Exact topological entropy for some non-hyperbolic maps

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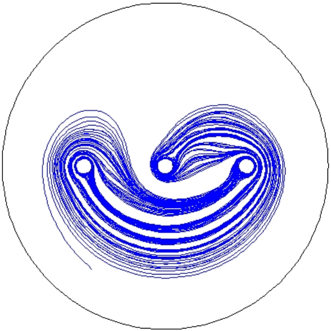
Stirring with rods

When stirring a viscous fluid with rods, a blob of dye gets ‘caught’ on the rods.



The rod motion can be connected to the isotopy class of the induced map [Boyland *et al.* (2000); Thiffeault & Finn (2006)].
[movie 1]

Describing the rod motion

Express in terms of generators of the braid group:

- σ_1 is the clockwise interchange of the first and second rods;
- σ_2 is the clockwise interchange of the second and third rods.

Any stirring protocol (rod motion) can be represented as a combination of these generators. We consider protocols of the form

$$\sigma_1^k \sigma_2^{-\ell}$$

Two types:

- counter-rotating ($k\ell > 0$);
- co-rotating ($k\ell < 0$).

The protocol on the previous slide had $k = \ell = 1$:

Action on homology

Find the growth rate of material lines — topological entropy.

Can use Burau matrix representation:

$$[\sigma_1] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$[\sigma_1^k \sigma_2^{-\ell}] = \begin{pmatrix} 1 + k\ell & k \\ \ell & 1 \end{pmatrix}$$

These matrices tell us about how loops are transformed.

$[\sigma_1^k \sigma_2^{-\ell}]$ is a hyperbolic matrix ($|\text{largest eigenvalue}| > 1$) if

$$|2 + k\ell| > 2$$

Topological entropy

If $[\sigma_1^k \sigma_2^{-\ell}]$ is hyperbolic, then the protocol is isotopic to a pseudo-Anosov mapping with positive entropy.

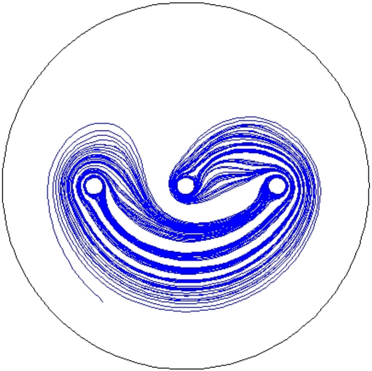
For instance, if $k = \ell = 1$ (counter-rotating) or $k = 1$, $\ell = -5$ (co-rotating), then $|2 + k\ell| = 3$, and

$$h = \log |\text{largest eigenvalue}| = \log(\tfrac{1}{2}(3 + \sqrt{5}))$$

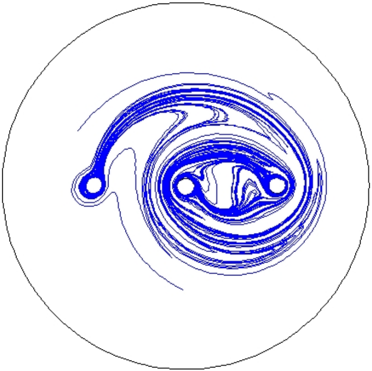
The only difference is that in the counter-rotating case the eigenvalue of the matrix $[\sigma_1^k \sigma_2^{-\ell}]$ is positive, while for the co-rotating case it is negative.

This h is a lower bound on the growth of material lines in the flow.

Stretching of material lines

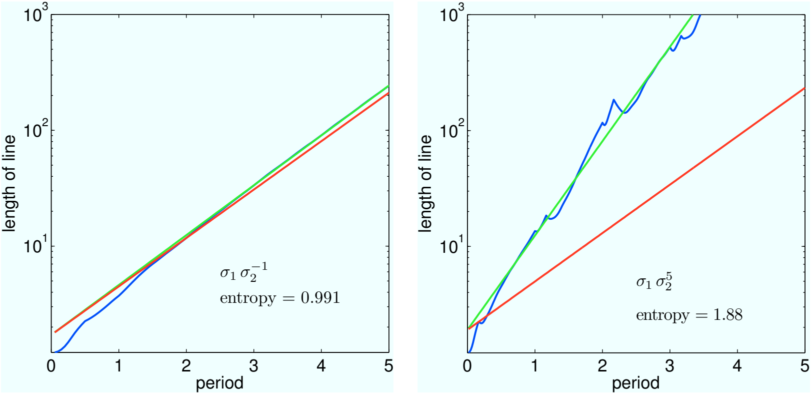


$\sigma_1\sigma_2^{-1}$



$\sigma_1\sigma_2^5$

Stretching of material lines: growth

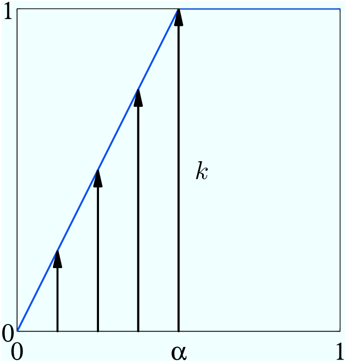


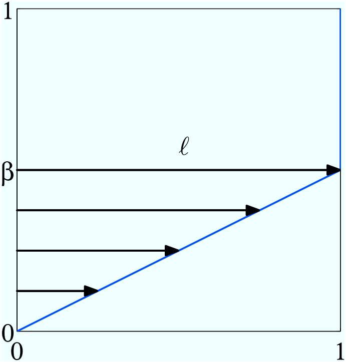
$\sigma_1\sigma_2^{-1}$ $\sigma_1\sigma_2^5$

Huge gap between lower bound and measured rate for the co-rotating case. Why?

