# Unsteady Flow Over an Aerofoil

Thomas Roger Brickell.
Supervisor Dr Jean - Luc Thiffeault.

September 23, 2004

Acknowledgements

I would like to thank my supervisor Dr Jean - Luc Thiffeault for his invaluable help and Professor Frank Leppington for notes and initial ideas on this project.

# Contents

**Abstract**

This project will start by first of all deriving and explaining the governing equations which come from a linearised model of the flow. Then we shall consider the well known case of a harmonically oscillating wing. Solving this in an attempt to show the general method to be used. Finally the case of a flat plate and sudden change of incidence shall be considered. The behaviour will be analysed both asymptotically as $t \to 0$ and as$t \to \infty$. Then the midsection when $t$ is neither large or small will be considered numerically.

# Chapter 1

# Background

## 1.1 Introduction

The problem of flow over an aerofoil is of both practical and industrial relevance. While most practical problems are solved using numerical methods it is still necessary to validate the results through a simplified model. By creating a simplified model and solving analytically it allows the possibility of gaining a better physical understanding of the system at 'ground level' which can not be done with a computational flow.

In this project we shall be looking at many ways in which to solve the problem of unsteady incompressible flow over an aerofoil. The flow being incompressible is a great simplifier to the problem, this allows to take many of the results of steady flow as read. It is still however, not a trivial problem.

The most common form of this problem was first solved back around the mid 1930's by several authors independently and has a full explanation in [1]. It considers that case of an eternally, harmonically oscillating wing. This gives rise to a solution in the form of Hankel functions. We shall consider this case and demonstrate the techniques used to solve this problem as a preliminary to the main subject of the project.

The main bulk of this project is concerned with the problem of a flat plate with a change of incidence angle. To solve this problem Laplace transforms will be utilized to remove the $t$ derivative from our integral equation. A solution for our circulation can then be found and inverted using the complex contour definition of the inverse Laplace transform. This leads to some interesting and not intuitive results by using a combination of numerical integration and asymptotic forms to derive analytical solutions.

Though out the project we shall be choosing the width of the wing, $a = 1$

the density $\rho = 1$ and the velocity $U = 1$ allowing us to work in normalised units. This means that an important ratio is $\frac{a}{U}$ as this represents the time. So all calculations involving time must be multiplied by this ratio.

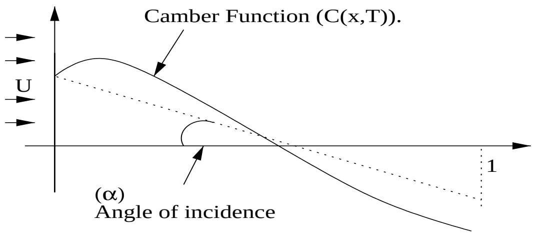


Figure 1.1: Aerofoil

## 1.2 Derivation of Governing Equations

The problem of incompressible flow over an aerofoil can take many results from steady flow. The governing equation is still Laplaceian of the potential flow as the flow around the wing is irrotational

$$\bigtriangledown^2 \phi. \tag{1.1}$$

We will be considering the anti-symmetric or lifting case where the aerofoil has zero thickness. Our Equation must obey three boundary conditions. Firstly that there is zero flux through the aerofoil given by 1.2

$$\frac{D(y-f)}{Dt} = 0. \tag{1.2}$$

Where $f$ and $y$ (the surface of the aero foil) are defined by

$$y = \epsilon\, c(x,t) \equiv\; f(x,t)$$

and $\frac{D}{Dt}$ is the material derivative. The surface of the aero foil is also shown in Fig 1.2.

We define $\underline{u} = (u, v)$ to be the velocity perturbation by the aerofoil and $\underline{U} = (U, V)$ our back ground velocity hence our total velocity $\underline{q} = (U + u, V + v)$ with $\underline{u} \ll \underline{U}$ and insert this in to 1.2 and considering only higher order terms,

$$v = Uc_x + c_t.$$

We the multiply this by 2 and re name it $\lambda$ which we take to be our boundary condition of zero flux through the wing,

$$\lambda = 2Uc_x + 2c_t. \tag{1.3}$$

The second condition is that any perturbation vanish at infinity i.e

$$\underline{u} \to 0, \; z \to \infty. \tag{1.4}$$

Seemingly obvious but still important.

To solve our problem we can replace the wing with series of source along the wing of strength $k$. We denote the complex potential by $W$ this can be written as

$$\delta W = \frac{-i}{2\pi} k(\zeta) \delta\zeta log(z - \zeta).$$

Differentiate with respect to $z$ and integrate over the period $(0, \infty)$:-

$$u - iv = \frac{-i}{2\pi} \int_0^\infty \frac{k(\zeta)}{z - \zeta} d\zeta.$$

We integrate over rather $(0, \infty$ than $(0, 1)$ as due to the unsteady flow a vortex sheet is shed off from the trailing edge of the aerofoil. Evaluating using Cauchy's Residue theorem and splitting in to real and imaginary parts we find

$$u(x, \pm 0) = \mp \frac{1}{2} k(x) \tag{1.5}$$

and

$$v(x, \pm 0) = \frac{1}{2\pi} ⨍_0^\infty \frac{k(\zeta)}{x - \zeta} d\zeta \;\; 0 < x < 1.$$

This then give us the integral equation (where $⨍$ is a principle value integral),

$$\lambda = \frac{1}{\pi} ⨍_0^1 \frac{k(\zeta)}{x - \zeta} d\zeta \; + \; \frac{1}{\pi} \int_1^\infty \frac{k(\zeta)}{x - \zeta} \, d\zeta \; 0 < x < 1. \tag{1.6}$$

This solution can alternatively be considered in a more mathematical sense in terms of Cauchy's integral formula by remembering the fact that $u - iv$ is an analytic function in the complex plain. Then considering a contour integral over the plain deformed by the aerofoil [2].

This integral equation is however incomplete as it only valid over $0 < x < 1$, so a further condition must be found. This condition is that after the trailing edge, the pressure must be continuous over the vortex sheet in the wake. Therefore we must consider the time dependent linearized bernoulli

$$\frac{P - P_\infty}{\rho} + \frac{\partial \phi}{\partial t} = -U\frac{\partial \phi}{\partial x}.$$

For the pressure to be continuous across the wake we require

$$P_- - P_+ = 0.$$

Using Bernoulli

$$p_+ - p_- = [-\phi_t + U\phi_x] \; x > 1$$

where,

$$[f] = f(x, +0) - f(x, -0)$$

this implies

$$[\phi(x,t)] = f(t - \frac{x}{U}) \; x > 1.$$

Using 1.5 we can see

$$k(x,t) = -[u] = -\frac{\partial[f(t - \frac{x}{U})]}{\partial x} \; x > 1.$$

Defining the strength of the vortex sheet to be

$$K(x,t) = \int_0^x k(x,t)dx = -[\phi(t - \frac{x}{U})] \; x \geq 1,$$

the bound vortex strength.

$$K(1,t) = K_1(t) = -[\phi(t - \frac{1}{U})]$$

as the vorticity must be continuous, finally we can say

$$K(x,t) = K_1(t - \frac{x-1}{U}) \; x \geq 1.$$

This makes sense as a wave of disturbance travelling away from the trailing edge of aerofoil at speed $U$.

So we now have a complete integral equation with enough conditions.

