pseudo-Anosovs with small or large dilatation

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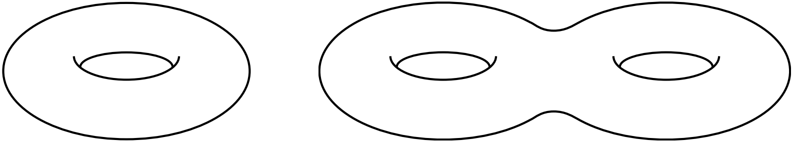
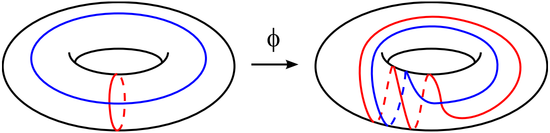
Surface homeomorphisms

homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a compact orientable surface without boundary, such as torus or 2-torus:



Can visualize by action on loops:

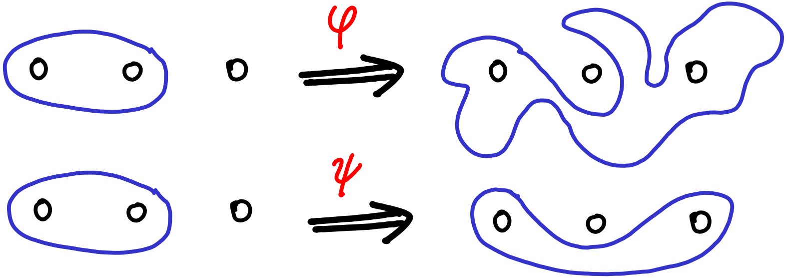


Isotopy

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Defines isotopy classes.

Again, convenient to think of isotopy in terms of loops:



(Loops will always mean essential loops.)

Thurston–Nielsen classification theorem

Theorem

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

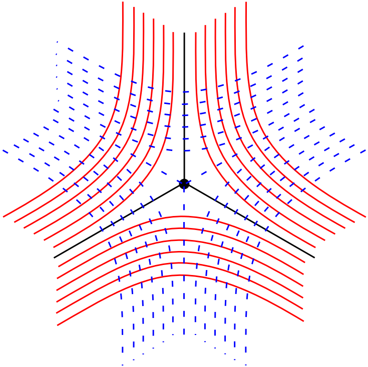
- **finite-order** *for some integer $k > 0$, $\psi^k \simeq$ identity;*
- **reducible** *ψ leaves invariant a disjoint union of essential simple closed curves, called reducing curves;*
- **pseudo-Anosov** *ψ leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\psi(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\psi(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda > 1$.*

The three categories characterize the isotopy class of φ.

pseudo-Anosov is the most interesting one.

A singular foliation

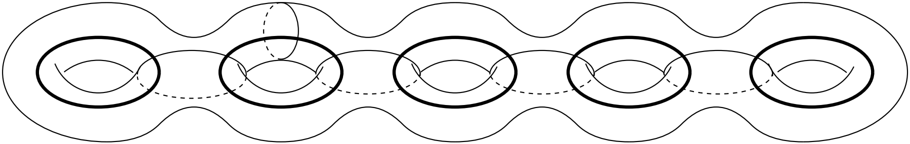
The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.



3-pronged singularity

Example: Dehn twists

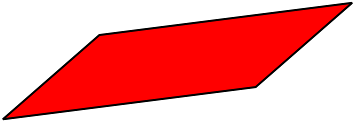
Two positive multi-twists (Dehn twists) around curves A, B (Thurston's construction).



[Leininger, C. J. (2004). *Geom. Topol.* **8**, 1301–1359]

Example: Translation surface

The diagonal 'Teichmüller flow' acts by $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ on translation surfaces.



The red polygon is a torus (parallel edges are identified by translation).

Its corners are at $(0,0)$, $(3\lambda,1)$, $(3+3\lambda,1+\lambda)$, $(3,\lambda)$.

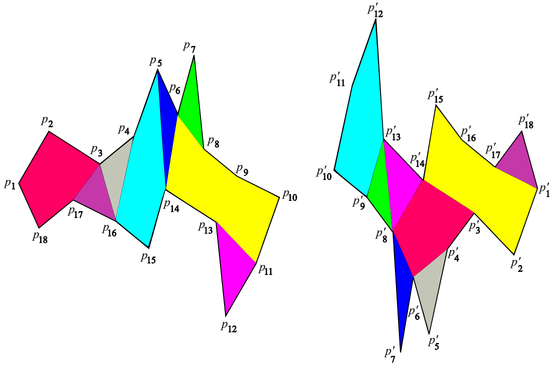
The pink polygon is the image.

