Scalar Decay in Chaotic Mixing

Local and Global Theory

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Overview

Part I: Local theory:

- How does a blob of dye behave in a steady flow?
- How does a blob behave in a random flow?
- How do a large number of blobs behave in a random flow?

Part II: Global theory:

- Get away from local (or blob) picture.
- Every detail matters (such as boundary conditions)!
- Fewer generic features.
- Focus is on eigenfunctions.

Prelude

The Advection–Diffusion Equation

The equation that is in the spotlight is the advection–diffusion equation

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta = \kappa \nabla^2 \theta$$

for the time-evolution of a distribution of concentration $\theta(\boldsymbol{x}, t)$, being advected by a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, and diffused with diffusivity κ.

We will restrict our attention to incompressible velocity fields, for which $\nabla \cdot \boldsymbol{v} = 0$.

We leave the exact nature of θ nebulous: it could be temperature, concentration of salt, dye, chemicals, isotopes, plankton. . . .

The only assumption for now is that this scalar is passive, which means that it does not affect the velocity field $\boldsymbol{v}$.

Some Properties

Define average over the domain V:

$$\langle\theta\rangle := \frac{1}{V}\int_V \theta \,\mathrm{d}V,$$

AD eq'n conserves the total quantity of θ, $\partial_t \langle\theta\rangle = 0$, for periodic or zero-flux ($\hat{\boldsymbol{n}} \cdot \nabla\theta = 0$) boundary conditions.

To measure the degree of mixing, define the variance,

$$\mathrm{Var} := \langle\theta^2\rangle - \langle\theta\rangle^2,$$

Then

$$\partial_t \mathrm{Var} = -2\kappa \left\langle |\nabla\theta|^2 \right\rangle \leq 0.$$

In bounded or periodic domains, we are guaranteed that variance will go to zero.

How is Mixing Enhanced?

There is an apparent problem with this:

The evolution equation for the variance no longer involves the velocity field. But if variance is to give us a measure of mixing, shouldn't its time-evolution involve the velocity field?

What's the catch?

We do not have a closed equation for the variance: the right-hand side involves $|\nabla\theta|^2$, which is not the same as θ^2. As we will see, the stirring velocity field can create very large gradients in the concentration field, which makes variance decrease much faster than it would if diffusivity were acting alone.

This, in a nutshell, is the essence of enhanced mixing.

Some Questions

Several important questions can now be raised:

- How fast is the approach to the perfectly-mixed state?
- How does this depend on diffusivity?
- What does the concentration field look like for long times? What is its spectrum?
- How does the probability distribution of θ evolve?
- Which stirring fields give efficient mixing?

The answers to these questions are quite complicated, and not fully known. I will attempt to give some hints of the answers.

Blobs, Part I:

Steady Flows

A Linear Velocity Field

What happens to a passive scalar advected by a linear velocity field? This is the starting point for what may be called the local theory of mixing.

The perfect setting to consider a linear flow is in the limit of large Schmidt number,

$$\mathrm{Sc} := \nu/\kappa$$

where ν is the kinematic viscosity of the fluid.

The scalar field has much faster spatial variations than the velocity field. Can focus on a region of the domain large enough for the scalar concentration to vary appreciably, but small enough that the velocity field appears linear.

This regime leads to the celebrated k^{-1} Batchelor spectrum [Batchelor, 1959].

Solution of the Problem

We choose a linear velocity field of the form

$$\boldsymbol{v} = \boldsymbol{x} \cdot \sigma(t), \qquad \mathrm{Tr}\,\sigma = 0.$$

We wish to solve the initial value problem

$$\partial_t \theta + \boldsymbol{x} \cdot \sigma(t) \cdot \nabla\theta = \kappa \nabla^2 \theta, \qquad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}).$$

We will follow closely the solution of Zeldovich et al. [1984], who solved this by the method of "partial solutions,"

$$\theta(\boldsymbol{x}, t) = \hat{\theta}(\boldsymbol{k}_0, t) \exp(\mathrm{i}\boldsymbol{k}(t) \cdot \boldsymbol{x}), \qquad \boldsymbol{k}(0) = \boldsymbol{k}_0, \quad \hat{\theta}(\boldsymbol{k}_0, 0) = \hat{\theta}_0(\boldsymbol{k}_0),$$

where $\boldsymbol{k}_0$ is some initial wavevector. We will see if we can make this into a solution by a judicious choice of $\hat{\theta}(\boldsymbol{k}_0, t)$ and $\boldsymbol{k}(t)$.

This gives the two evolution equations

$$\partial_t \boldsymbol{k} = -\sigma \cdot \boldsymbol{k} \, , \tag{1}$$

$$\partial_t \hat{\theta} = -\kappa \, k^2 \hat{\theta}. \tag{2}$$

We can write the solution to (1) in terms of the fundamental solution $\mathcal{T}_t$ as

$$\boldsymbol{k}(t) = \mathcal{T}_t \cdot \boldsymbol{k}_0 \, ,$$

where

$$\partial_t \mathcal{T}_t = -\sigma(t) \cdot \mathcal{T}_t, \qquad \mathcal{T}_0 = \mathrm{Id}$$

and Id is the identity matrix. We can then use the same fundamental solution for all initial conditions $\boldsymbol{k}_0$.

If σ is not a function of time, then the fundamental solution is simply a matrix exponential,

$$\mathcal{T}_t = \exp(-\sigma\, t),$$

but in general the form of $\mathcal{T}_t$ is more complicated.

Note that because $\mathrm{Tr}\,\sigma = 0$, we have

$$\det \mathcal{T}_t = 1,$$

which expresses volume conservation (incompressibility).

Can express the solution to (1) and (2) as

$$\boldsymbol{k}(t) = \mathcal{T}_t \cdot \boldsymbol{k}_0 \,,$$

$$\hat{\theta}(\boldsymbol{k}_0, t) = \hat{\theta}_0(\boldsymbol{k}_0) \exp\left\{-\kappa \int_0^t \left(\mathcal{T}_s \cdot \boldsymbol{k}_0\right)^2 \mathrm{d}s\right\}.$$

We can think of $\mathcal{T}_t$ as transforming a Lagrangian wavevector $\boldsymbol{k}_0$ to its Eulerian counterpart $\boldsymbol{k}$.

$\hat{\theta}$ decays diffusively at a rate determined by the cumulative norm of the wavenumber $\boldsymbol{k} = \mathcal{T}_s \cdot \boldsymbol{k}_0$ experienced during its evolution.

The full solution to the AD eq'n is now given by superposition of the partial solutions,

$$\begin{aligned}\theta(\boldsymbol{x},t) &= \int \hat{\theta}(\boldsymbol{k}_0,t)\exp(\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x})\,\mathrm{d}^3k_0 \\ &= \int \hat{\theta}_0(\boldsymbol{k}_0)\exp\left\{\mathrm{i}\,\boldsymbol{x}\cdot\mathcal{T}_t\cdot\boldsymbol{k}_0 - \kappa\int_0^t \left(\mathcal{T}_s\cdot\boldsymbol{k}_0\right)^2\mathrm{d}s\right\}\mathrm{d}^3k_0\,,\end{aligned}$$

where $\hat{\theta}_0(\boldsymbol{k}_0)$ is the Fourier transform of the initial condition $\theta_0(\boldsymbol{x})$.

Two effects: stretching of the initial wavenumber and decay of the initial amplitude.

Stretching Flow in 2D

Take an even more idealised approach: consider the case where the velocity gradient matrix σ is constant and two-dimensional.

After a coordinate change, the traceless matrix σ can take one of two possible forms,

$$\sigma^{(2a)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad \text{and} \qquad \sigma^{(2b)} = \begin{pmatrix} 0 & 0 \\ U' & 0 \end{pmatrix}.$$

Case (2a) is a uniformly stretching flow that stretches exponentially in one direction, and contracts in the other.

Case (2b) is a linear shear flow in the x_1 direction.

We assume without loss of generality that $\lambda > 0$ and $U' > 0$.

The corresponding fundamental matrices $\mathcal{T}_t = \exp(-\sigma\, t)$ are easy to compute.

For Case (2a) we merely exponentiate the diagonal elements.

$$\mathcal{T}_t^{(2\mathrm{a})} = \begin{pmatrix} \mathrm{e}^{-\lambda t} & 0 \\ 0 & \mathrm{e}^{\lambda t} \end{pmatrix}$$

For Case (2b) the exponential power series terminates after two terms, because $\sigma^{(2b)}$ is nilpotent.

$$\mathcal{T}_t^{(2\mathrm{b})} = \begin{pmatrix} 1 & 0 \\ -U' t & 1 \end{pmatrix}.$$

Consider Case (2a), a flow with constant stretching. The action of the fundamental matrix on $\boldsymbol{k}_0$ is

$$\mathcal{T}_t^{(2a)} \cdot \boldsymbol{k}_0 = \left(\mathrm{e}^{-\lambda t}\, k_{01}\,,\ \mathrm{e}^{\lambda t}\, k_{02} \right),$$

with norm

$$\left(\mathcal{T}_t^{(2a)} \cdot \boldsymbol{k}_0\right)^2 = \mathrm{e}^{-2\lambda t}\, {k_{0}}_1^2 + \mathrm{e}^{2\lambda t}\, {k_{0}}_2^2\,.$$

The wavevector $\boldsymbol{k}(t) = \mathcal{T}_t^{(2a)} \cdot \boldsymbol{k}_0$ grows exponentially in time, which means that the length scale is becoming very small.

This only occurs in the direction x_2, which is sensible because that direction corresponds to a contracting flow.

For one Fourier mode, we have

$$\hat{\theta}(\boldsymbol{k}_0, t) = \hat{\theta}_0(\boldsymbol{k}_0) \exp\left\{-\kappa \int_0^t \left(\mathrm{e}^{-2\lambda s}\, {k_0}_1^2 + \mathrm{e}^{2\lambda s}\, {k_0}_2^2\right) \mathrm{d}s\right\} .$$

The time-integral can be done explicitly, and we find

$$\hat{\theta}(\boldsymbol{k}_0, t) = \hat{\theta}_0(\boldsymbol{k}_0) \exp\left\{-\frac{\kappa}{2\lambda} \left(\left(\mathrm{e}^{2\lambda t} - 1\right) {k_0}_2^2 - \left(\mathrm{e}^{-2\lambda t} - 1\right) {k_0}_1^2\right)\right\} .$$

For moderately long times ($t \gtrsim \lambda^{-1}$), we can surely neglect $\mathrm{e}^{-2\lambda t}$ compared to 1, and 1 compared to $\mathrm{e}^{2\lambda t}$,

$$\hat{\theta}(\boldsymbol{k}_0, t) \simeq \hat{\theta}_0(\boldsymbol{k}_0) \exp\left\{-\frac{\kappa}{2\lambda} \left(\mathrm{e}^{2\lambda t}\, {k_0}_2^2 + {k_0}_1^2\right)\right\} .$$

This assumption of moderately long time is easily justified physically.