1. No flux through surface of the aerofoil.

2. Induced disturbances tend to zero as we move away from the aerofoil.

3. Continuous pressure across the wake.

We will now consider the force on the wing and calculate the lift coefficient. To do this we will consider bernoulli again and calculate the pressure on either side of the aerofoil

$$\frac{p_+ - p_-}{\rho} = [\phi_t + U\phi_x]$$

remembering

$$[\phi] = -\int_0^x k(x,t)dx$$

hence

$$p_+ - p_- = \rho\left[\frac{\partial K}{\partial t} + U\frac{\partial K}{\partial x}\right].$$

So this then allows us to define our lift coefficient as

$$C_L = -\frac{1}{(1/2)\rho U^2}\int (p_+ - p_-)\, dx = -\frac{2}{U^2}\int \frac{\partial K}{\partial t} + \frac{\partial K}{\partial x} dx. \tag{1.7}$$

Armed with these four facts we can now tackle the problem it's self.

# Chapter 2

# Oscillating Wing

We will first consider the problem of an oscillating wing. This is a well know solution to unsteady, incompressible flow which was first solved back in the mid 1930's almost simultaneously by several authors.

## 2.1 Governing Equation

We have our governing integral equation 1.6. We will consider the time dependance to be of the form $e^{i\omega t}$. This will result in all $K, k$ & $v \propto e^{i\omega t}$. Consider the strength of the vortex sheet around the wing

$$K_1(t) = K_1 e^{i\omega t}.$$

Where $K_1$ is an unknown constant.
From our derivation of the boundary conditions we can then say

$$K(x,t) = K_1 e^{i\omega t} e^{-i\omega \frac{x-1}{U}} \quad x > 1$$

hence

$$k(x,t) = \frac{\partial K}{\partial x} = -\frac{i\omega}{U} K_1 e^{-i\omega \frac{x-1}{U}} e^{i\omega t} \quad x > 1.$$

Thus we may cancel the $e^{i\omega t}$ factor through the whole of our integral equation.

$$\lambda = \frac{1}{\pi} \underbrace{\fint_0^1 \frac{k(\zeta)}{x-\zeta} d\zeta}_{=(i)} - \frac{i\omega K_1}{U\pi} \underbrace{\int_1^\infty \frac{k(\zeta)}{x-\zeta} d\zeta}_{=(ii)} \quad 0 < x < 1. \tag{2.1}$$

Solving by multiplying through by $\left(\frac{x}{1-x}\right)^{\frac{1}{2}}$ and integrating over the period $(0,1)$[?, AandLp91] The left hand side is a know function of x and can be evaluated. If we now consider the right hand side, each term in turn

$$(i) \rightarrow \int_0^1 \fint_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \frac{k(\zeta)}{\zeta - x} d\zeta dx.$$

Making use of the substitutions

$$x = \frac{1}{2} + \frac{1}{2} cos\ \psi$$

and

$$\zeta = \frac{1}{2} + \frac{1}{2} cos\ \theta$$

and using Glauerts first integral we get

$$-\pi \int_0^1 k(\zeta) d\zeta \quad 0 < \zeta < 1.$$

Taking the second term

$$(ii) \rightarrow \int_1^\infty e^{-i\frac{\omega}{U}(x-1)} \int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \frac{dx}{(\zeta - x)} d\zeta \tag{2.2}$$

using Glauerts integral again

$$= -\pi \int_1^\infty e^{-i\frac{\omega}{U}(x-1)} \left(1 - \left(\frac{\zeta}{1-\zeta}\right)^{\frac{1}{2}}\right) d\zeta.$$

Now our integral equation becomes

$$\int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \lambda(x) dx =$$

$$\frac{-\omega K_1}{U} \int_1^\infty \underbrace{e^{-i\frac{\omega}{U}(x-1)} \left(1 - \left(\frac{\zeta}{1-\zeta}\right)^{\frac{1}{2}}\right)}_{=(iii)} d\zeta \tag{2.3}$$

$$- \int_0^1 k(\zeta) d\zeta.$$

## 2.2 Strength of the vortex sheet

This is calculated by following on from above. The first term of (iii) is easily evaluated but the second requires a little more thought.

Using the change of variable $x = 2\zeta - 1$ and the substitution $\Omega = \frac{\omega}{2U}$.

$$\int_1^\infty e^{-\frac{i\omega\zeta}{U}} \left(\frac{\zeta}{\zeta - 1}\right)^{\frac{1}{2}} d\zeta = \frac{1}{2}\int_1^\infty e^{-\frac{i\omega(x+1)}{U}} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} dx$$
$$= \frac{1}{2}\int_1^\infty e^{-\frac{i\omega(x+1)}{U}} \left(\frac{x+1}{(x^2-1)^{1/2}}\right) dx.$$

We can then use the substitution $x = sin(\eta)$ and using some trig identities we arrive at

$$\frac{-ie^{-i\Omega}}{2}\int e^{-i\Omega sin(\eta)}(sin(\eta)+1)d\eta.$$

Then using the substitution $t = e^{i\eta}$ we get

$$-\frac{e^{-i\Omega}}{2}\int e^{-\frac{\omega}{2}(t-\frac{1}{t})}\left[\frac{1}{2i}\left(t+\frac{1}{t}\right)\right]dt$$
$$= -\frac{e^{-i\Omega}}{2}\left[\frac{e^{-i\Omega}}{i\Omega} - \left[\frac{\pi}{2}H_1^{(2)} + i\frac{\pi}{2}H_0^{(2)}\right]\right]$$

Now remembering from 2.3 we have

$$\int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \lambda(x)dx = K_1\left(-1 + i2\Omega\int_1^\infty e^{-i2\Omega(\zeta-1)}\left(1 - \left(\frac{\zeta}{1-\zeta}\right)^{\frac{1}{2}}\right)d\zeta\right) \tag{2.4}$$

and using our result from above and Mathematica to put integrals in to the from of Hankel functions we get

$$\int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \lambda(x)dx = K_1 i\Omega\frac{\pi}{2}e^{i\Omega}\left[H_1^{(2)}(\Omega) + iH_0^{(2)}(\Omega)\right].$$

From steady flow theory we can recall that if $\lambda(x)$ is redefined via $x = \frac{1}{2} + \frac{1}{2}\cos(\psi)$ then$\lambda(x)$ is an odd periodic function and hence can be presumed to have a cosine Fourier expansion.

$$\lambda(\psi) = \lambda_0 + \sum_{n=0}^{\infty} \lambda_n \cos(n\psi)$$

with $\lambda_n \frac{2}{\pi} \int_0^\pi \lambda(\psi)\cos(n\psi)d\psi$ and $\lambda_1 \frac{1}{\pi} \int_0^\pi \lambda(\psi)d\psi$.