Linked toral twist maps (LTTMs)

Consider a simpler problem: 'Dehn twists' on a strip on the torus.



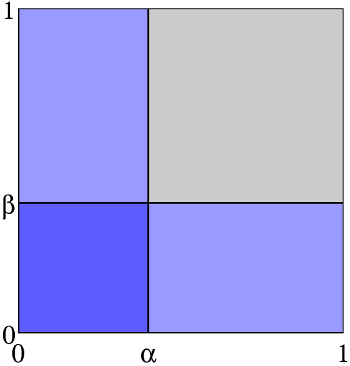


Compose these two maps together: Linked Toral Twist Map.

[Devaney, Przytycki, Burton & Easton,. . . see Sturman *et al.* (2006).]

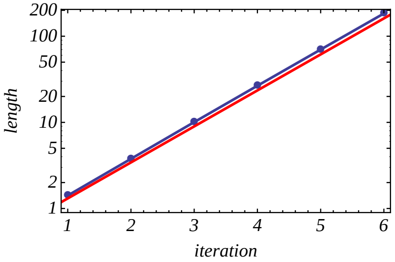
Linked toral twist maps (LTTMs)

Regions: fixed (gray), one map (light blue), both maps (dark blue).

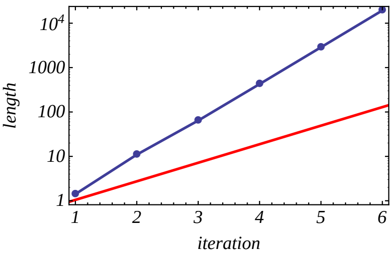


$\alpha = \beta = 1$: recover Anosov homeomorphism (same isotopy class).

Line growth for LTTMs



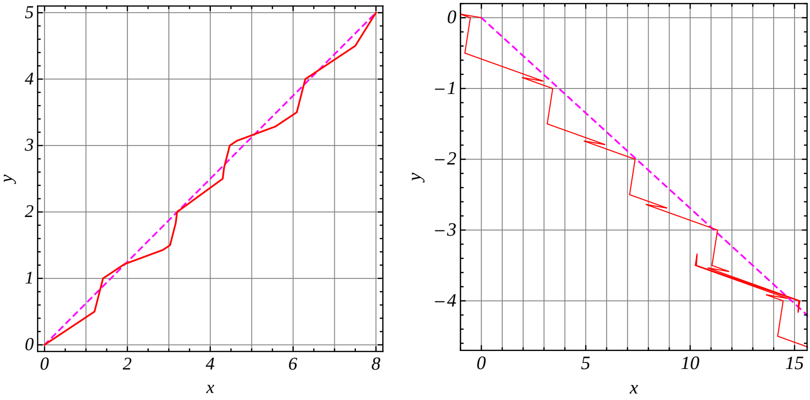
counter-rot. $(k\ell = 1 > 0)$
entropy $\simeq$ .962
(same as Anosov class)



co-rot. $(k\ell = -5 < 0)$
entropy $\simeq$ 1.91 $>$ .962

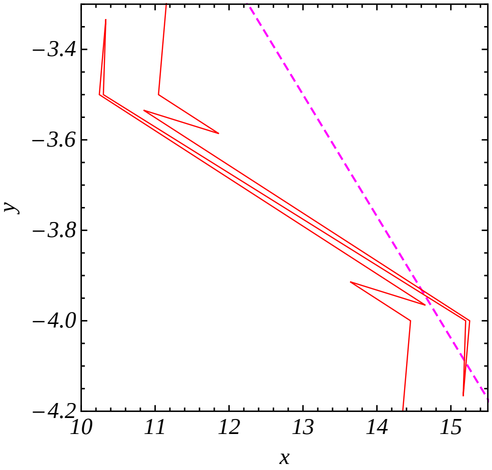
Line growth for LTTMs: Unstable manifold

Unstable manifold on the universal cover:



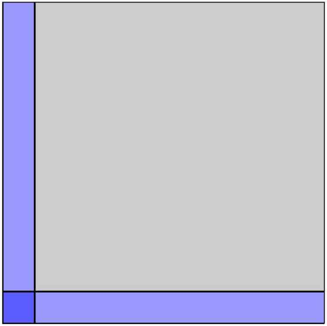
counter-rot. $(k\ell = 1 > 0)$ co-rot. $(k\ell = -5 < 0)$

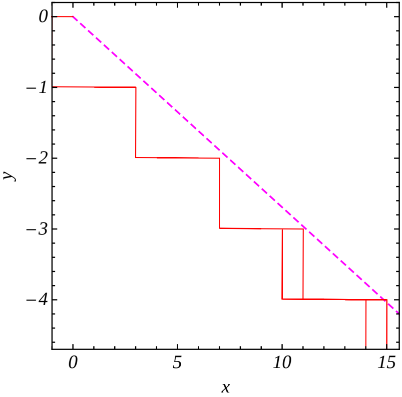
Extra growth comes from ‘folds’



Spine

For small α, β, squish strip on 'spine' (co-rotating, $k\ell < 0$):





Unstable manifold is then a 'staircase,' with some defects.

Spine: entropy

On the spine, can see the extra entropy comes from material lines not being ‘pulled tight’.

$\Rightarrow$ when looking at the action of the map on words in π_1 (loops), we shouldn't cancel loops with their inverses (leave them dangling).

This is the same as taking absolute values for the action on π_1:

$$\begin{pmatrix} 1 + |k\ell| & |k| \\ |\ell| & 1 \end{pmatrix}$$

For $k = 1$, $\ell = -5$, this gives an entropy of 1.92. Compare to the numerically-computed value 1.98 (as α, $\beta \to 0$).

So there's a bit more entropy, but close! Not clear yet where the extra growth comes from...

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