For $\lambda =$ (Golden ratio)^2, we get an isometry. This corresponds to a pseudo-Anosov homeomorphism.

[See Mathematica applet.]

Example: Translation surface (2)

Map all the points on the left by $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, with $\lambda^8 + \lambda^5 - \lambda^4 + \lambda^3 + 1 = 0$.



The image is on the right, which has been cut up to exhibit the isometry of the two surfaces.

In this ‘flat surface’ picture, the foliations consist of straight horizontal/vertical lines.

The singularities in the foliation live at the corners. There are two, with angles 4π and 12π. Gauss–Bonnet then tells us this is a surface of genus four. [Lanneau & J-LT (2011a)]

The minimizer problem ('systole')

- On a given surface $\mathcal{S}$, which pA has the least λ?
  ('Shortest closed geodesic of Teichmüller flow')
- The minimum is known to exist (Thurston);
- Punctured discs: Known for $n = 3$ to 7 [Song *et al.* (2002); Ham & Song (2007); Lanneau & J-LT (2010, 2011a,b)];
  - Minimizer is simple for n odd [Hironaka & Kin (2006)], though not proved in general;
- Closed surfaces: known for genus 2 [Zhirov (1995); Cho & Ham (2008); Lanneau & J-LT (2011a)].

Simpler problem: Orientable minimizer

- No punctures: surface of genus g;
- If the foliation is orientable, then things are much simpler;
- Action of the pA on first homology captures dilatation λ;
- Polynomials of degree $2g$;
- Procedure:
  - We have a guess for the minimizer;
  - Find all integer-coefficient, reciprocal polynomials that have largest root smaller than λ;
  - Show that they can't correspond to pAs;
  - For the smallest one that can, construct pA.
- [Lanneau, E. & J-LT (2011a). *Ann. Inst. Fourier,* **61** (1), 105–144. See also article in Dynamical Systems Magazine.]

Newton–Girard formulas

We need an efficient way to bound the number of polynomials with largest root smaller than λ. Given a reciprocal polynomial

$$P(x) = x^{2g} + a_1\, x^{2g-1} + \ldots + a_2\, x^2 + a_1\, x + 1$$

we have the Newton–Girard formulas for the traces,

$$\mathrm{Tr}(\phi_*^k) = -\sum_{m=1}^{k-1} a_m \mathrm{Tr}(\phi_*^{k-m}) - k a_k,$$

where

- ϕ is a (hypothetical) pA associated with $P(x)$;
- ϕ_* is the matrix giving the action of the pA ϕ on first homology;
- $\mathrm{Tr}(\phi_*)$ is its trace.

Bounding the traces

The trace satisfies

$$|\mathrm{Tr}(\phi_*^k)| = \left|\sum_{m=1}^{g}(\lambda_m^k + \lambda_m^{-k})\right| \leq g(r^k + r^{-k})$$

where λ_m are the roots of ϕ_*, and $r = \max_m(|\lambda_m|)$.

- Bound $\mathrm{Tr}(\phi_*^k)$ with $r < \lambda$, $k = 1, \ldots, g$;
- Use these g traces and the Newton–Girard formulas to construct candidate $P(x)$;
- Overwhelming majority have fractional coeffs $\rightarrow$ discard!
- Carefully check the remaining polynomials:
  - Is their largest root real?
  - Is it strictly greater than all the other roots?
  - Is it really less than λ?
- Largest tractable case: $g = 8$ (10^{12} polynomials).

Lefschetz's fixed point theorem

This procedure still leaves a fair number of polynomials — though not enormous (10's to 100's, even for $g = 8$.)