If $\kappa k_0^2/\lambda \ll 1$, where k_0 is the largest initial wavenumber (that is, the smallest initial scale), then can neglect diffusion unless

$$\mathrm{e}^{2\lambda t} \gtrsim \mathrm{Pe}^{-1}, \quad \text{or} \quad \lambda\, t \gtrsim \log \mathrm{Pe}^{-1/2}$$

where the Péclet number is

$$\mathrm{Pe} = \lambda/\kappa\, k_0^2 \,.$$

The Péclet number influences this time scale only weakly. Diffusivity has only a logarithmic effect. Thus vigorous stirring always has a chance to overcome a small diffusivity: we need just stir a bit longer.

Roughly, for one Fourier mode our solution predicts

$$\hat{\theta} \sim \exp\left\{-\mathrm{Pe}^{-1}\,\mathrm{e}^{2\lambda t}\right\}$$

for $\lambda t \gg 1$: a superexponential decay (unreasonable).

This decay comes from the factor

$$\exp(-(\kappa/2\lambda)\,\mathrm{e}^{2\lambda t}\,k_{0}{}_{2}^{2})\,.$$

There is an exponential increase in the wavenumber. This is exactly the mechanism for enhanced mixing we advertised earlier: very large gradients of concentration are being created, exponentially fast. This mechanism is just acting too quickly for our taste!

So what's the problem? Clearly the concentration in most wavenumbers gets annihilated almost instantly, once enough time has elapsed.

Blow up the k_{02} integration by making the coordinate change $\widetilde{k}_{02} = k_{02}\,\mathrm{e}^{\lambda t}$,

$$\theta(\boldsymbol{x},t) = \mathrm{e}^{-\lambda t} \int_{-\infty}^{\infty} \mathrm{d}k_{01} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{k}_{02}\; \hat{\theta}_0(k_{01}, \widetilde{k}_{02}\,\mathrm{e}^{-\lambda t})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}} \times \exp\left\{-\frac{\kappa}{2\lambda}\left(\widetilde{k}_{02}^{\,2} + k_{01}^{\,2}\right)\right\},$$

For large times, dominated by very small wavenumbers in x_2 direction.

For small κ, we can neglect the $k_{01}^{\,2}$ term in the exponential.

Taking the inverse Fourier transform,

$$\boxed{\theta(\boldsymbol{x},t) \simeq \mathrm{e}^{-\lambda t}\, G\big(x_2\,;\,\ell\big) \int_{-\infty}^{\infty} \theta_0(\mathrm{e}^{-\lambda t}x_1, \tilde{x}_2)\,\mathrm{d}\tilde{x}_2\,,}$$

where

$$G(x\,;\,\sigma) := \frac{1}{\sqrt{2\pi\ell^2}}\,\mathrm{e}^{-x^2/2\ell^2}$$

is a normalised Gaussian with standard deviation ℓ, and

$$\ell := \sqrt{\kappa/\lambda}\,.$$

The x_1 dependence is given by the stretched initial distribution, averaged over x_2. The x_2 dependence is always Gaussian.

The important thing to notice is that

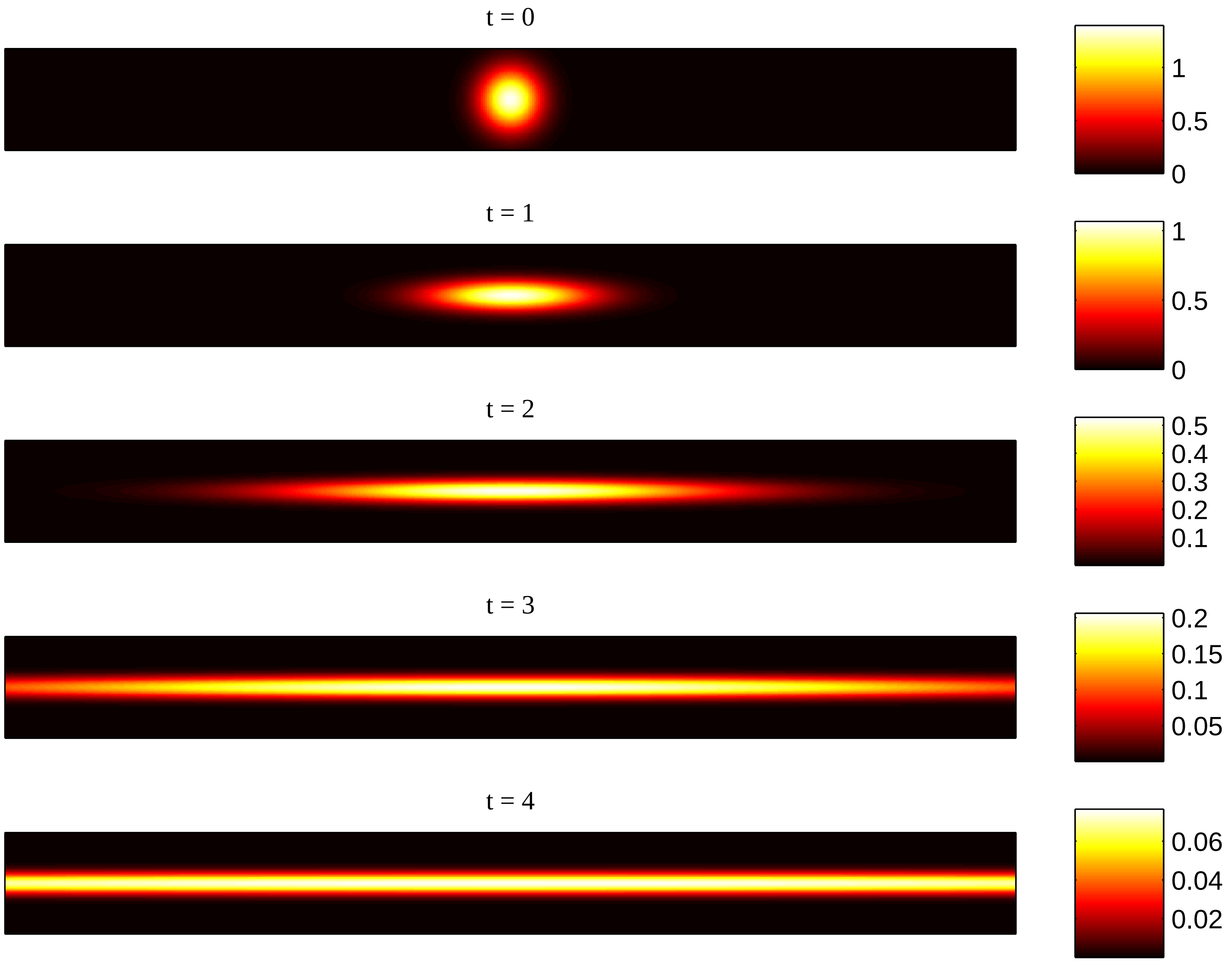
$$\theta(\boldsymbol{x}, t) \sim \mathrm{e}^{-\lambda t}\,.$$

This is a much more reasonable estimate for the decay of concentration than superexponential! The concentration thus decays exponentially at a rate given by the stretching rate of the flow.

The asymptotic decay rate tends to be independent of diffusivity.

But note that a nonzero diffusivity is crucial in obtaining this result. The only direct effect of the diffusivity is to lengthen the wait before exponential decay sets in. But this is only logarithmic in the diffusivity.

Numerical Example



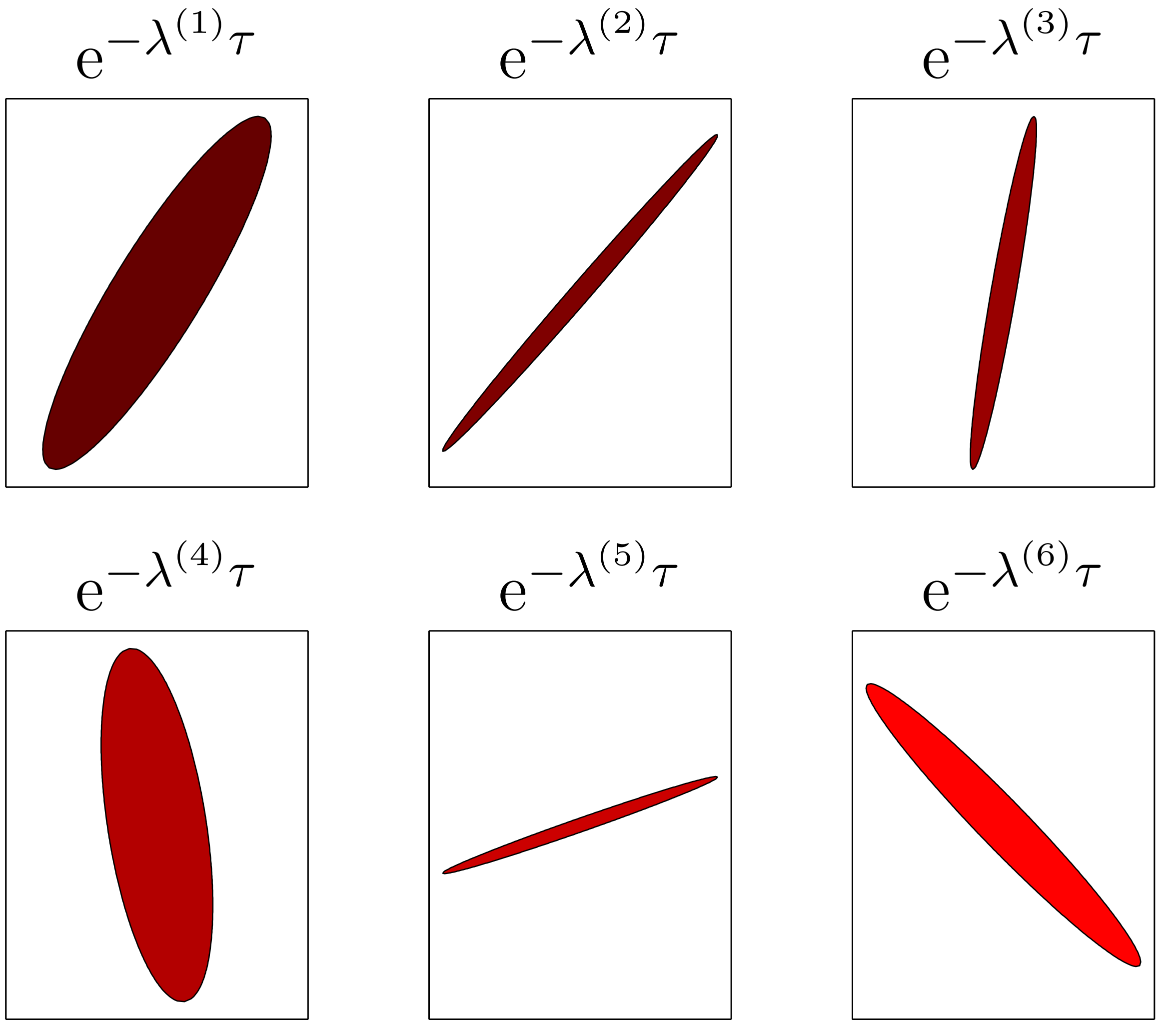
Blobs, Part II:

Random Strain

A Single Blob

- We have thus far analysed the deformation of a patch of concentration field (a 'blob') in a linear velocity field.
- We will now inch slightly closer to the real world by giving a random time dependence to our velocity field.
- As before, consider a single blob in a two-dimensional constant-stretching velocity field, but assume the orientation and stretching rate λ of the flow change randomly every time τ.
- We assume that the time τ is much larger than a typical stretching rate $\lambda^{(i)}$ at the ith period, so that there is sufficient time for the blob to be deformed into its asymptotic form.
- The results presented are the culmination of a flurry of activity in the late 90's [Antonsen, Jr. et al., 1996, Balkovsky and Fouxon, 1999, Son, 1999, Falkovich et al., 2001].

A Single Blob



The amplitude of the concentration field decays by $\exp(-\lambda^{(i)}\tau)$ at each period.

Decay of Concentration

The concentration field after n periods will thus be proportional to the product of decay factors,

$$\begin{aligned}\theta &\sim \mathrm{e}^{-\lambda^{(1)}\tau}\mathrm{e}^{-\lambda^{(2)}\tau}\cdots\mathrm{e}^{-\lambda^{(n)}\tau},\\ &= \mathrm{e}^{-(\lambda^{(1)}+\lambda^{(2)}+\cdots+\lambda^{(n)})\,\tau}.\end{aligned}$$

We may rewrite this as

$$\theta \sim \mathrm{e}^{-\Lambda_n t},$$

where $t = n\tau$, and

$$\Lambda_n := \frac{1}{n}\sum_{i=1}^{n}\lambda^{(i)}$$

is the 'running' mean value of the stretching rate at the nth period.

Asymptotic Behaviour

- As we let n become large, how de we expect the concentration field to decay?
- We might expect that it would decay at the mean value $\bar{\lambda}$ of the stretching rates $\lambda^{(i)}$.
- This is not the case: the running mean Λ_n does not converge to the mean $\bar{\lambda}$.
- Rather, by the central limit theorem its expected value is $\bar{\lambda}$, but its fluctuations around that value are proportional to $1/\sqrt{t}$. These fluctuations have an impact on the decay rate of θ.

Average over Realisations

The set of variables $\lambda^{(i)}$ is known as a realisation.

Now let us imagine performing our blob experiment several times, and averaging the resulting concentration fields: this is known as an ensemble average over realisations.

Ensemble-averaging smooths out fluctuations present in each given realisation.

We may then replace the running mean Λ_n by a sample-space variable Λ, together with its probability distribution $P(\Lambda, t)$. The mean (expected value) $\overline{\theta^\alpha}$ of the αth power of the concentration field is then proportional to

$$\overline{\theta^\alpha} \sim \int_0^\infty \mathrm{e}^{-\alpha\Lambda t}\, P(\Lambda, t)\, \mathrm{d}\Lambda\,.$$

The PDF of Λ

The form of the probability distribution function (PDF) $P(\Lambda, t)$ is given by the central limit theorem:

$$P(\Lambda, t) \simeq G\big(\Lambda - \bar{\Lambda}; \sqrt{\nu/t}\big),$$

that is, a Gaussian distribution with standard deviation $\sqrt{\nu/t}$.

Actually, the central limit theorem only applies to values of Λ that do not deviate too much from the mean. A more general form of the PDF of Λ comes from large deviation theory,

$$P(\Lambda, t) \simeq \sqrt{\frac{t\, S''(0)}{2\pi}}\, \mathrm{e}^{-tS(\Lambda-\bar{\Lambda})}.$$

The function $S(x)$ is known as the rate function, the entropy function, or the Cramér function.

Large Deviation Theory

S is a time-independent convex function with a minimum value of 0 at 0:

$$S(0) = S'(0) = 0.$$

If Λ is near the mean, we have

$$S(\Lambda - \bar{\Lambda}) \simeq \tfrac{1}{2}\, S''(0)(\Lambda - \bar{\Lambda})^2,$$

which recovers the Gaussian result with $\nu = 1/S''(0)$.

The Gaussian or Large Deviation asymptotic forms are only valid for large t.

(Which in our case means $t \gg \tau$, or equivalently $n \gg 1$.)

The Decay Rate of $\overline{\theta^\alpha}$

We can now evaluate our integral with the asymptotic PDF,

$$\overline{\theta^\alpha} \sim \int_0^\infty \mathrm{e}^{-\alpha\Lambda t}\,\mathrm{e}^{-tS(\Lambda-\bar{\Lambda})}\,\mathrm{d}\Lambda \sim \int_0^\infty \mathrm{e}^{-tH(\Lambda)}\,\mathrm{d}\Lambda \sim \mathrm{e}^{-\gamma_\alpha t}\,,$$

where we have omitted the nonexponential prefactors, and defined

$$H(\Lambda) := \alpha\Lambda + S(\Lambda - \bar{\Lambda}).$$

Since t is large, the integral is dominated by the minimum value of $H(\Lambda)$: we can use the saddle-point approximation. The decay rate is then given by

$$\gamma_\alpha = H(\Lambda_{\mathrm{sp}}), \quad \text{with} \quad H'(\Lambda_{\mathrm{sp}}) = 0,$$

where Λ_{sp} is the saddle-point.

The Decay Rate of $\overline{\theta^\alpha}$ (cont'd)

There's a caveat to this: for α large enough the saddle point $\Lambda_{\rm sp}$ is negative. This is not possible: the stretching rates are defined to be nonnegative.

The best we can do is to choose $\Lambda_{\rm sp} = 0$: the ensemble average is dominated by realisations with no stretching.

In that case,

$$\gamma_\alpha = H(0) = S(-\bar{\Lambda}),$$

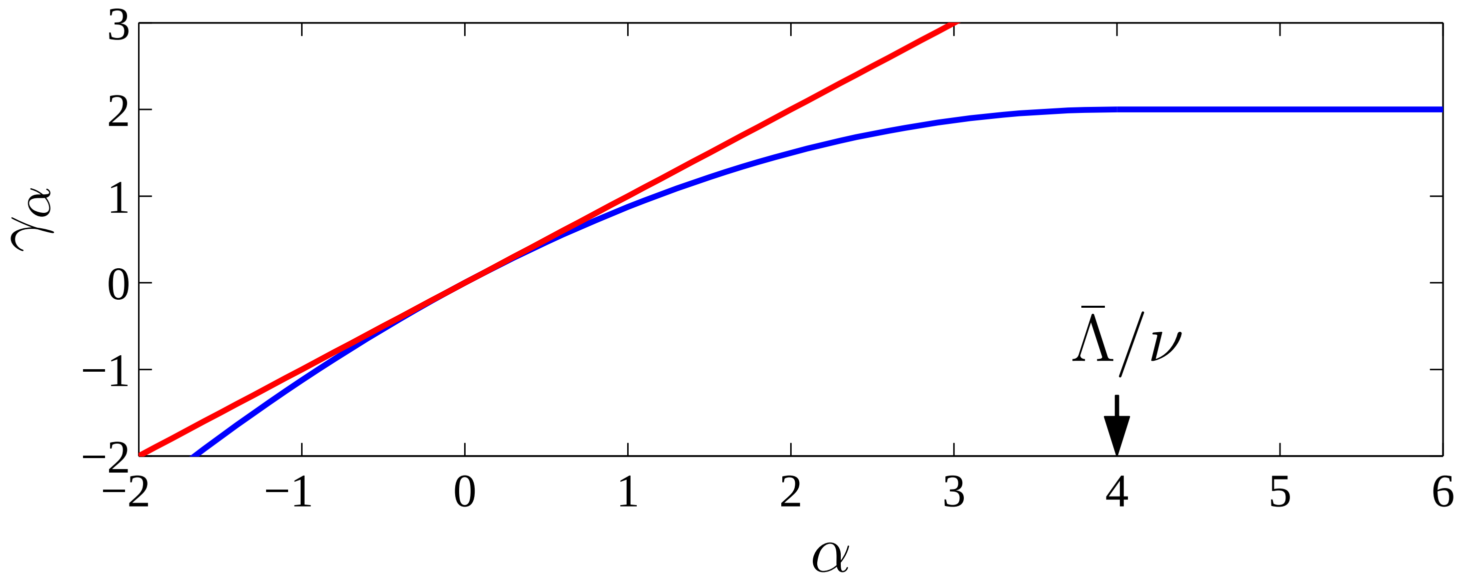
independent of α!