Considering this fact and making the change of variable and only keeping relevant terms due to orthogonality of functions we have

$$\begin{aligned}\int_0^1 \lambda(x)dx &= \frac{1}{2}\int_0^\pi \lambda_0 + \lambda_0\cos(\psi) + \lambda_1\cos(\psi) + \lambda_1\cos(\psi)^2 d\psi \\ &= \frac{\pi}{2}\left(\lambda_0 + \frac{\lambda_1}{2}\right).\end{aligned}$$

Inserting this into 2.4 and rearranging yields

$$K_1 = \frac{\lambda_0 + \frac{\lambda_0}{2}}{i\Omega e^{i\Omega}\left[H_1^{(2)}(\Omega) + iH_0^{(2)}(\Omega)\right]}.$$

## 2.3 Lift Coefficient

Now consider the lift coefficient. Returning to our inter integral equation

$$\lambda(x) + \mu(x) = \frac{1}{\pi} \fint_0^1 \frac{k(\zeta)}{\zeta - x} d\zeta$$

with

$$\mu(x) = -\frac{i\omega}{U\pi} K_1 \int_1^\infty \frac{e^{-i\frac{\omega}{U}(\zeta-1)}}{\zeta - x} d\zeta.$$

The standard solution for this integral equation:

$$k(x) = -\frac{1}{\pi}\left(\frac{1-x}{x}\right)^{1/2} \fint_0^1 (\lambda(\zeta) + \mu(\zeta))\left(\frac{\zeta}{1-\zeta}\right)\frac{d\zeta}{\zeta - x}. \tag{2.5}$$

This comes via Cauchy's integral formula [2].

Considering the lift coefficient 1.7 then in our case of an oscillating aerofoil

$$\begin{aligned}C_L &= -\frac{2}{U^2}\int_0^1 \left[Uk(x) + i\omega \int_0^x k(x')dx'\right] dx \\ &= -\frac{2}{U^2}\left[\int_0^1 k(x)dx + 2i\Omega \int_0^1 (1-x)k(x)dx\right].\end{aligned}$$

Leaving us with two terms to evaluate.
i) $\int_0^1 k(x)dx$
ii) $\int_0^1 xk(x)dx$
Consider i)

$$\int_0^1 k(x)dx = -\frac{1}{\pi}\fint_0^1 (\lambda(\zeta)+\mu(\zeta))\left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta \underbrace{\fint_0^1 \left(\frac{1-x}{x}\right)^{1/2}\frac{dx}{1-x}}_{=\pi}$$

(via the substitutions $x = \frac{1}{2}+\frac{1}{2}\cos(\psi)$, $\zeta = \frac{1}{2}+\frac{1}{2}\cos(\theta)$ and Glauerts integral)

$$= -\fint_0^1 (\lambda(\zeta)+\mu(\zeta))\left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta.$$

Now considering ii)

$$\int_0^1 xk(x)dx = -\frac{1}{\pi}\fint_0^1 (\lambda(\zeta)+\mu(\zeta))\left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta \underbrace{\fint_0^1 \left(\frac{1-x}{x}\right)^{1/2}\frac{x}{1-x}dx}_{=\pi\zeta-\frac{\pi}{2}}.$$

Finally

$$C_L = \frac{2}{U}\left[(1+3i\Omega)\int_0^1 (\lambda+\mu)\left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta \right.$$
$$\left. - 2i\,\Omega\int_0^1 (\lambda+\mu)\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\zeta d\zeta\right].$$

Leaving us with two more calculations to consider, i) $\int_0^1 \mu\left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta$ and ii) $\int_0^1 \mu\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\zeta d\zeta$. With $\mu(\zeta) = \frac{2i\Omega K_1}{\pi}\int_1^\infty \frac{e^{-2i\Omega(x-1)}}{x-\zeta}dx$.
Considering i)

$$i) = 2i\Omega\frac{K_1}{\pi}\int_0^1 \left(\frac{\zeta}{1-\zeta}\right)^{1/2}\int_0^1 \frac{e^{-2i\Omega(x-1)}}{x-\zeta}dxd\zeta$$
$$= 2i\Omega\frac{K_1}{\pi}\int_1^\infty e^{-2i\Omega(x-1)}(-\pi)\left(1-\left[1-\left(\frac{x}{x-1}\right)^{1/2}\right]\right)dx\ x>1,$$

this is just like 2.2 and so we can say

$$= -i\Omega K_1 e^{i\Omega}\left[\frac{e^{-i\Omega}}{i\Omega} + \frac{\pi}{2}H_1^{(2)} + i\frac{\pi}{2}H_1^{(2)}\right].$$

Now considering ii)

$$2i\Omega\frac{K_1}{\pi}\int_1^\infty\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\zeta\int_0^1\frac{e^{-2i\Omega(x-1)}}{x-\zeta}dxd\zeta,$$

then

$$2i\Omega K_1\int_1^\infty e^{-2i\Omega(x-1)}\left(-\frac{1}{2} - x\left[1-\left(\frac{x}{x-1}\right)^{1/2}\right]\right)dx.$$

.

Now if we preform the change of variable $s = 2x-1$ this gives us

$$= 2i\Omega K_1\frac{e^{i\Omega}}{4}\left[\left[\frac{2e^{-\Omega s}}{i\Omega}\right]_1^\infty + \left[\frac{e^{-\Omega s}}{i\Omega}\right]_1^\infty + \left[\frac{e^{-\Omega s}}{(-i\Omega)^2}\right]_1^\infty + \frac{\pi}{2i}\left(H_0^{(2)} - \frac{1}{\Omega}H_1^{(2)}\right) - \pi H_1^{(2)} + \frac{\pi}{2i}H_0^{(2)}\right],$$

hence we can say

$$(ii) = i\Omega K_1\frac{e^{i\Omega}}{2}\left[\frac{-3e^{-i\Omega}}{i\Omega} + \frac{e^{-i\Omega}}{\Omega^2} + \frac{\pi}{i}H_0^{(2)} - \frac{\pi}{2i\Omega}H_1^{(2)} - \pi H_1^{(2)}\right].$$

So finally we can write down a formula for $C_L$ by multiplying by$e^{i\Omega t}$ (the original time dependance) and take the real part.

$$C_L = Re\left[e^{i\Omega t}\left(\frac{2}{U}\left[(1+3i\Omega)\int_0^1\lambda\left(\frac{\zeta}{1-\zeta}\right)^{1/2}d\zeta - 2i\Omega\int_0^1\lambda\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\zeta d\zeta\right]\right.\right.$$
$$\left.\left.+K_1e^{i\Omega}\frac{2}{U}\left[\frac{3\Omega^2\pi}{2}H_1^{(2)}(\Omega)(1+i) + \frac{\Omega\pi}{2}H_1^{(2)}(\Omega)(1-2\Omega) + \frac{\Omega^2\pi}{i}H_0^{(2)}(\Omega)\right]\right)\right]. \tag{2.6}$$

This then means we can work out the strength of the vortex sheet and lift for any camber function in theory. In practice the integrals required may be very difficult and best done numerically. Still, the general essence of the solution is maintained.

## 2.4 Summary

For the case of an oscillating wing a complete solution is well know for all camber functions. In practice however it may be necessary to compute numerically the value of the integrals involved. The solution does exist though even if it is potentially difficult to compute in practise.

# Chapter 3

# Sudden Change of Incidence

## 3.1 Basic Problem

Here we follow the same method as in oscillating wing case but now $e^{i\omega t}$ is replaced by $e^{st}$. The disturbance starts at $t = 0$, there is a time derivative in our governing equation, hence why we will be using Laplace transforms to solve this problem. This also affects our wake effect as now the wake only starts when the disturbance starts and hence travels back at a speed $U.t$. The effect on the integral in 1.6 is that the limits of integration goes from being $(1, \infty)$ to $(1, U.t)$.