The next step involves using Lefschetz's fixed point theorem to eliminate more polynomials:

$$L(\phi) = 2 - \mathrm{Tr}(\phi_*) = \sum_{p \in \mathrm{Fix}(\phi)} \mathrm{Ind}(\phi, p)$$

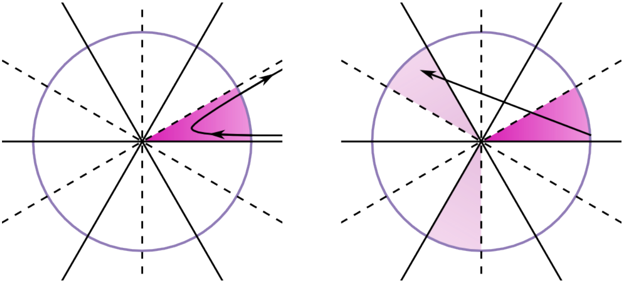
where

- $L(\phi)$ is the Lefschetz number;
- $\mathrm{Fix}(\phi)$ is set of fixed points of ϕ;
- $\mathrm{Ind}(\phi, p)$ is index of ϕ at p.

Given a polynomial we can easily compute $L(\phi^k)$ for every iterate using the Newton–Girard formulas. We don't need to know ϕ itself.

Topological index at a fixed point

The index is defined as the number of revolutions of a vector joining x to $\phi(x)$ as x travels counterclockwise around a small circle.



For this case, each sector can map to itself (left, index $1 - 6 = -5$) or to one of two other sectors (right, index $+1$).

Eliminating polynomials

Outline of procedure: for a surface of genus g,

- Use the Euler–Poincaré formula to list possible singularity data for the foliations;
- For each singularity data, compute possible contributions to the index (depending on how the singularities and their separatrices are permuted);
- Check if index is consistent with Lefschetz's theorem.

With this, we can reduce the number of polynomials to one or two!

Minimizers for orientable foliations

g	polynomial	minimizer
2	$X^4 - X^3 - X^2 - X + 1$	$\simeq 1.72208$ †
3	$X^6 - X^4 - X^3 - X^2 + 1$	$\simeq 1.40127$
4	$X^8 - X^5 - X^4 - X^3 + 1$	$\simeq 1.28064$
5	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	$\simeq 1.17628$ $*$
6	$X^{12} - X^7 - X^6 - X^5 + 1$	$\gtrsim 1.17628$
7	$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$	$\simeq 1.11548$
8	$X^{16} - X^9 - X^8 - X^7 + 1$	$\simeq 1.12876$

- † Zhirov (1995)'s result; also for nonorientable [Lanneau–T];
- $*$ Lehmer's number; realized by Leininger (2004)'s pA;
- For genus 6 we have not explicitly constructed the pA;
- Genus 6 is the first nondecreasing case.
- Genus 7 and 8: pA's found by Aaber & Dunfield (2010) and Kin & Takasawa (2010b) [$g = 7$]; Hironaka (2010) [$g = 8$].

Question

Examining the cases with even g leads to a natural question (Lanneau & J-LT, 2011a):

> *Is the minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface, for g even, with orientable invariant foliations, equal to the largest root of the polynomial $X^{2g} - X^{g+1} - X^{g} - X^{g-1} + 1$?*

This would imply that the minimum dilatation asymptotes to (Golden ratio)$^{2/g}$ for $g \gg 1$.

This appears to be the 'sparsest' reciprocal polynomial that also satisfies the Lefschetz formula.

Recent progress

- Recently, Hironaka (2014) has constructed a family of pseudo-Anosov mappings for surfaces of even genus which have the characteristic polynomial $X^{2g} - X^{g+1} - X^g - X^{g-1} + 1$.
- This proves existence of such a pA, but doesn't prove it has the minimum dilatation.
- McMullen (2013) proves the following:

Theorem (McMullen (2013))

The minimum value of the spectral radius $\rho(A)$ over all reciprocal Perron–Frobenius matrices $A \in M_{2n}(\mathbb{Z})$, $n \geq 2$, is given by the largest root of the polynomial

$$L_{2n}(t) = t^{2n} - t^{n+1} - t^n - t^{n-1} + 1.$$

Consequently $\rho(A)^n \geq (3 + \sqrt{5})/2$ for all such A.

The taffy puller

[Photo and movie by M. D. Finn.]