All this is best illustrated by quoting the Gaussian result,

$$\boxed{\gamma_\alpha = \begin{cases} \alpha\left(\bar{\Lambda} - \frac{1}{2}\,\alpha\nu\right), & \alpha < \bar{\Lambda}/\nu; \\ \bar{\Lambda}^2/2\nu, & \alpha \geq \bar{\Lambda}/\nu. \end{cases}}$$

Decay Rate of $\overline{\theta^\alpha}$ for Gaussian PDF

Decay rate γ_α for the moments of concentration $\overline{\theta^\alpha}$ of a blob (blue) in a Gaussian random stretching flow:



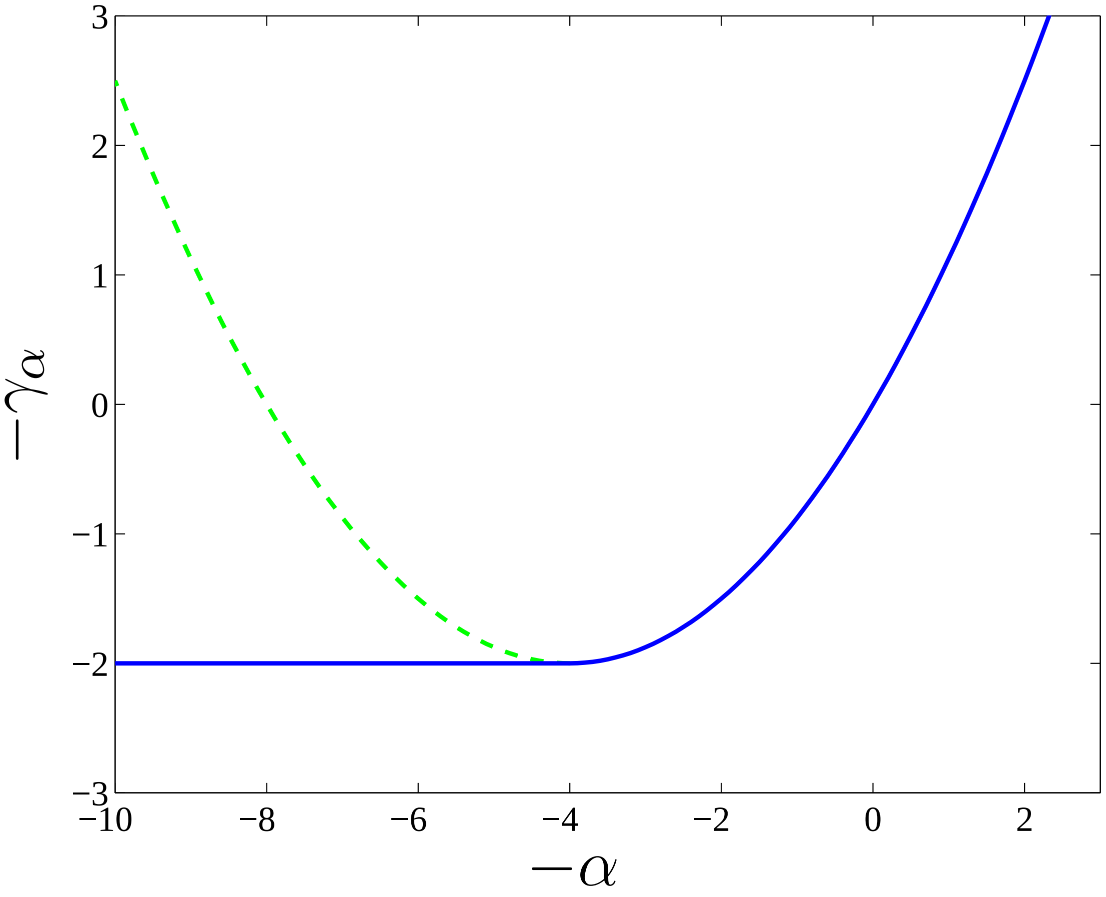
The red line is for a fixed, nonrandom flow.

The curve and its first derivative are always continuous.

Notice that the blue curve (for a random flow) lies below the red curve (for a nonrandom flow).

A More Familiar Form?

Plot this upside down and reversed:



Plateau shows the difference between line- and blob-stretching.

Fluctuations about the Mean

This is a general result: if $f(x)$ is a convex function and x a random variable, Jensen's inequality says that

$$\overline{f(x)} \geq f(\overline{x}).$$

Now, $\mathrm{e}^{-\alpha t \Lambda}$ is a convex funtion of Λ, so we have

$$\overline{\mathrm{e}^{-\alpha t \Lambda}} \geq \mathrm{e}^{-\alpha t \bar{\Lambda}},$$

which implies that the decay rate satisfies

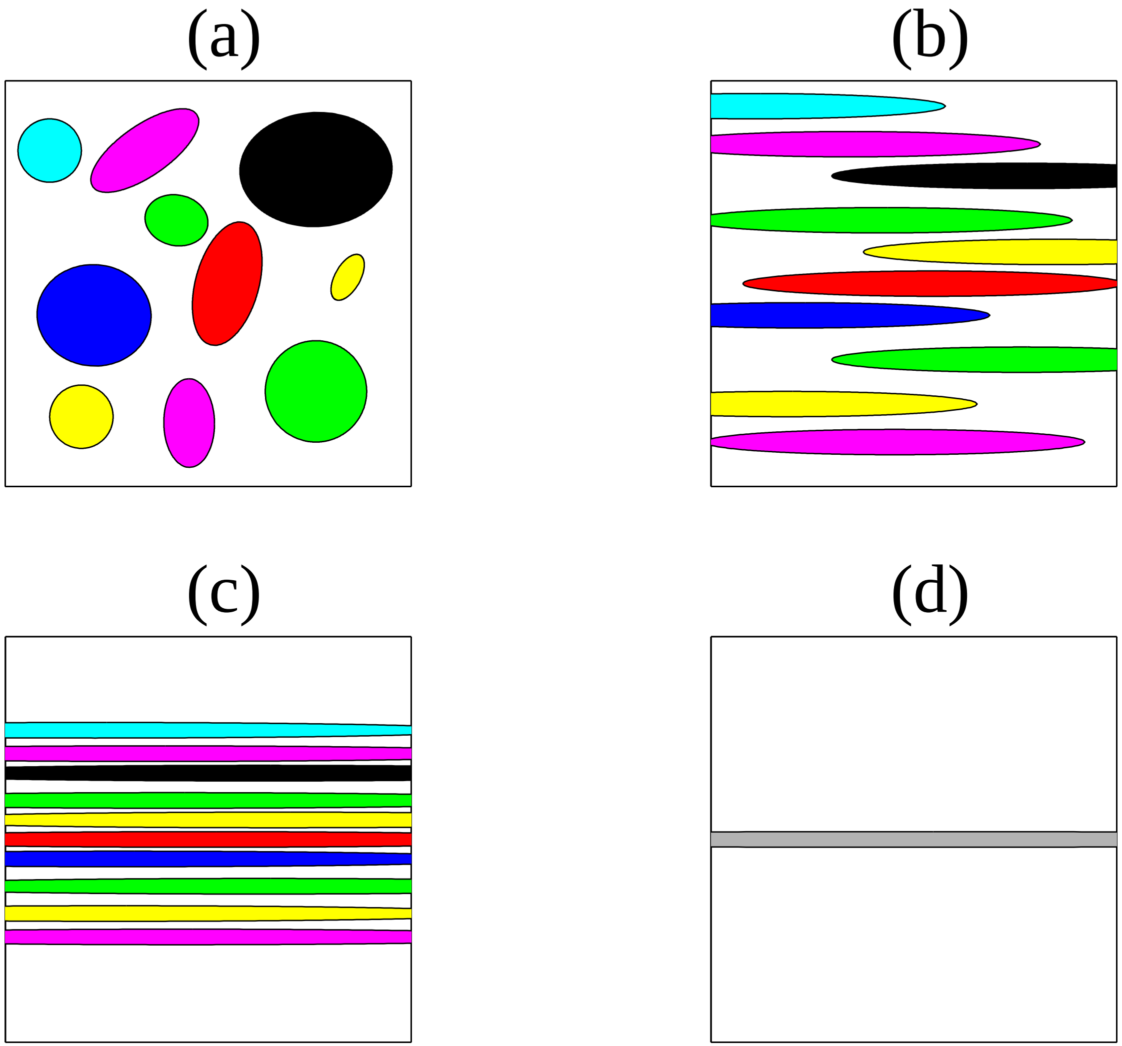
$$\boxed{\gamma_\alpha \leq \alpha \, \bar{\Lambda} \, .}$$

Thus, fluctuations in Λ inevitably lead to a slower decay rate γ_α.

Many Blobs

- Consider now a large number of blobs, homogeneously and isotropically distributed, with random concentrations. We assume that the mean concentration over all the blobs is zero.
- If we now apply a uniform stretching flow, the blobs are all stretched horizontally and contracted in the vertical direction.
- They are squished together in the vertical direction until diffusion becomes important.
- The effect of diffusion is to homogenise the concentration field until it reaches a value which is the average of the concentration of the individual blobs.
- This is depicted by the long gray blob in which will itself keep contracting until it reaches the diffusive length ℓ.

Many Blobs



Overlap of Blobs

The expected value of the concentration at a point $\boldsymbol{x}$ on the gray filament consisting of N overlapping blobs is zero.

By the central limit theorem, the fluctuations in θ are

$$\langle \theta^2(\boldsymbol{x}, t) \rangle_{\text{blobs}} \sim N \mathrm{e}^{-2\Lambda t} \left(\frac{1}{N} \sum_{i}^{N} {\theta_0^{(i)}}^2 \right)$$

where $\theta_0^{(i)}$ is the initial concentration of the ith blob, and $\langle \cdot \rangle_{\text{blobs}}$ denotes a sum over the overlapping blobs at point $\boldsymbol{x}$.

But the number of overlapping blobs N is proportional to $\mathrm{e}^{\Lambda t}$: as time increases more and more blobs converge and interact diffusively.

Overlap of Blobs: Decay of Moments

Overall, then

$$\left\langle\theta^2(\boldsymbol{x},t)\right\rangle_{\text{blobs}}^{1/2} \sim \mathrm{e}^{-\Lambda t/2}\,.$$

Compare this to $\overline{\theta^2} \sim \mathrm{e}^{-\Lambda t}$ for the single-blob case: the overlap between blobs has led to an extra square root.

Thus, the ensemble averages $\overline{\langle\theta^2(\boldsymbol{x},t)\rangle_{\text{blobs}}^{\alpha}}$ for the overlapping blobs are computed exactly as for the single blob case.