We start by taking the Laplace transform of integral equation 1.6 with modified limits of integration by multiplying by $e^{-st}$ and $\int_0^\infty dt$. If we consider our wing to be a flat plate with zero incidence initially and angle$\alpha$ after $t = 0$ this greatly simplifies the taking of laplace transforms.
Let

$$\lambda(x,t) \to \tilde{\lambda}(x,s) \;\; k(x,t) \to \tilde{k}(x,s).$$

Our camber function is a flat plate so there is no initial condition. So if we apply Laplace transforms to our integral equation we get

$$\begin{aligned}\tilde{\lambda}(x,s) = {} & \frac{1}{\pi} \int_0^1 \frac{\tilde{k}(\zeta,s)}{\zeta - x} d\zeta \\ & - \frac{1}{\pi U} \int_0^\infty \underbrace{\int_1^{1+Ut} \frac{\delta_t K_1 \left(t - \left(\frac{\zeta - 1}{U}\right)\right) e^{-st}}{\zeta - x} d\zeta dt}_{=(i)} .\end{aligned}$$

Considering(i) and integrating by parts (upper limit of integration changed by use of a heavy side step function)

$$(i) = \underbrace{\left[K_1\left(t-\left(\frac{\zeta-1}{U}\right)\right)e^{-st}\right]_0^\infty}_{=0}$$
$$+\,s\int_0^\infty K_1\left(t-\left(\frac{\zeta-1}{U}\right)\right)e^{-st}dt.$$

As $K_1 = 0$ at $t = 0$ by initial conditions of our camber function. Now if we consider the change of variables $\tau = t - \left(\frac{\zeta-1}{U}\right)$ and remember that before $t = 0$ there is no disturbance so we can further adjust the lower limit of integration from $\tau = \frac{-(\zeta-1)}{U}$ to $\tau = 0$ we finally get

$$\tilde{\lambda}(x,s) = \frac{1}{\pi}⨍_0^1 \frac{\tilde{k}(\zeta,s)s}{\zeta-x}d\zeta - \frac{\tilde{K_1}(s)}{\pi U}\int_1^\infty \frac{e^{s(\zeta-1)/U}}{\zeta-x}d\zeta. \tag{3.1}$$

As in the case of the oscillating wing we multiply by $\left(\frac{x}{1-x}\right)^{1/2}$ and $\int_0^1$. Then after some final evaluation and use of Glauert's integrals, like in chapter 2 we get,

$$\int_0^1 \tilde{\lambda}(x,s)\left(\frac{x}{1-x}\right)^{1/2}dx =$$
$$-\tilde{K_1}(s) + \frac{s}{U}\tilde{K_1}(s)\int_1^\infty \left(1-\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\right)e^{-s\frac{\zeta-1}{U}}d\zeta.$$

Evaluating the first term in the integral and using the change of variables $\eta = \zeta - 1$ and some help for Mathematica,

$$\int_0^1 \tilde{\lambda}(x,s)\left(\frac{x}{1-x}\right)^{1/2}dx = -\frac{s\tilde{K}(s)}{U}\int_0^1\left(\frac{\eta+1}{\eta}\right)^{\frac{1}{2}}$$
$$= -\frac{s\tilde{K}(s)\sqrt{\pi}}{U}\mathbf{U}(\frac{1}{2},2,s).$$

Here $\mathbf{U}$ in the Hypergeometric U function. This finally leads us to

$$\tilde{K}(s) = -\frac{U}{s\sqrt{\pi}\mathbf{U}\left(\frac{1}{2},2,s\right)}\int_0^1\left(\frac{x}{1-x}\right)^{\frac{1}{2}}\tilde{\lambda}(x,s)dx. \tag{3.2}$$

Now remember that our camber function is a flat plate that rises to an incidence angle of $\alpha$ at time $t = 0$. So $c(x) = 0 \;\forall\; t \leq 0$ and $c(x) = \alpha(1-x)\theta(t) \;\forall\; t > 0$. Hence $c_x = -\alpha\theta(t)$ and $c_t = \alpha(1-x)\delta(t) \;\forall t > 0$. Then

$$\tilde{\lambda}(x,s) = \int_0^\infty -2U\{c_x + c_t\} = -2U\alpha \int_0^\infty e^{-st}\left[-\theta(t) + (1-x)\delta(t)\right] dt.$$

(The integral of our delta function is 1 as in reality the disturbance happens after $t = 0$ so our delta function could be considered as $\delta(t+\epsilon)$.) If from here on in we let $U = 1$.

$$\tilde{\lambda}(x,s) = 2\alpha\left(\frac{1}{s} - (1-x)\right)$$

Using this and Glauert's integral we can say

$$\int_0^1 \tilde{\lambda}(x,s)\left(\frac{x}{1-x}\right)^{1/2} dx = \alpha\pi\left(\frac{1}{s} - \frac{1}{4}\right),$$

for our case of a flat plat that gives us

$$\tilde{K}(s) = -\frac{\alpha\pi}{\sqrt{\pi}s\mathbf{U}(\frac{1}{2},2,s)}\left(\frac{1}{s} - \frac{1}{4}\right).$$

Now we have the answer in terms of Laplace transformed functions. This means all we have to do is the trivial task of inverting the Laplace transform. This is necessary as otherwise we can't truly understand what's happening.

## 3.2 Inverse Laplace Transform

To do this we shall consider the complex contour definition of a inverse Laplace transform [4]

$$f(t) = \frac{1}{2\pi i}\int_{\mathbb{C}} \tilde{f} e^{st} ds.$$

We need to consider the branch cuts and poles in our function. There is a branch cut on $(-\infty, 0)$ and a simple pole at $s = 0$ on $\mathbf{U}$. Our contour of integration is shown on fig 3.2, where the radius of the semi circles $\to \infty$.

To evaluate this we must consider the limits of $\mathbf{U}$. These definitions can be found on the function.wolfram.com web site [3].

$$\lim_{\epsilon \to^+ 0} \mathbf{U}(a,b,x+i\epsilon) = \mathbf{U}(a,b,x)$$

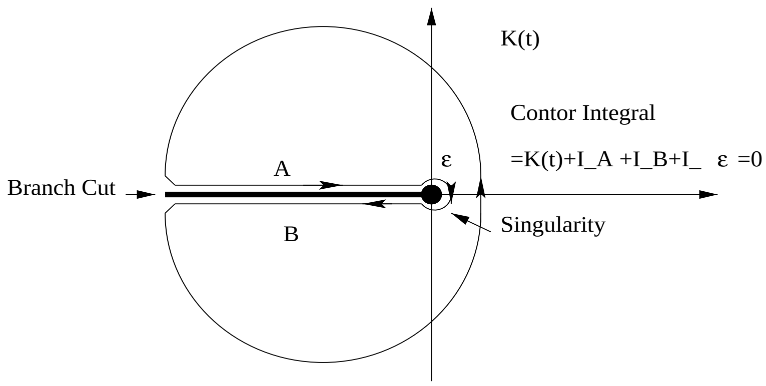


Figure 3.1: Contour Integral for Inverse Laplace

and

$$\lim_{\epsilon \to^{-} 0} \mathbf{U}(a,b,x-i\epsilon) = e^{2\pi i b}\mathbf{U}(a,b,x) - \frac{2\pi i e^{ib\pi}}{\Gamma(a-b+1)\gamma(b)}\,{}_1\mathbf{F}_1(a;b;x).$$