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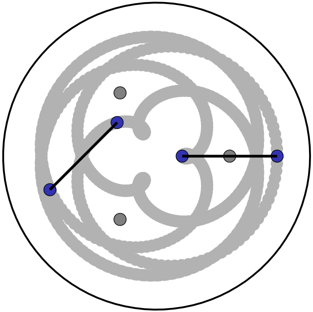
The mixograph

Experimental device for kneading bread dough:



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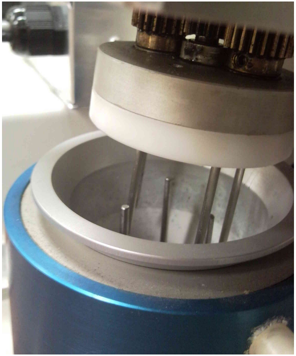

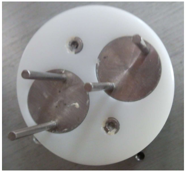

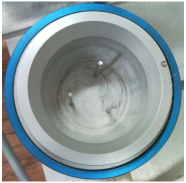

[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Experiment of Boyland, Aref & Stremler

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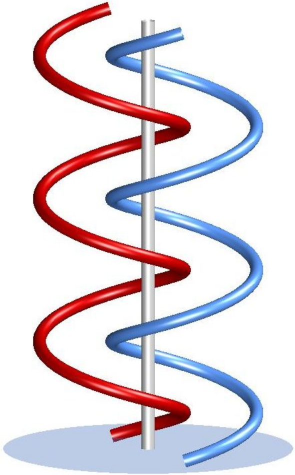
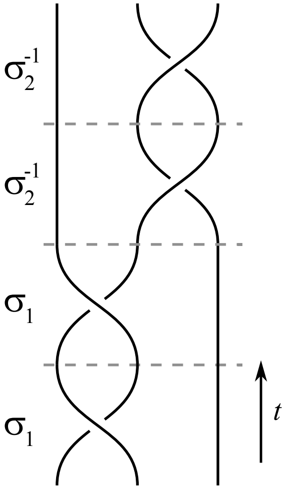
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

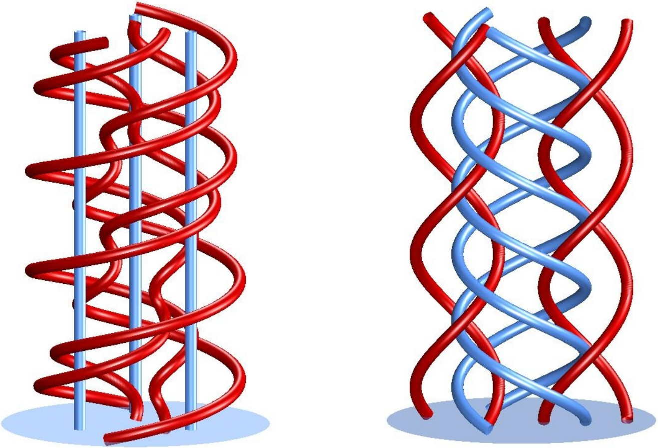
Braid description of taffy puller





The three rods of the taffy puller in a space-time diagram. Defines a braid on $n = 3$ strands, $\sigma_1^2\sigma_2^{-2}$ (three periods shown).

Braid description of mixograph



$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$

braid on B_7, the braid group on 7 strands.

Topological entropy of a braid

Burau representation for 3-braids:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$[\sigma_1^{-1}\,\sigma_2] = [\sigma_1^{-1}] \cdot [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This matrix has spectral radius $(3+\sqrt{5})/2$ (Golden Ratio^2), and hence the topological entropy is $\log[(3+\sqrt{5})/2]$.

This is the growth rate of a ‘rubber band’ caught on the rods.

This matrix trick only works for 3-braids, unfortunately.

Optimizing over generators

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1^{-1}\,\sigma_2$ has entropy $\log[(3+\sqrt{5})/2]$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2}\log[(3+\sqrt{5})/2] = \log[\text{Golden Ratio}]$.
- Assume all the generators are used (stronger: irreducible).

Optimal braid

- In B_3 and B_4, the optimal TEPG is log[Golden Ratio].
- Realized by $\sigma_1^{-1}\sigma_2$ and $\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$, respectively.
- In B_n, $n > 4$, the optimal TEPG is $<$ log[Golden Ratio].

