003)].

In area-preserving form:

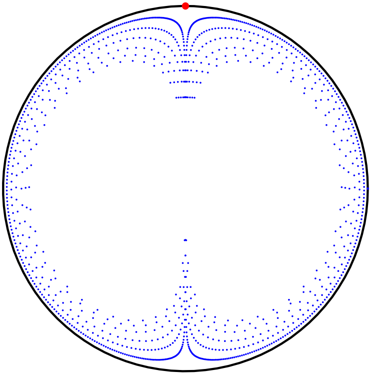
$$\bar{x}_{\parallel} = x_{\parallel} + \Omega + a(x_{\parallel})\,\bar{x}_{\perp}$$
$$\bar{x}_{\perp} = \left(\sqrt{1 + 2a'(x_{\parallel})\,x_{\perp}} - 1\right)/a'(x_{\parallel})$$

Here Ω is the wall rotation rate.

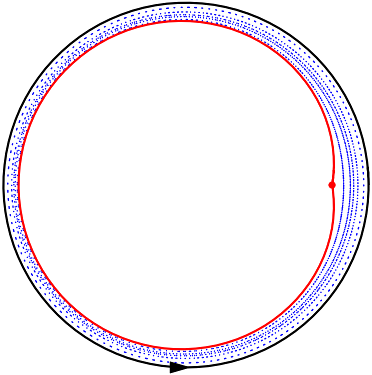
The function $a(x_{\parallel})$ can be measured from numerics (or even perhaps experiments), but is not too far from $\sin x_{\parallel}$.

[Thiffeault et al. (2011)]

Iterates of the Edge Map



Fixed wall: parabolic separation point (algebraic)



Moving wall: hyperbolic fixed point (exponential)

[Thiffeault et al. (2011)]

Conclusions

- If the chaotic region extends to the walls, then the decay of concentration is algebraic (typically t^{-3} for variance).
- The no-slip boundary condition at the walls is to blame.
- Would recover a strange eigenmode for very long times, once the mixing pattern is within a Batchelor length from the edge (not very useful in practice!).
- The decay is well-predicted by a baker's map with a parabolic point.
- We can shield the mixing region from the walls by wrapping it in a regular island (transport barrier).
- We then recover exponential decay.
- How to control this in practice? Is it really advantageous? Is scraping the walls better?
- See Gouillart et al. (2007, 2008, 2009); Thiffeault et al. (2011).

Chernykh, A. & Lebedev, V. 2008 Passive scalar structures in peripheral regions of random flows. *JETP Lett.* **87**, 682–686.

Chertkov, M. & Lebedev, V. 2003 Boundary Effects on Chaotic Advection-Diffusion Chemical Reactions. *Phys. Rev. Lett.* **90**, 134501.

Gouillart, E., Dauchot, O., Dubrulle, B., Roux, S. & Thiffeault, J.-L. 2008 Slow decay of concentration variance due to no-slip walls in chaotic mixing. *Phys. Rev. E* **78**, 026211. `http://arxiv.org/abs/0803.0709`.

Gouillart, E., Dauchot, O., Thiffeault, J.-L. & Roux, S. 2009 Open-flow Mixing: Experimental Evidence for Strange Eigenmodes. *Phys. Fluids* **21**, 022603.

Gouillart, E., Kuncio, N., Dauchot, O., Dubrulle, B., Roux, S. & Thiffeault, J.-L. 2007 Walls Inhibit Chaotic Mixing. *Phys. Rev. Lett.* **99**, 114501.

Lebedev, V. V. & Turitsyn, K. S. 2004 Passive scalar evolution in peripheral regions. *Phys. Rev. E* **69**, 036301.

Pierrehumbert, R. T. 1994 Tracer microstructure in the large-eddy dominated regime. *Chaos Solitons Fractals* **4**, 1091–1110.

Rothstein, D., Henry, E. & Gollub, J. P. 1999 Persistent patterns in transient chaotic fluid mixing. *Nature* **401**, 770–772.

Salman, H. & Haynes, P. H. 2007 A numerical study of passive scalar evolution in peripheral regions. *Phys. Fluids* **19**, 067101.

Thiffeault, J.-L., Gouillart, E. & Dauchot, O. 2011 Moving walls accelerate mixing. *Phys. Rev. E* **84**, 036313. `arXiv:1106.3945`.

Voth, G. A., Saint, T. C., Dobler, G. & Gollub, J. P. 2003 Mixing rates and symmetry breaking in two-dimensional chaotic flow. *Phys. Fluids* **15**, 2560–2566.

Wang, Y., Haller, G., Banaszuk, A. & Tadmor, G. 2003 Closed-loop Lagrangian separation control in a bluff body shear flow model. *Phys. Fluids* **15**, 2251–2266.