. For our case of a flat plate where $a = \frac{1}{2},\ b = 2$ we consider

$$\lim_{\epsilon \to 0_+} \mathbf{U}\left(\frac{1}{2}, 2, x+i\epsilon\right) - \mathbf{U}\left(\frac{1}{2}, 2, x-i\epsilon\right) = -\sqrt{\pi} i\,{}_1\mathbf{F}_1\left(\frac{1}{2}, 2, x\right)\ x < 0.$$

We know that our whole compleat integral must be zero. So we can say $K(t)+I_A+I_B+I_\epsilon = 0$ where the subscripts refer to the part of the contour. We use the substitution $s = x+i\epsilon$ for $I_A$ then

$$I_A = -\frac{\alpha}{2i\sqrt{\pi}} \int_{-\infty}^{0} \frac{e^{xt}dx}{\mathbf{U}\left(\frac{1}{2}, 2, x+i\epsilon\right) x^n}$$

and $s = x - i\epsilon$ for $I_B$

$$I_B = -\frac{\alpha}{2i\sqrt{\pi}} \int_{0}^{-\infty} \frac{e^{xt}dx}{\mathbf{U}\left(\frac{1}{2}, 2, x-i\epsilon\right) x^n}.$$

Now adding the two contours

$$\begin{aligned} I_A + I_B &= -\frac{\alpha}{2i\sqrt{\pi}} \int_{-\infty}^{0} \frac{e^{xt}dx}{x^n} \left[ \frac{1}{\mathbf{U}\left(\frac{1}{2}, 2, x+i\epsilon\right)} - \frac{1}{\mathbf{U}\left(\frac{1}{2}, 2, x-i\epsilon\right)} \right] \\ &= -\frac{\alpha}{2i\sqrt{\pi}} \int_{-\infty}^{0} \frac{e^{xt}dx}{x^n} \\ &\underbrace{\left[ \frac{1}{\mathbf{U}\left(\frac{1}{2}, 2, x+i\epsilon\right)} - \frac{1}{\mathbf{U}\left(\frac{1}{2}, 2, x+i\epsilon\right) + \sqrt{\pi} i \, {}_1\mathbf{F}_1\left(\frac{1}{2}, 2, x\right)} \right]}_{=(i)}. \end{aligned} \tag{3.3}$$

We shall refer to 3.3 as $I_n$ from here on.

### 3.2.1 Large t

Unless we use a numerical approximation we can not continue any further analytically. However if we just consider the behaviour as $t \to \infty$ we can make progress. If $t \to \infty$ for our integral to still converge we must let $s \to 0$. From looking in the wolfram web site [3] it can be confirmed that

$$\mathbf{U}\left(\frac{1}{2}, 2, s\right) \simeq \frac{1}{\sqrt{\pi} s}.$$

Using this in (i) gives

$$(i) = \sqrt{\pi} x - \frac{1}{(\sqrt{\pi} x)^{-1} + \sqrt{\pi} i \, {}_1\mathbf{F}_1\left(\frac{1}{2}, 2, x\right)}$$

using binomial expansion

$$\begin{aligned} &= \sqrt{\pi} x - \sqrt{\pi} x + \pi x^2 \sqrt{\pi} i \underbrace{{}_1\mathbf{F}_1\left(\frac{1}{2}, 2, x\right)}_{=1} + O\left(x^3\right) \\ &= \pi\sqrt{\pi} i x^2 + O\left(x^3\right). \end{aligned}$$

Now using this back in 3.3 and using the change of variable $\eta = -|x|t$

$$\begin{aligned} I_n \simeq -\frac{\alpha\pi}{2} \int_{-\infty}^{0} \frac{e^{xt}}{x^{n-2}} dx &= -\frac{\alpha\pi}{2} (-1)^n t^{n-3} \int_0^{\eta_0} e^{-\eta} \eta^{2-n} d\eta \\ &= -\frac{\alpha\pi}{2} (-1)^n t^{n-3} \Gamma(3-n). \end{aligned}$$

For the flat plate case we have to consider the case $n = 2$ and also $n = 1$ then multiplied by $\frac{1}{4}$. So
$n = 2$

$$I_n = -\frac{\alpha\pi}{2t}$$

and $n = 1$

$$-\frac{1}{4}I_n = -\frac{\alpha\pi}{8t^2}.$$

This looks good physically at it is saying the strength of the vortex sheet tails off with a $\frac{1}{t}$ and $\frac{1}{t^2}$ dependance.

The singularity at the origin still has to be considered in our line integral. For this we shall use the parameterisation $s = \epsilon e^{i\theta}$ for $\epsilon \ll 1$. Using our approximation for $\mathbf{U}\left(\frac{1}{2}, 2, x\right)$ near $x = 0$ this gives

$$I_\epsilon \simeq -\frac{\alpha}{2}\int_{-\pi}^{\pi} \underbrace{e^{\epsilon e^{i\theta}}}_{=1\ \epsilon\to 0}\, d\theta = \pi\alpha.$$

So using this and our other results from the contour integration we can say

$$K(t) = I = -I_A - I_B - I_\epsilon$$

i.e

$$K_{wing}(t) = -\frac{\pi\alpha}{2}\left(\frac{1}{t} + \frac{1}{4t^2} + 2\right)\ t \gg 1. \tag{3.4}$$

This is validated by the fact that if we let $t \to \infty$ then we regain the steady state solution for strength of the vortex sheet around a wing.

### 3.2.2 Complete Solution

The solution for $t \gg 1$ however tells nothing about what happens the rest of the time. To get around this problem we shall split our region of integration of the inverse Laplace transform in to three regions. Considering analytically the asymptotic as $s \to 0$ and $s \to -\infty$ then consider the middle region numerically, which we shall later show has little significance compared to the two values of the asymptotic expansions.

If we divide up the area to be integrated over in to three regions. Region 1 $[0, s_0]$ region 2 $(s_0, s_\infty)$ and region 3 $[s_\infty, -\infty]$, as shown in Figure 3.2.2.

Region 1 can be solved relatively easily as we have already done the necessary approximations with our large $t$ approach. All we have to do is

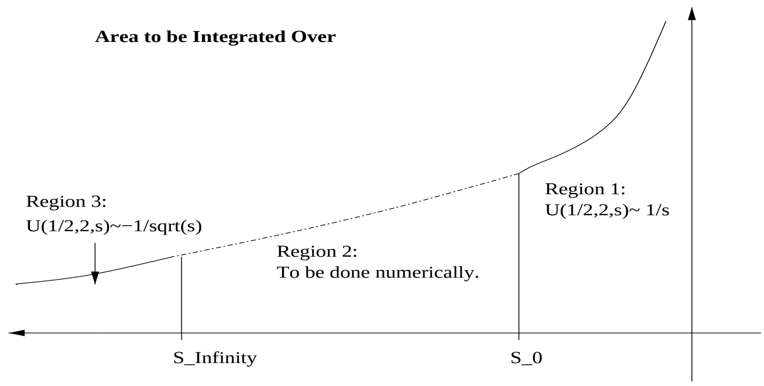


Figure 3.2: Inversion of $\tilde{K}$ in three sections

change the limits of integration. Consider

$$I_n = -\frac{\alpha\pi}{2}\int_{-s_0}^{0}\frac{e^{xt}}{x^{n-2}}dx$$

using the change of variables $\eta = -|x|t$

$$\begin{aligned} I_n =& -\frac{\alpha\pi}{2}(-1)^n t^{n-3}\int_0^{\eta_0} e^{-\eta}\eta^{2-n}d\eta = -\frac{\alpha\pi}{2}(-1)^n t^{n-3}\gamma(3-n,\eta_0) \\ =& -\frac{\alpha\pi}{2}(-1)^n t^{n-3}\gamma(3-n,S_0 t) \end{aligned}$$

where $\gamma$ is the lower incomplete gamma function.