Because of the assumption of homogeneity, the point-average is the same as the average over the whole domain, and we have

$$C_{2\alpha} = \left\langle\theta^2\right\rangle^{\alpha} \sim \mathrm{e}^{-\gamma_\alpha t}\,,$$

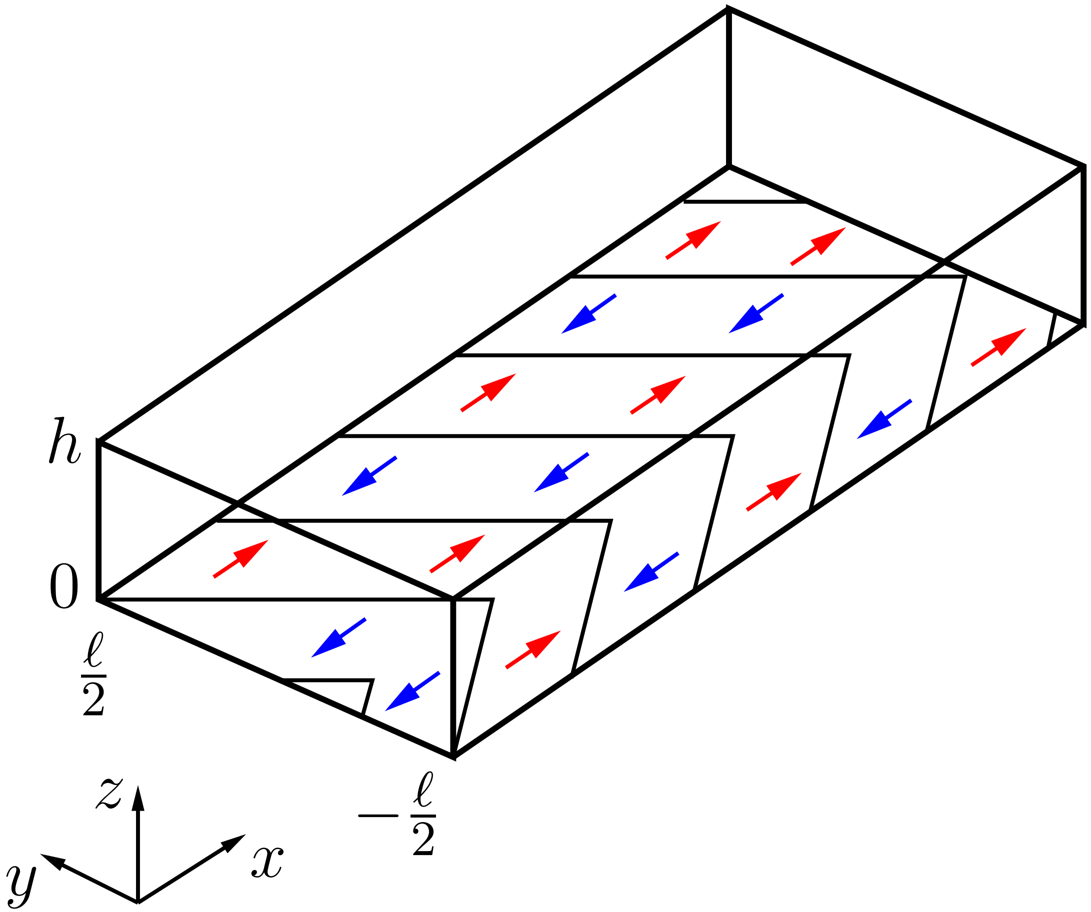
with γ_α the same as before.

“Reality”

Practical Considerations

- The "many blobs" picture has a chance of depicting reality.
- Two important questions:
  - Where does the ensemble average come from?
  - What gives the stretching rates?
- The many initial random blobs provide the ensemble: each blob is a "realisation".
- The mean stretchings Λ are given by the finite-time Lyapunov exponents.
- These give the mean stretching experienced by a fluid element, and account for reorientations of the blobs.

Example: Microchannel Mixer

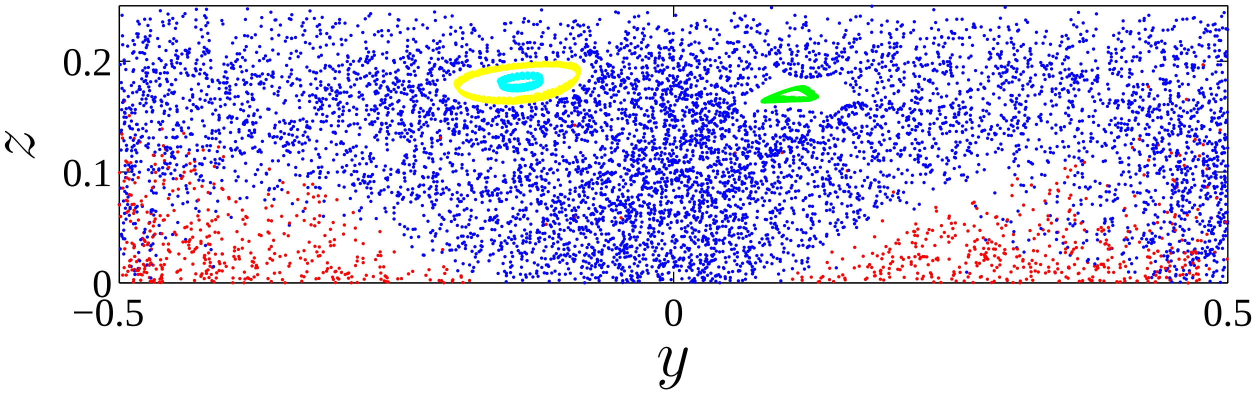


- Periodic electro-osmotic potential at the bottom (moving wall).
- Width $\sim 100\,\mu$m, height $\sim$ 10–50 μm.
- A typical mean fluid velocity is 10^2–10^3 $\mu\text{m}/\text{s}$.

Use Stokes flow and lubrication approximations to derive analytical solutions (with M. A. Ewart).

Poincaré Section

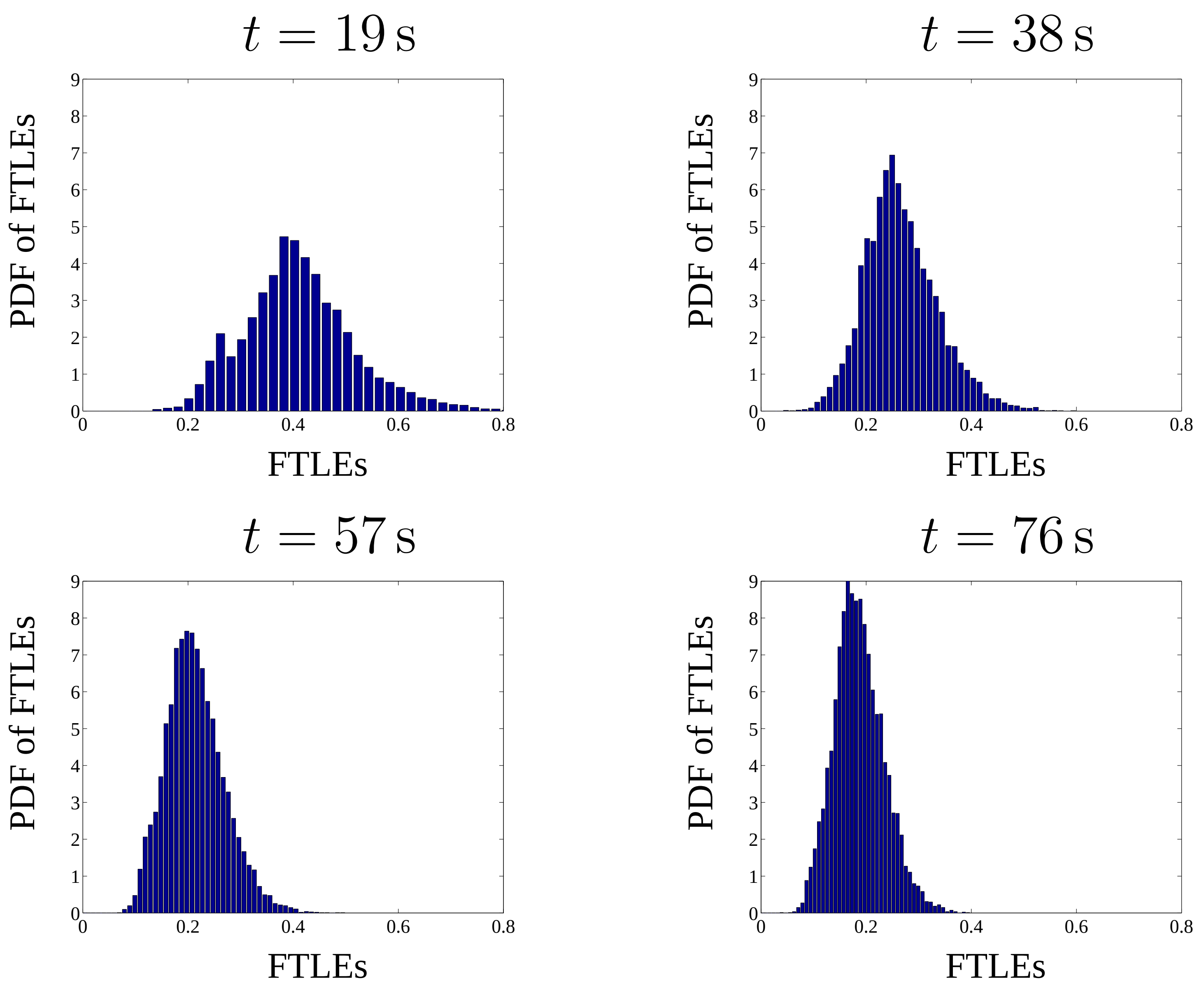
Section through vertical plane at $x = 0$



