ENERGY CONSERVING TRUNCATIONS IN THERMAL CONVECTION

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In a horizontal layer of fluid heated from below and subject to gravity, a state of cellular convective flow called Rayleigh–Bénard convection[1] occurs. Experiments have shown that in such a system, shear flow modes (spanning the horizontal dimension of the system) can be destabilized[2]. In tokamak plasmas, it is thought that a shear flow in the edge layer may be responsible for the so-called H–mode of confinement: convection cells form as a result of the nonlinear development of the Rayleigh–Taylor instability in regions of unfavorable magnetic curvature and lead to the generation of shear flow, creating a barrier to particle transport.

In this work we make a Fourier expansion of the stream function and temperature fields, and then make a finite truncation of the infinite set of ODE's obtained. We then examine the conditions under which the truncations preserve the invariants of the full PDE's in the dissipationless limit.

EQUATIONS OF THE SYSTEM

The problem consists of a 2-D incompressible fluid subject to gravity and lying between two horizontal plates with fixed temperature difference ΔT, with a hotter bottom plate. The velocity and temperature fields are periodic in the x (horizontal) direction, and at the walls we use stress-free boundary conditions. In the Boussinesq approximation, the dimensionless equations governing the flow are

$$\begin{aligned} \frac{\partial \nabla^2 \chi}{\partial t} + \left\{ \chi \,, \nabla^2 \chi \right\} &= \nu \, \nabla^4 \chi + \frac{\partial T}{\partial x}, \\ \frac{\partial T}{\partial t} + \left\{ \chi \,, T \right\} &= \kappa \, \nabla^2 T + \frac{\partial \chi}{\partial x}, \end{aligned} \tag{1}$$

where χ is the stream function (with $\mathbf{v} = (-\partial_{\mathbf{z}}\chi, \partial_{\mathbf{x}}\chi)$), T is the temperature deviation from a linear profile, ν is the kinematic viscosity, and κ is the thermal conductivity. The Poisson bracket is defined as $\{A \,, B\} \equiv \partial_x A \, \partial_z B - \partial_z A \, \partial_x B$.

In the dissipationless limit, where ν and κ are set equal to zero, (1) conserves the total energy:

$$E = K + U = \frac{1}{2} \left\langle \, (\nabla \chi)^2 \, \right\rangle - \langle \, zT \, \rangle \,, \tag{2}$$

where the angle brackets denote an average over the domain. Equations (1) also conserve $\langle \, T \, \rangle$ and some quadratic invariants which are easily preserved by truncations, so we will not deal with them here.

MODAL EXPANSION

We now expand the two fields in Fourier modes:

$$\left\{ \begin{array}{c} \chi \\ T \end{array} \right\} = \sum_{m,n} \left\{ \begin{array}{c} \chi_{mn}(t) \\ T_{mn}(t) \end{array} \right\} e^{i(mz+nkx)}, \tag{3}$$

where k is the ratio of the height of a convection cell to its width. This expansion is more general than traditional ones[3] in that it allows for a variable phase in the rolls and admits a non-vanishing shear flow part (the χ_{m0} modes). Expansion (3) can then be inserted into (1) to yield an infinite sequence of coupled nonlinear ODE's[4]. A truncation is made by including only the finite set of modes A_χ for the stream function and A_T for the temperature. If, say, A_χ contains χ_{mn}, then it must also contain $\chi_{m,-n}$, $\chi_{-m,n}$, and $\chi_{-m,-n}$, since these are related by the boundary conditions. We define M to be the largest mode number m included in A_χ.

Such a truncation will preserve the quadratic invariants of the full PDE's, but $\langle \, T \, \rangle$ and the energy (2) will in general not be conserved. To see how we can preserve the energy (2), we look at the expansion for its kinetic and potential part:

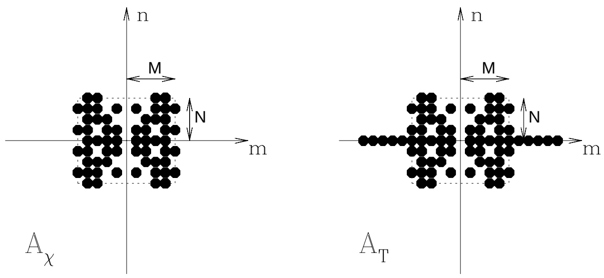

FIG. 1. Modes that must be included in a truncation to preserve the invariants.

$$K = \frac{1}{2}\sum_{m,n}\rho_{mn}\left|\chi_{mn}\right|^2 \tag{4}$$

$$U = -2\sum_{p>0}\frac{(-1)^p}{p}\,T^i_{p0}\ . \tag{5}$$

After taking the time derivative of (4) and (5) and using the equations of motion for the modes, we obtain the following condition for the conservation of E:

$$T^*_{mn} + \sum_{p>0}(-1)^p\,\mathrm{sgn}(m-p)\,T^*_{|m-p|,n} + \sum_{p'>0}(-1)^{p'}\,T^*_{m+p',n} = 0. \tag{6}$$

The T^*_{mn} $(m,n>0)$ term cannot be canceled by a term from the sum over p', since $m+p'>m$. Only the term in the first sum with $p=2m$ can cancel it. The largest m is M, and the sums over p and p' came from the time derivative of U. From (5) we conclude that we need to include all modes of the form $T^i_{p0},\ p=1,\dots 2M$, the odd modes helping to conserve $\langle\,T\,\rangle$ (see Fig. 1). The truncation then preserves all of the invariants of the full PDE's.

More importantly, truncations that preserve the invariants in the dissipationless limit can be shown to be bounded when the dissipation is included[4]. This is not true of arbitrary truncations, as can be seen in Fig. 2, where a comparison of the total heat flux is made between a 6–mode truncation popular in the literature[5] and its energy-conserving counterpart (7 modes). The curve of the 7–ODE model has a slope closer to experiment.

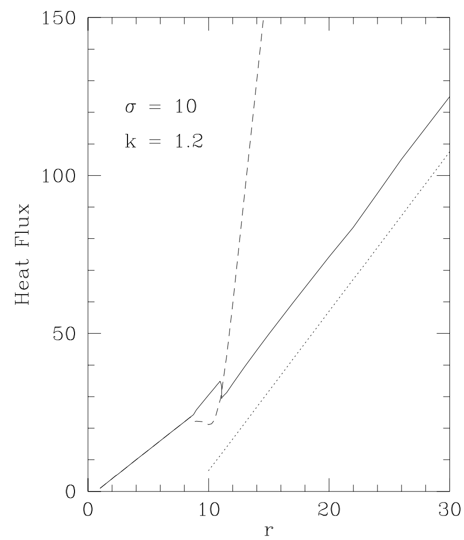

FIG. 2. Plot of the total heat flux vs the reduced Rayleigh number r for the 6-ODE (dashed line) and the 7-ODE (solid line) models. For this graph, $k=1.2$, $\sigma=10$. The dotted line has a slope of 5.05, corresponding to the experimental result[2] for $\sigma=7$.

In a steady-state, the energy-conserving truncations display an x-averaged heat flow which is independent of z, as physically expected[4]. The cascade of energy is modeled without extraneous terms adding unphysical dissipation or sources to the system. The boundedness property, the proof of which depends strongly on the energy conserving modes, is also desirable from a physical standpoint. For these reasons, energy-conserving truncations are a more reasonable form of truncation to use in modeling the Rayleigh–Bénard system.

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