For our flat wing case we must consider the cases $n=1$ and $n=2$.

$n=1$

$$I_1 = \frac{\alpha\pi}{2t^2}\gamma(2,-s_0 t)$$

and $n=2$

$$I_1 = -\frac{\alpha\pi}{2t}\gamma(1,-s_0 t)$$

hence

$$I_A + I_B = -\frac{\alpha\pi}{2}\left[\frac{1}{t}\gamma(1,-s_0 t) + \frac{1}{4t^2}\gamma(2,-s_0 t)\right].$$

### Small t

Now we shall consider the other asymptotic extreme of $s \to -\infty$, corresponding to $t \to 0$. The wolfram web site [3] shows us that as $s \to -\infty$

$$\mathbf{U}\left(\frac{1}{2},2,s\right) \sim -\frac{1}{\sqrt{s}}$$

and

$${}_1\mathbf{F}_1\left(\frac{1}{2},2,s\right) \sim \frac{2}{(-s\pi)^{\frac{1}{2}}}.$$

Substituting this into $I_n$

$$-\frac{\alpha}{2i\sqrt{\pi}}\int_{-\infty}^{s_\infty}\frac{e^{xt}}{x^n}\left[-\sqrt{x}-\frac{1}{(-\sqrt{x})^{-1}+i\sqrt{\pi}2i(\sqrt{-x\pi})^{-1}}\right]dx$$

and then cancelling the $i$ and $\sqrt{\pi}$

$$I_n = -\frac{\alpha}{2\sqrt{\pi}i}\int_{-\infty}^{s_\infty}\frac{e^{xt}}{x^{n-\frac{1}{2}}}dx.$$

Just like in the case for $t \gg 1$ we shall use the change of variables $eta = -|x|t$ giving

$$= -(-1)^n\frac{\alpha}{2\sqrt{\pi}}t^{n-\frac{3}{2}}\int_{\eta_\infty}^{\infty}e^{-\eta}\eta^{\frac{1}{2}-n}d\eta = -(-1)^n\frac{\alpha}{2\sqrt{\pi}}t^{n-\frac{3}{2}}\Gamma\left(\frac{3}{2}-n,\eta_\infty\right).$$

Where $\Gamma$ is the incomplete gamma function. Once again for the case of a flat plate we have to consider the case of $n = 1$ and $n = 2$.

Considering $n = 1$

$$I_1 = \frac{\alpha}{2\sqrt{\pi t}}\Gamma\left(\frac{1}{2},-s_\infty t\right)$$

and $n = 2$

$$I_2 = -\frac{\alpha}{2\sqrt{\pi}}\sqrt{t}\Gamma\left(-\frac{1}{2},-s_\infty t\right).$$

So

$$I_A + I_B = -\frac{\alpha}{2\sqrt{\pi}}\left[\sqrt{t}\Gamma\left(-\frac{1}{2},-s_\infty t\right)+\frac{1}{4\sqrt{t}}\Gamma\left(\frac{1}{2},-s_\infty t\right)\right].$$

As $t \to 0$ it not clear that region 1 is negligible. If $t \to 0$ then $s_0 t = \eta_0 \to 0$ so we can use taylors expansion of $e^\eta$. If we apply this to $I_n$

$$I_n \sim t^{n-3}\int_0^{\eta_0}\eta^{2-n}\left(1+O\eta\right)d\eta.$$

If we look at the $n = 1$ case

$$I_1 \sim \frac{\eta_0^2}{2t^2} = \frac{t^2 s_0^2}{2t^2}$$

this obviously does not contribute to the solution as $t \to 0$. If we now look at $n = 2$. Similarly

$$I_2 \sim \frac{\eta_0}{t} = \frac{ts_0}{t}$$

and therefore contribution from this case also. So we can conclude that as $t \to 0$ that region 1 makes no contribution.

When we consider the case that $t \to \infty$ we must also consider whether region 3 makes a significant contribution. This is also easily solved as if $t \to \infty$ then$\eta_\infty \to -\infty$ and hence

$$I_n = \frac{\alpha}{i} \int_{-\infty}^{\eta_\infty} \frac{e^{xt}}{x^{n-\frac{1}{2}}} dx \to 0.$$

**Numerical Solution and Comparison to Analytical Solutions**

Now all we have left to consider is region 2. It needs to be shown that region 2 has little dependance on what value of t we choose. Using mathematica we choose $t = 1$ as this is neither big or small. We define a integral constructed as show 3.2.2 with the asymptotic form representing regions 1 and 3. Region 2 is then just numerically integrated between our two bounds $s_0$ and $s_\infty$. Then a function is created to model our specific case of a flat plate, i.e $I_2 - \frac{1}{4} I_1$. Two cases where considered to find what values of $s_0$ and $s_\infty$ the integral converged for. This was achieved by varying $s_0$ and $s_\infty$ independently. The integral converged for $s_0 = 10^{-4}$ and $s_\infty = 10^1$ but to be safe we took $s_\infty = 10^2$. Once this is set we let $t \in \left(10^{-8}, 10^5\right)$, with the asymptotic forms of our function for very small and very large $t$ and the split region form of the integral for the intermediate values of $t$. This gave us

$$I_{A2} + I_{B2} - \frac{1}{4}\left(I_{A1} + I_{B1}\right)$$

which tends to zero from below as shown by 3.2.2. Remembering that $K(t) = -I_A - I_B - I_\epsilon$ and that $I_\epsilon = \pi\alpha$. Then we can produce a graph of $K(t)$ 3.2.2. It can seen that for a short amount of time, till say $t_1$, $K(t)$ is positive producing a **down force** on the plate. This is only true for a finite $t$ and hence this area is the initial downward impulse. Using an iterative root finding process we found $t_1 \simeq 0.12$ and our initial impulse ($\mathbf{I}$ say) is $\mathbf{I} \simeq= -0.318101$. This down force is unintuitive but only lasts for a very short amount of time. The infinite $K$ at $t = 0$ is due to the fact that we have to consider $n = 1$ (producing the $\frac{1}{\sqrt{t}}$ term) for the case of an instantaneously changing angle flat plate but if the plate changes over a finite time this would