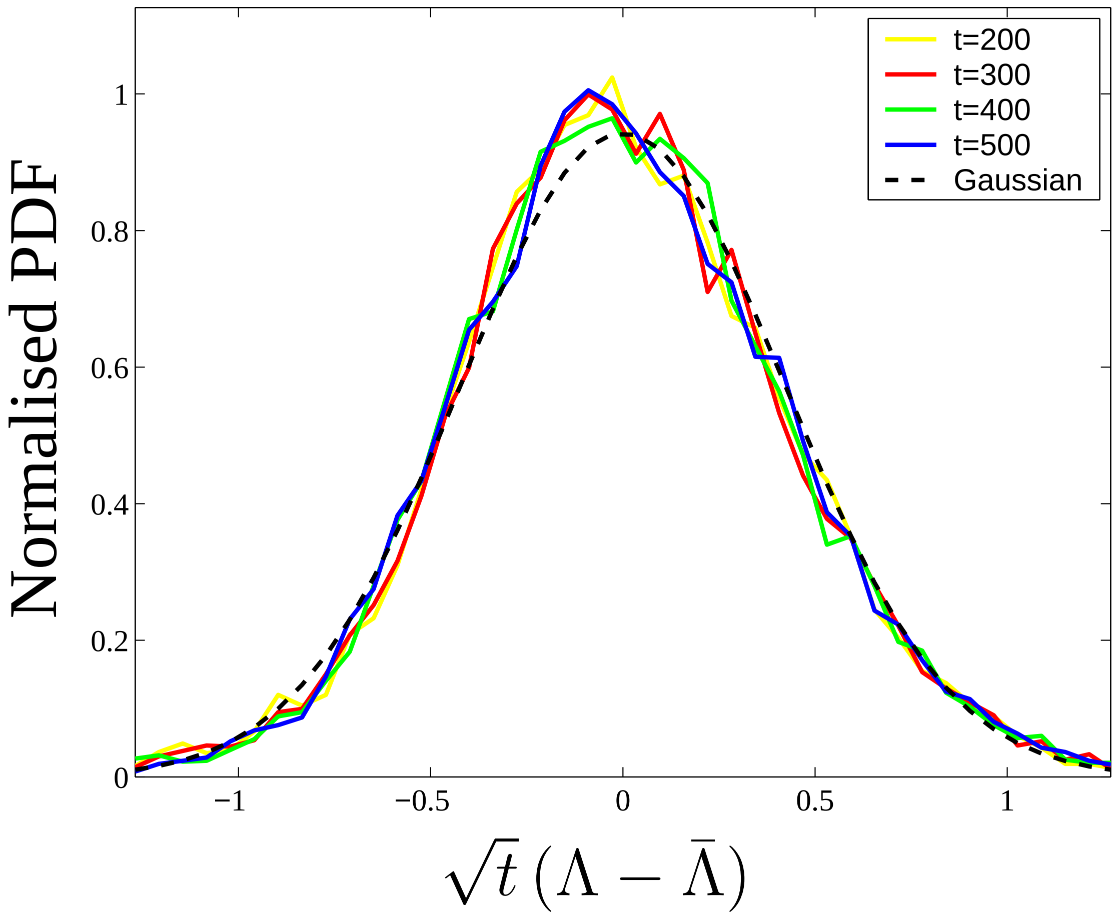
The red and blue dots represent the same trajectory periodically puncturing two vertical planes many times over (blue if in the same direction as the flow, red otherwise).

The green and yellow dots show two trajectories in regular, nonmixing regions.

PDF of Finite-time Lyapunov Exponents (FTLEs)



Rescaled Distribution of FTLEs



The mean Lyapunov exponent is $\bar{\Lambda} \simeq 0.116\,\mathrm{s}^{-1}$, and the standard deviation of the Gaussian is $\sqrt{\nu/t}$, with $\nu \simeq 0.168\,\mathrm{s}^{-1}$.

Mixing Time

From the "many blobs" theory:

$$C_2 = \langle \theta^2 \rangle \sim \mathrm{e}^{-\gamma_1 t}$$

$$\begin{aligned} \gamma_1 &= \bar{\Lambda}^2/2\nu \quad \text{since } \nu > \bar{\Lambda} \\ &= (0.116)^2/(2 \cdot 0.168) \simeq 0.040\,\mathrm{s}^{-1}. \end{aligned}$$

- The "mixing time" is $\gamma_1^{-1} \simeq 25$ seconds.
- Fluctuations triple the mixing time compared to $\bar{\Lambda}^{-1}$!
- We don't really know if this is right…
- Not spectacular improvement over diffusion time for, say, DNA molecules, but still pretty good (factor of four).

Summary: Local Theory

- The decay rate for the passive scalar depends on the distribution of finite-time Lyapunov exponents.
- The fluctuations in the Lyapunov exponents tend to work against good mixing.
- There may be regions of poor mixing (regular regions).
- This local regime is not always valid: must also understand the role of strange eigenfunctions.
- The breakdown is associated with blobs feeling higher-order moments (curvature) of the velocity field and beginning to bend and fold.
- But range of validity is poorly understood (and controversial).
- Comparison to direct solution is needed (but difficult).

Limitations of the Local Theory

- Agreement of decay rate with Cramér function prediction is uncertain.
- The Cramér function is extremely difficult to obtain accurately, even in two dimensions.
- There is evidence that moments don't behave as predicted for longer times. They show a linear increase with α. This is consistent with $\theta \sim \mathrm{e}^{-\gamma t}$ everywhere [Fereday and Haynes, 2003].
- Boundary conditions: Results often change dramatically between periodic vs no-flux [Gilbert, 2004].
- Some systems have a decay rate that is completely independent of stretching [Fereday et al., 2002, Wonhas and Vassilicos, 2002, Thiffeault and Childress, 2003, Thiffeault, 2004].

Global Theory

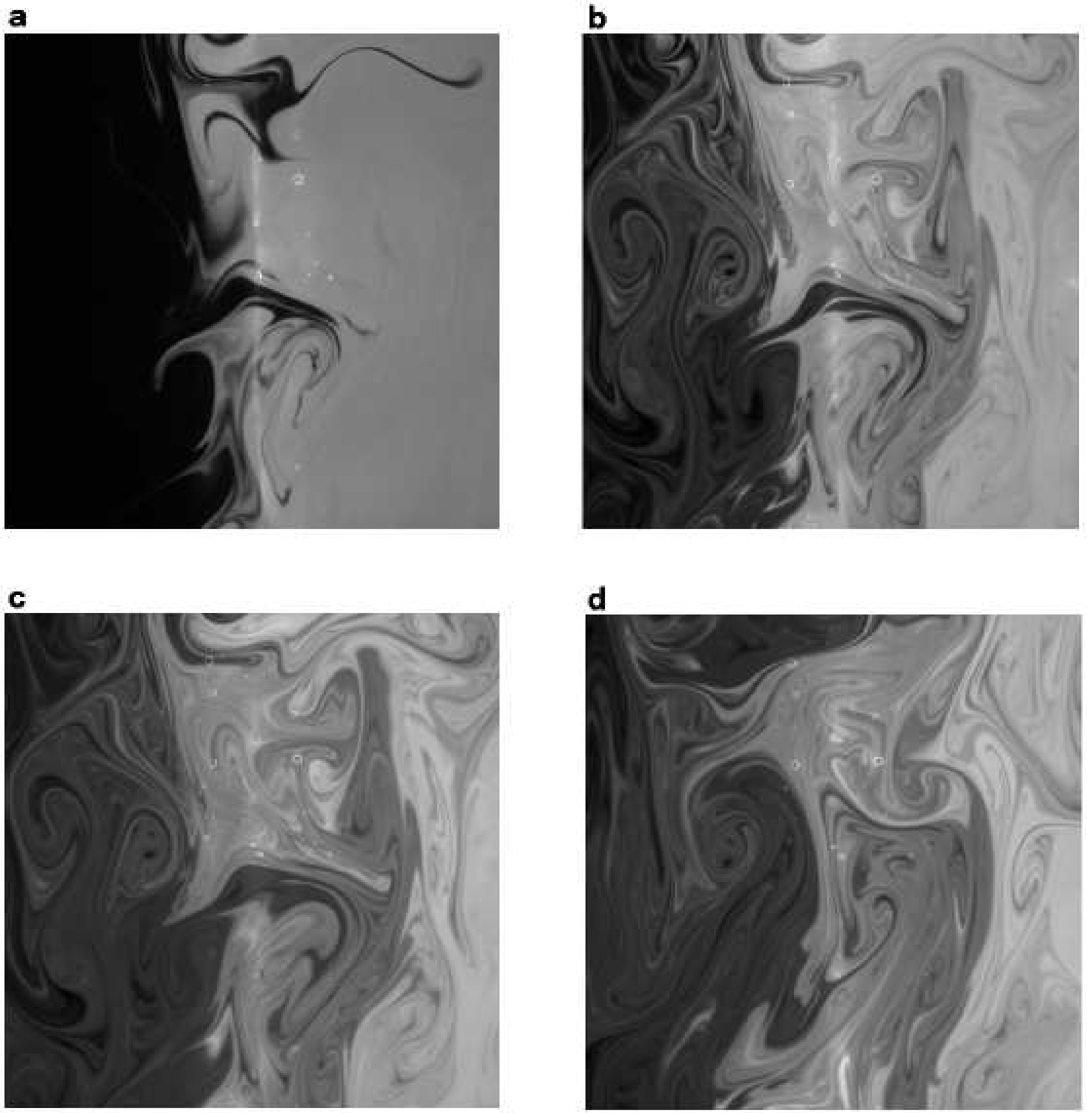
Let $\theta \sim \mathrm{e}^{\gamma t}$ and rewrite AD equation as

$$(-\boldsymbol{v} \cdot \nabla + \kappa \nabla^2)\theta = \gamma\,\theta\,.$$

This is a linear eigenvalue problem for the AD operator.

- Difficult! Global problem. Boundary conditions matter.
- All eigenvalues have negative real part.
- But one (or several) eigenvalues must be largest.
- This eigenfunctions will dominate at long times.
- Diffusion is crucial in regularising at small scales (arrest cascade, which allows existence of eigenmode).
- Called "strange eigenmode" [Pierrehumbert, 1994].