Why? Recall Burau representation:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

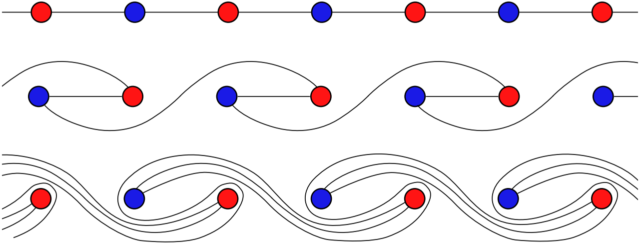
Its spectral radius provides a lower bound on entropy. However,

$$|[\sigma_1]| = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad |[\sigma_2]| = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

provides an upper bound! Need to find Joint Spectral Radius.

Periodic array of rods

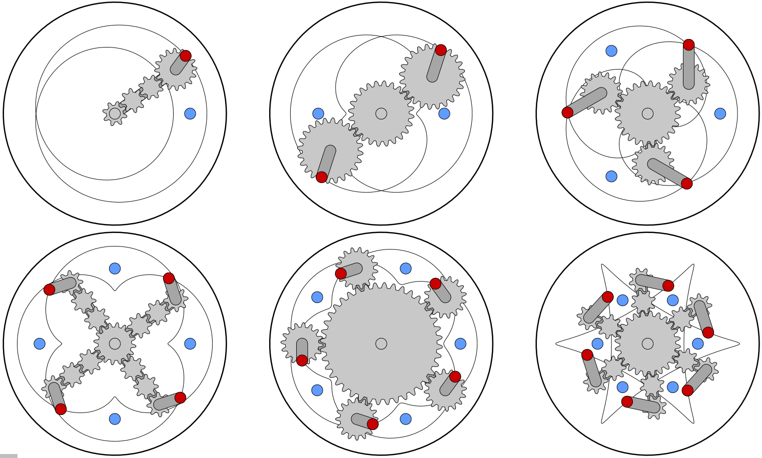
- Consider periodic lattice of rods.
- Move all the rods such that they execute the Boyland *et al.* (2000) rod motion (J-LT & Finn, 2006; Finn & J-LT, 2011).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro *et al.* (1999)).

Silver mixers

- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.



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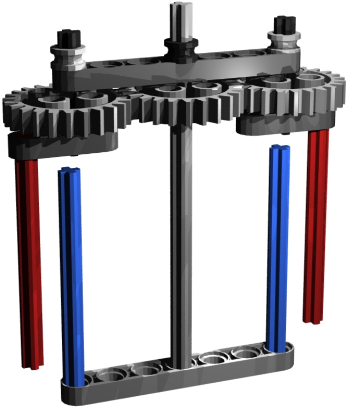
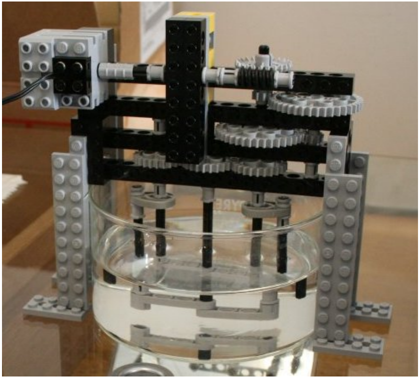
Build it!





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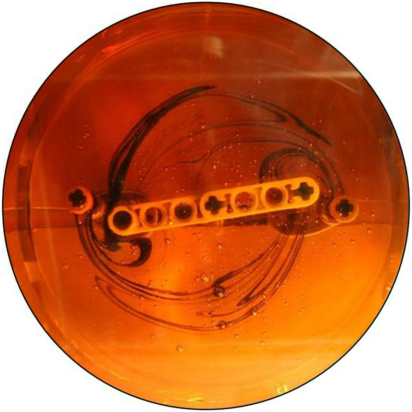
[M. D. Finn and J-LT, *SIAM Review* **53**, 723 (2011)]

Experiment: Silver mixer with four rods



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[See Finn, M. D. & J-LT (2011). *SIAM Rev.* **53** (4), 723–743 for proofs, heavily influenced by work on π_1-stirrers of Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology,* **11** (4), 2265–2296.]

Conclusions

- Having rods undergo 'braiding' motion guarantees a minimal amound of entropy (stretching of material lines).
- Can optimize to find the best rod motions, but depends on choice of 'cost function.'
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!
- Found orientable minimizer on surfaces of genus $g \leq 8$; only known nonorientable case is for genus 2.

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