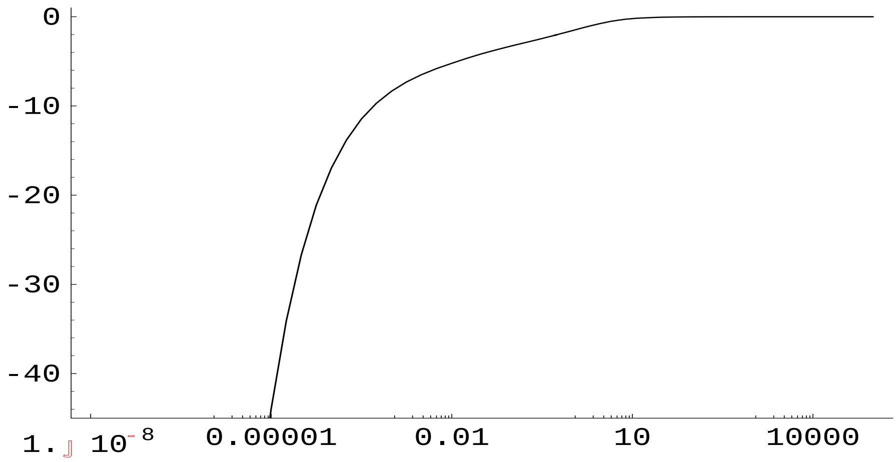


Figure 3.3: Numerical Integration of flat plate

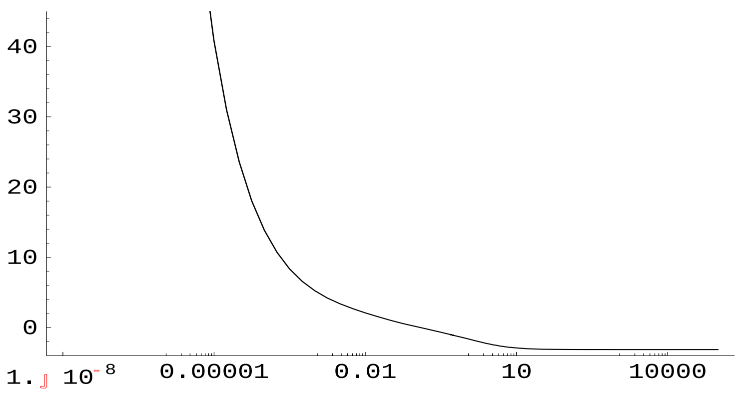


Figure 3.4: K(t) calculated numerically

not exist and hence these would be no infinite lift. This singularity is however not as bad as it first seems as it is integrable, hence our finite value for **I**.

When time has progressed a little we can se that the strength of the vortex sheet becomes negative and tends towards a constant value as we would expect. In the case of a flat plate this is $-\pi$ as we have to multiply our result by the angle of incidence to gain our final $K$. To validate our split range numerical model with our asymptotic solutions we refer to 3.2.2 This fig shows graphically how the two asymptotic representations match the numerical solution for their appropriate limits of $t$. In 3.2.2 you will see how the asymptotic solutio for $t \to 0$ (green) does not meet our numerical solution (blue) but only tends towards it. This is only due to the range of the graph as at this scale it is clear that there is divergence of the two solutions as $t$ grows. The solution for $t \gg 1$ (red) clearly joins our split region solution with values as small as $t = 1$. Due to the similarity of the split region solutio and the $t \ll 1$ solution it can be seen that a fair approximation to the problem could be if the two asymptotic solutions where joined neglecting the numerical region.

Taking this fact if and remembering $s_0 = -10^{-4}$, $s_\infty = -10^2$ and letting $t1 = 10$ be the cross over point. We can now use our definitions from the earlier sections

$$K(t) = \frac{\alpha}{2\sqrt{\pi}}\left[\sqrt{t}\Gamma\left(-\frac{1}{2}, -s_\infty\, t\right) + \frac{1}{4\sqrt{t}}\Gamma\left(-\frac{1}{2}, -s_\infty\, t, -\right)\right] - \alpha\pi \; t \leq 10$$

and

$$K(t) = \frac{\alpha\pi}{2}\left[\frac{1}{t}\gamma\left(1, -s_0\, t\right) + \frac{1}{t^2}\gamma\left(2, -s_0\, t\right)\right] - \alpha\pi \; t > 10.$$

This can be said to be a reasonable approximation to the complete answer for the case of a flat plate.

## 3.3 Vorticity of Sheet

We would like also to find out the strength of the vortex sheet. This can be calculated from the definition

$$K(t) = \int_1^{1+Ut} k(x,t)dx.$$

The boundary condition of constant pressure across the vortex sheet told us that $k(x,t) = k\left(1, t - \frac{x-1}{U}\right)$ and remembering we set $U = 1$ we can say

$$K(t) = \int_1^{1+t} k\left(1, t - \frac{x-1}{U}\right)dx.$$

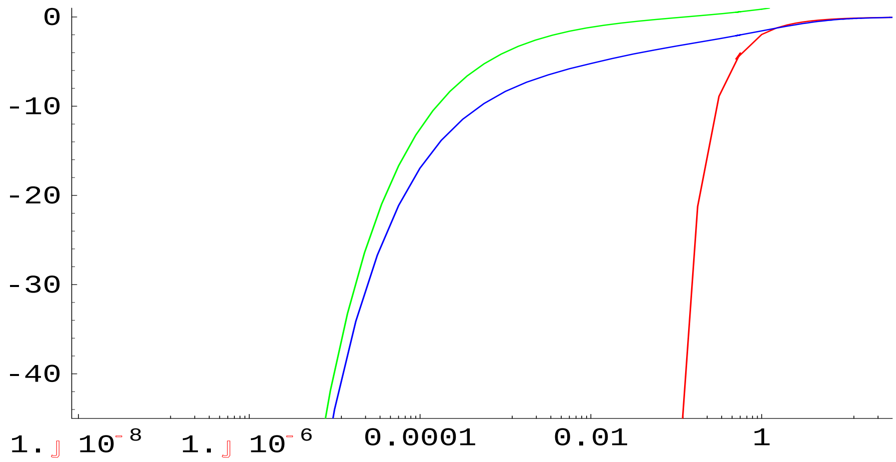


Figure 3.5: Comparison of Asymptotic forms to Numerical Solution

Using the change of variables $\eta = t - (x-1)$ and differentiating

$$k(1,t) = K'(t).$$

So hence

$$k(x,t) = k\left(1, t - \frac{x-1}{U}\right) = K'\left(t - \frac{x-1}{U}\right)$$

(remember though we have chosen $U = 1$). This is easily evaluated as to calculate $K'$ we only need to consider the same calculation as before as a derivative in Laplace is defined as

$$\tilde{f}'(x,s) = \tilde{f}(x,0) + s\tilde{f}(x,t).$$

The implications of this for the flat plate is that as (no initial condition as zero angle of incidence at $t = 0$ so no flow disturbance)

$$K(t) = -\left(I_2 - \frac{1}{4}I_1\right) - I_\epsilon$$

then we have to let $n \to n-1$, hence

$$k(t) = -\left(I_1 - \frac{1}{4}I_0\right).$$

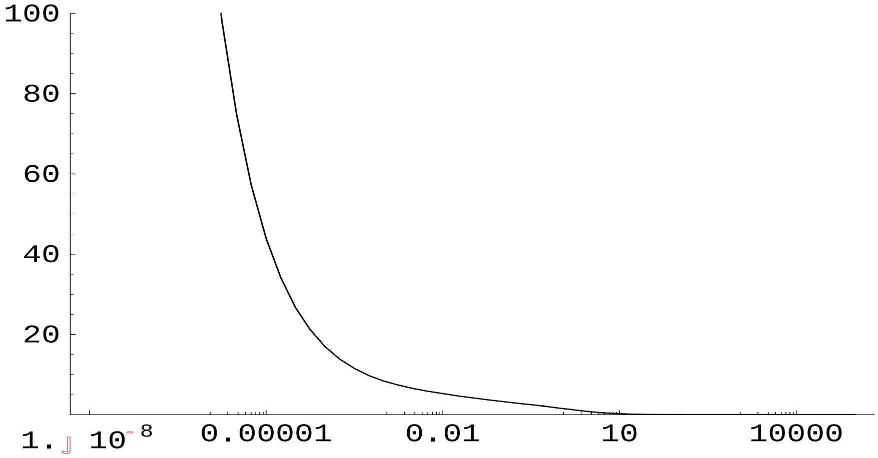


Figure 3.6: Vortex Sheet Strength

Applying this to our split region solution gives a solution as show in 3.3. For $t \ll 1$