Experiment of Rothstein *et al.*: Persistent Pattern



Disordered array of magnets with oscillatory current drive a thin layer of electrolytic solution.

periods $2, 20, 50, 50.5$

[Rothstein et al., 1999]

The Modified Cat Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0,1]^2$,

$$\mathcal{M}(\boldsymbol{x}) = \mathbb{M} \cdot \boldsymbol{x} + \phi(\boldsymbol{x}),$$

where

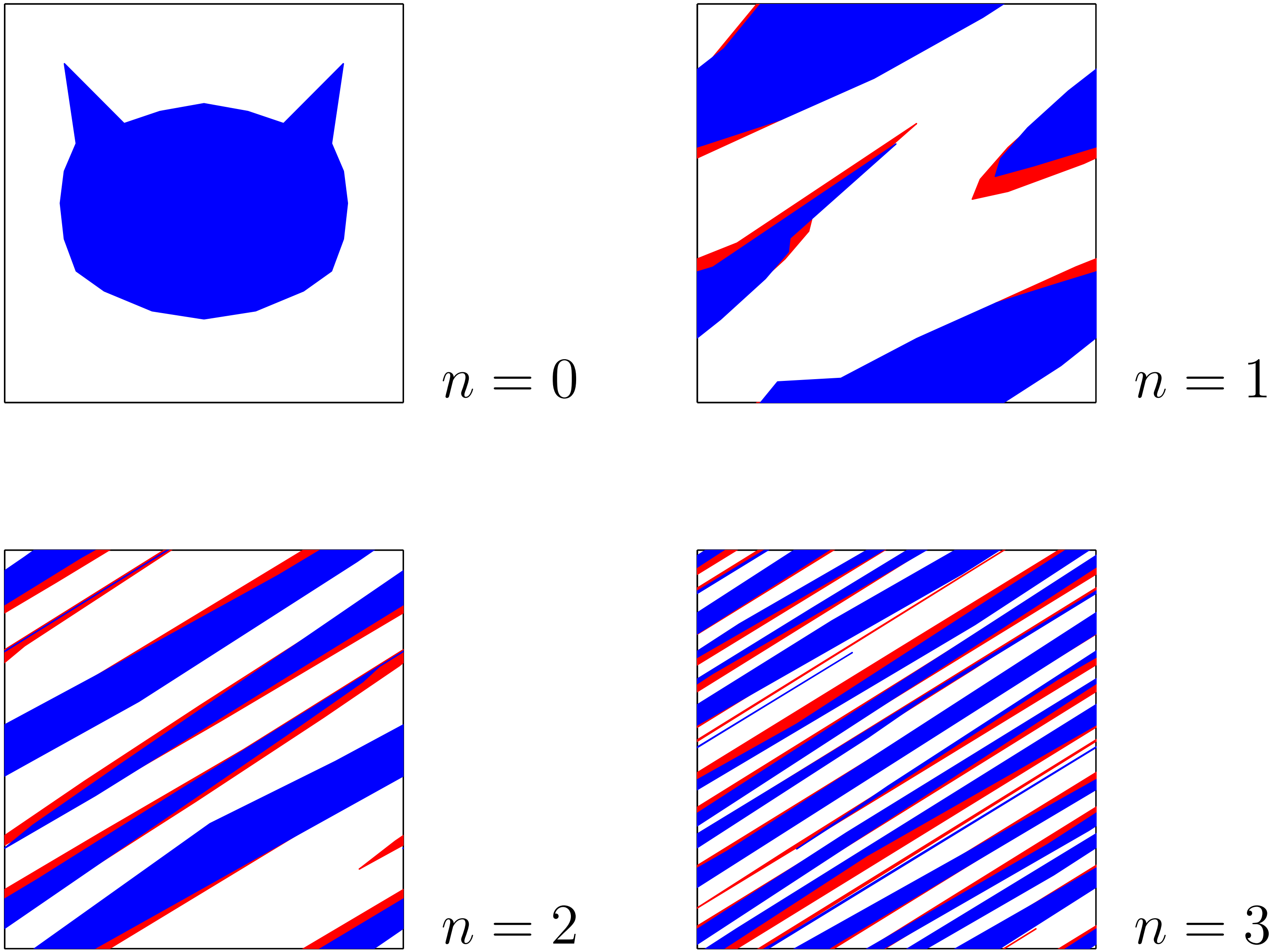
$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \phi(\boldsymbol{x}) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot \boldsymbol{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $\varepsilon = 0$ the stretching of fluid elements is homogeneous in space.
For small ε the system is still uniformly hyperbolic.

Action of the Modified Cat Map



Advection and Pulsed Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(n-1)}(\boldsymbol{x})$,

$$\theta^{(n)}(\boldsymbol{x}) = \mathcal{H}_\kappa\, \theta^{(n-1)}(\mathcal{M}^{-1}(\boldsymbol{x}))$$

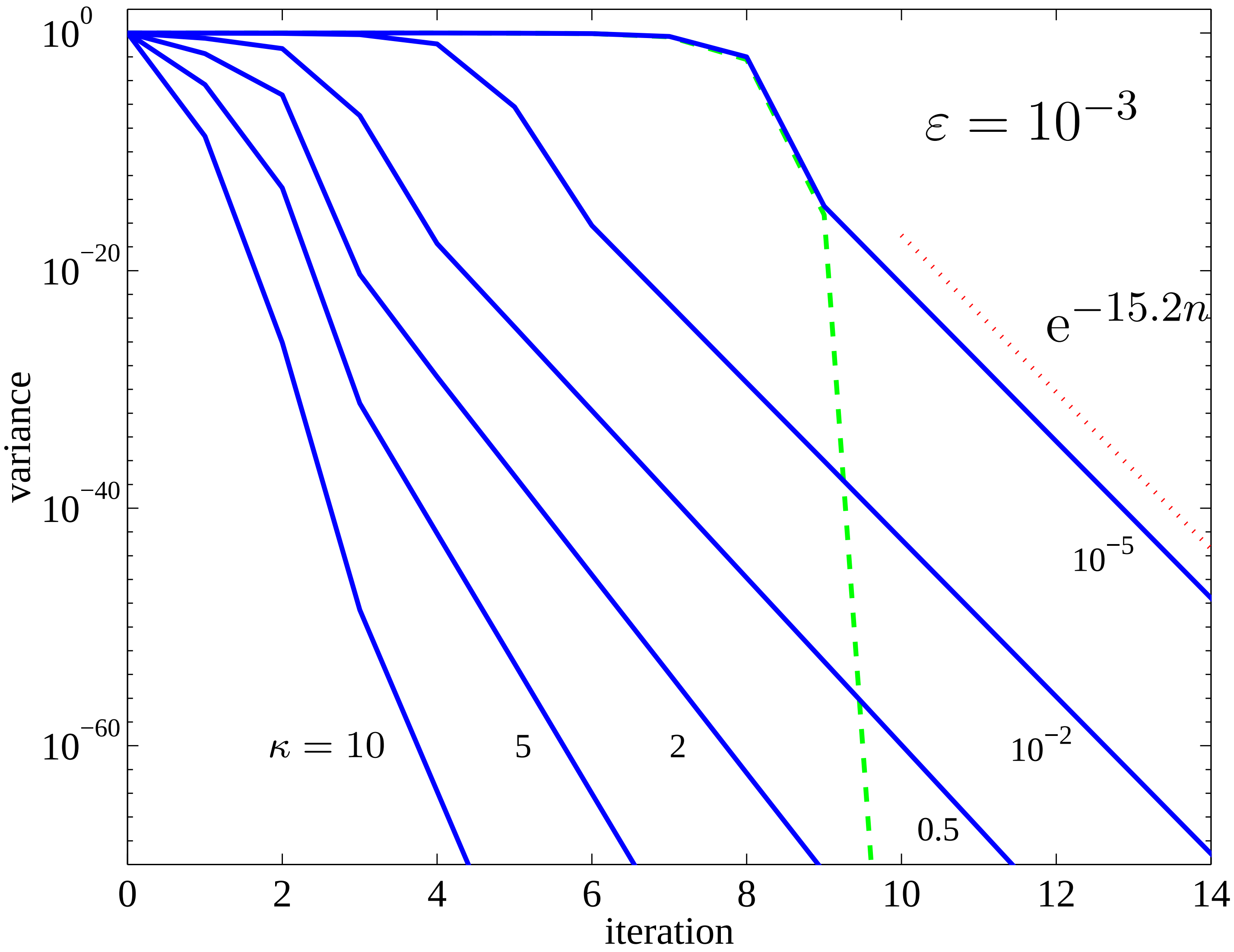
where κ is the diffusivity, with the heat operator $\mathcal{H}_\kappa$ and kernel h_κ

$$\mathcal{H}_\kappa \theta(\boldsymbol{x}) := \int_{\mathbb{T}^2} h_\kappa(\boldsymbol{x} - \boldsymbol{y})\theta(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y};$$

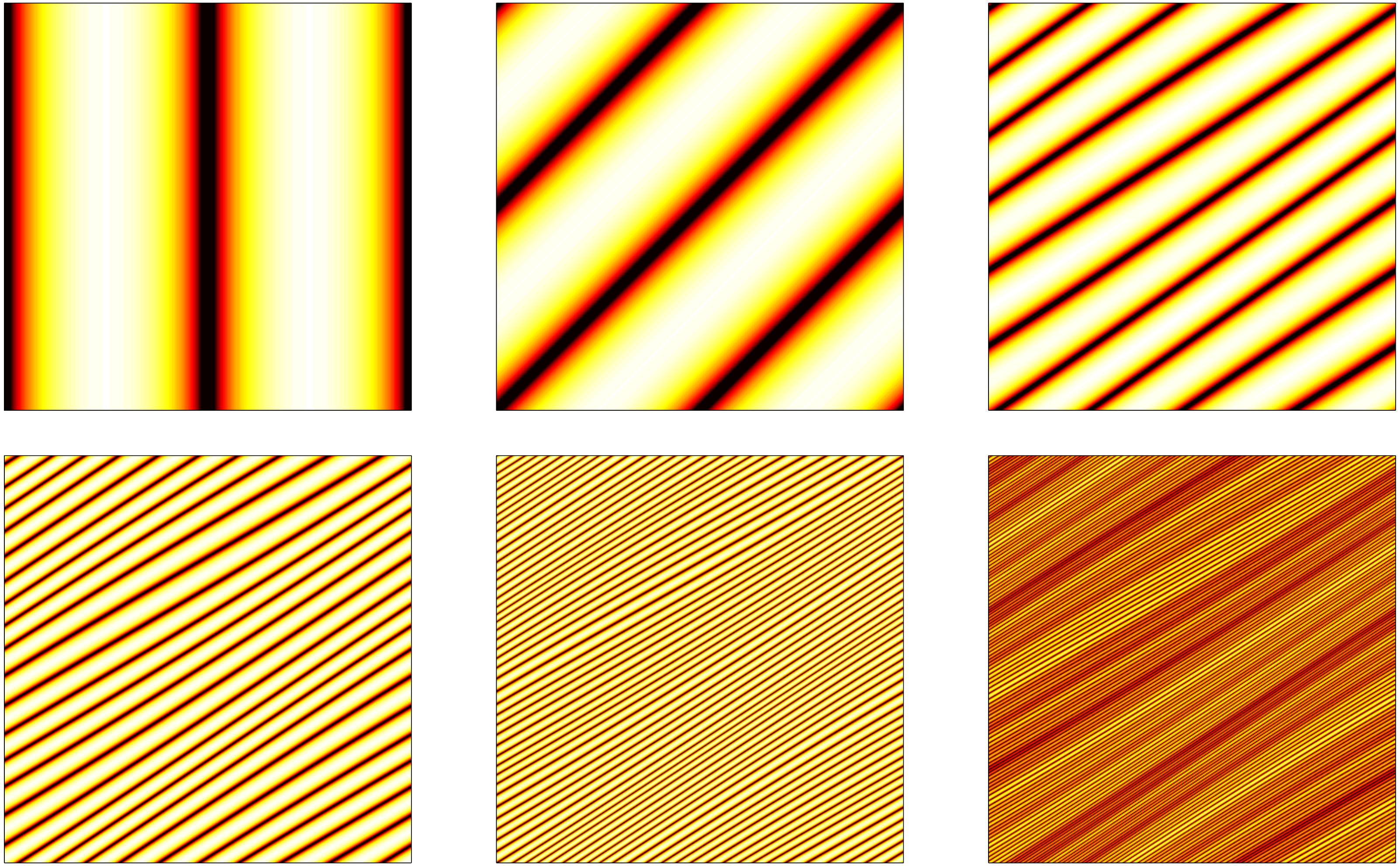
$$h_\kappa(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}^2\kappa).$$