$$
\begin{aligned}
k(x,t) &= k\left(1, t-(x-1)\right) \\
&= \frac{\alpha}{2\sqrt{\pi}}\left[\sqrt{t-(x-1)}\,\Gamma\left(\frac{1}{2}, -s_\infty(t-(x-1))\right)\right. \\
&\left. + \frac{1}{4\sqrt{t-(x-1)}}\Gamma\left(\frac{3}{2}, -s_\infty(t-(x-1))\right)\right]
\end{aligned}
$$

and for $t \gg 1$

$$
\begin{aligned}
k(x,t) &= k\left(1, t-(x-1)\right) \\
&= \frac{\alpha\pi}{2}\left[\frac{1}{t-(x-1)}\gamma(2, -s_0(t-(x-1))\right. \\
&\left. + \frac{1}{4(t-(x-1))^2}\gamma(3, -s_0(t-(x-1))\right].
\end{aligned}
$$

This shows what we expect that the strength of the sheet tends to zero as we approach the steady state solution. Comparing the asymptotic forms to the split region it is obvious that neither case can be said to be domineering 3.3.

### 3.3.1 Quick Note on Scaling

Though out this chapter dimensionless variables have been used. This means that if you require a answer with dimensions you must multiply your answer

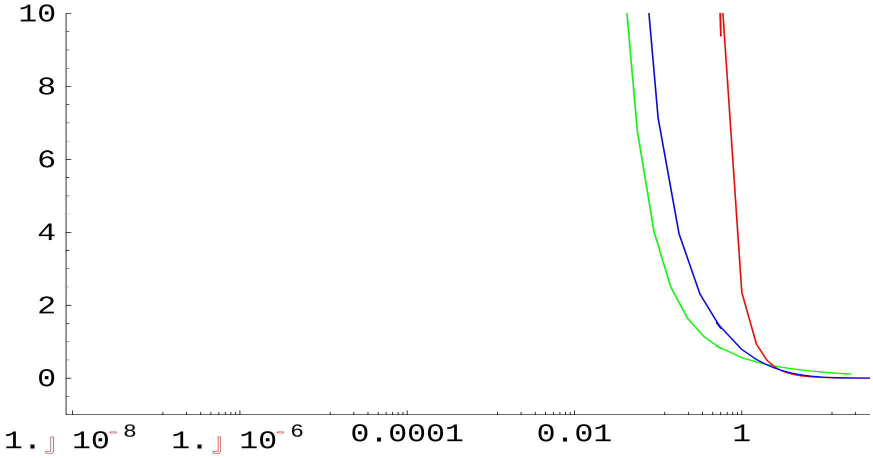


Figure 3.7: Comparison of Asymptotic to Split Region Form

by $\frac{a}{U}$ where $a$ is the width of the aerofoil and $U$ is the speed of the oncoming flow.

## 3.4 Summary

The problem of a sudden change of incidence can be fully solved in the cases of asymptotic $t$, the problem comes with 'reasonable sized' values of $t$. It turns out how ever that by splitting the region of the inverse Laplace transform and using a small amount of numerical integration we can gain a complete solution. If further relaxations are allowed it turns out that the numerical region can be neglected and a good approximation to this solution can be made with just the two asymptotic forms as a piece wise function.

The strength of the vortex sheet is an added bonus but has little surprises when considering the strength of the vortex sheet strength $K$.

A interesting point is to consider how long in reality dose the down force last for? From Mathematica we found a value of $t_1 = 0.12$ but this is not multiplied by our ratio of $\frac{a}{U}$. If we take a small plane as an example with $a = 1$m and $U = 50$m/s then that means $t_1 = 0.0024$sec, or a very short amount of time.

The second numerical value from the calculations was the initial impulse $\mathbf{I} = -0.32$. Impulse ($\Delta p$) has units kg m/s so if we take the density of air$\rho \approx 1.25$kg/m$^3$ then $\Delta p = -0.31 \times \rho \times U \approx 20$kgm/s. To have something to compar this against we consider gravity $g \approx 10$m/s if we take a small plane weighing 3000kg then $\Delta p_g \approx mg \times 0.12 \times \frac{a}{U}$ so we get $\Delta p_g \approx 60$. The result of this is that our impulse is less than the effect of gravity and hence would not be notice especially over a short time.

# Chapter 4

# Conclusion

For the case of an oscillating wing the solution is well know and fully solvable even if practically some of the integrations could well be impossible to do and have to be numerically evaluated. The only real weakness of this solution is that it assumes the disturbance has been and will be in existence for all time.

The case of a sudden change of incidence is a more complicated problem and in the end has to resort to some numerical integration, although only for a small and largely insignificant region. Finding the solution found for the case of a flat plate changing angle instantaneously uncovers some interesting properties. There is a slight down ward impulse which is definitely counter intuitive but in practise would be very hard to measure or feel due to the linearisation assumptions made (small angle e.t.c). The strength of the vortex sheet is initially infinitely positive as well which is unphysical, this is however due to the instant change in angle and would disappear if a more general case of wing changing over a finite but non zero time span. The singularity causing this infinite amount of positive strength of the vortex sheet is given by the $\frac{1}{\sqrt{t}}$, and whilst this gives a singularity at $t = 0$ it is an integrable singularity and hence why the initial impulse is finite. The strength of the vortex sheet $k$ is constant with how $K$ behaves but having this option allows for some interesting extensions to the problem.

If we had more time there are several ways in which we could extend or generalise this project. Consideration of a more general case of how the flat plate moves from zero angle to angle $\alpha$. This would be a simple extension in which the heavy side step function $\theta(t)$ would be replaced by a general function. This would show how the initial infinite $K$ is removed if the transition time from one angle to the next is not zero. A very interesting extension to the solution is that once $k(x, y)$ is know is to to replace the vortex sheet withe a single point vortex depending on $t$ at the trailing edge of the aerofoil or several point vortices over the vortex sheet depending on the solution. This is being done by J. B. Keller.

If we consider that a wing takes say 0.5sec to change angle of incidence and as we calculated earlier our initial down force time is $t_1 = 0.0024$ this makes our model a bad approximation in reality. However it could very well be a good approximation for a sudden gust of wind such as in turbulence, a flap moving quickly or a mechanical failure.

In essence a largely analytical solution can be found, like in many areas of maths for a simple case. The problem comes when generalisations are made. This however should not put us off even if the project was extended for more general cases it appears that significant progress could be made by analytical methods. Important applications of an analytical solution is to allow a better understanding of system and giving a solution which numerical simulations can be bench marked against for validation.

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Appendix A

# Mathematica Work Sheet