In other words: advect instantaneously and then diffuse for one unit of time.

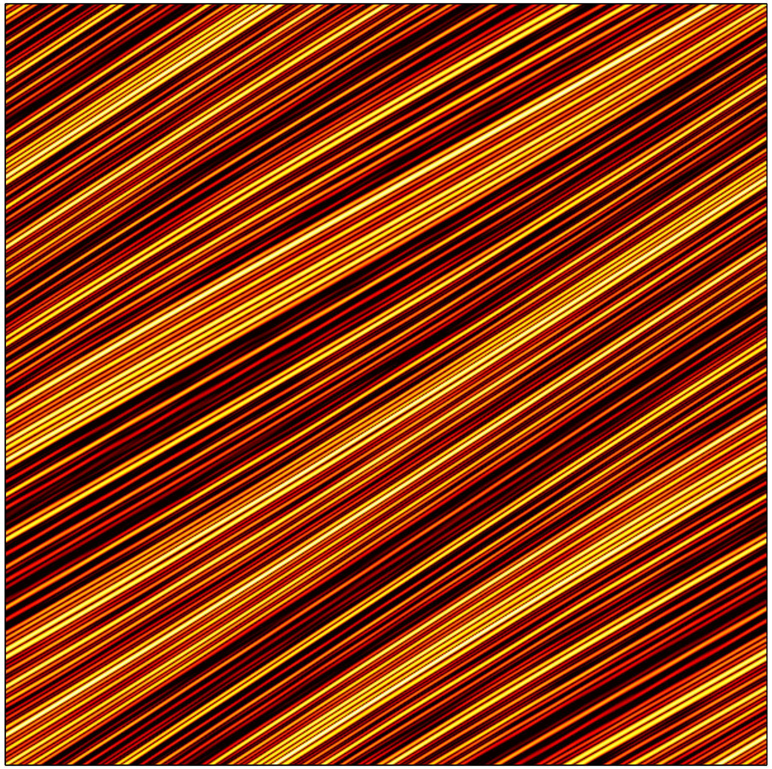
Decay of Variance



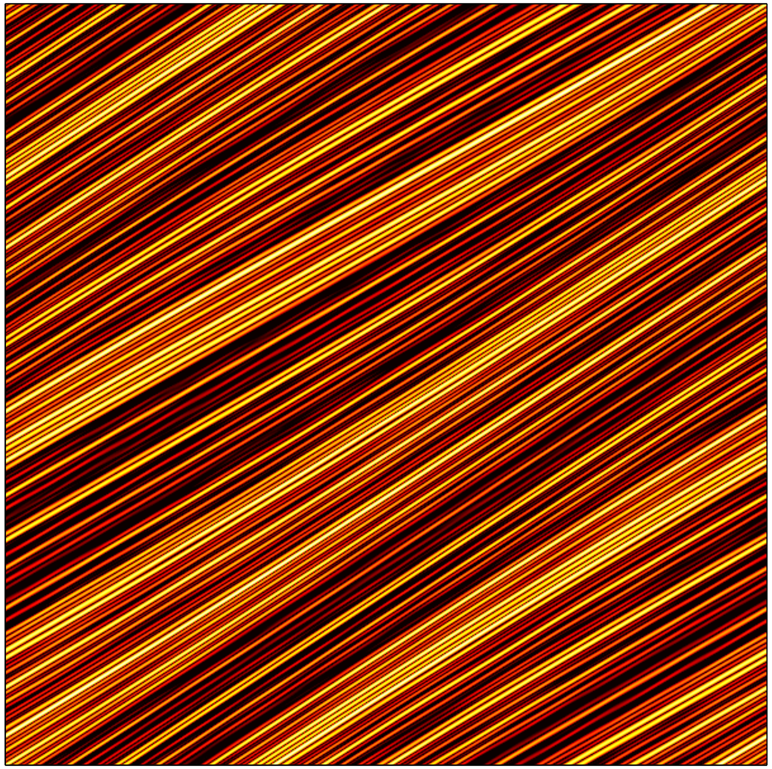
Variance: 5 iterations for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$



The Strange Eigenmode

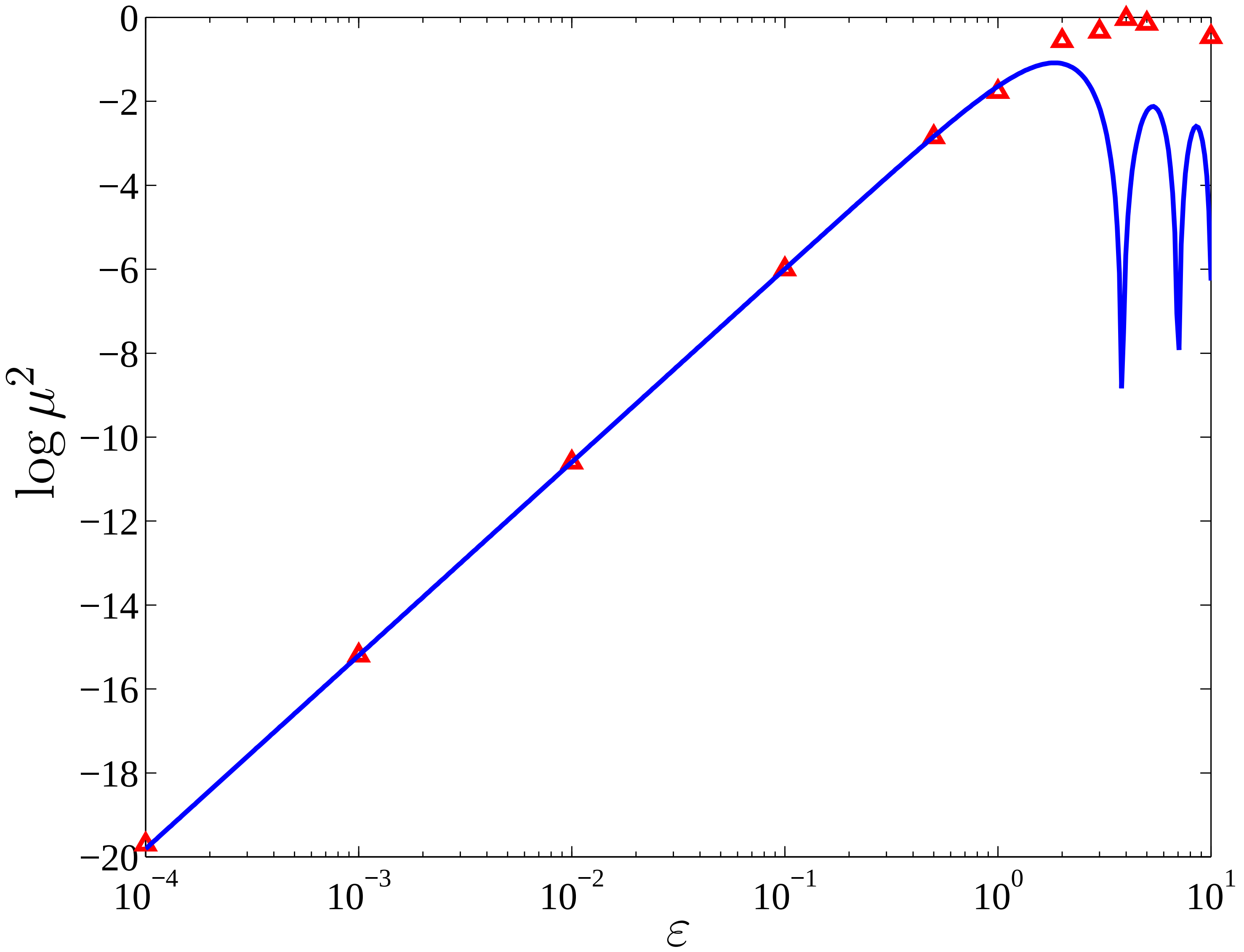


Iteration 28



Iteration 30

Decay Rate as $\kappa \to 0$



Summary: Global Theory

- Advection and diffusion conspire to create eigenfunctions. The slowest-decaying ones survive longest.
- Difficult and non-generic.
- In some cases the local and global theories are clearly different. Pathological?
- The dynamo theory literature contains many relevant results (cancellation exponents, . . .).
- The ergodic theory literature is also promising (Pollicott–Ruelle resonances), if it could be understood by mortals (and if one cares mostly about hyperbolic systems).
- The approach to zero diffusivity is cool [Hascoët and Eckhardt, 2004].
- Time-aperiodic systems a challenge (statistical eigenmodes).

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