# TOPOLOGICAL CHAOS IN FLOWS ON SURFACES OF ARBITRARY GENUS

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*Summary* The emerging field of topological fluid kinematics is concerned with design and analysis of effective fluid mixers based on the topology of the motion of stirring apparatus and other periodic flow structures. Knowing even a small amount of flow topology often permits very powerful diagnoses, such as proving existence of chaotic dynamics and a lower bound on mixing measures based on material stretching. In this paper we present a canonical method for examining flows on surfaces of arbitrary genus given the flow topology encoded as a braid. The method may be used to study fluid mixing driven by an arbitrary number of stirrers in either bounded or spatially-periodic fluid domains. Additionally, and unlike previous techniques, the current work may also be applied to flows on manifolds of higher genus.

## INTRODUCTION

Over the last eight years, a number of authors have investigated how efficient laminar fluid mixing can be achieved by engineering fluid flows with a favourable topology. The idea of applying topological tools to fluid mixing was proposed by Boyland, Aref and Stremler [1]. They explain that certain time-periodic stirring motions give rise to a mapping of the fluid domain onto itself that has the topology of a pseudo-Anosov map. These maps are desirable for fluid mixing as they have chaotic dynamics, and a positive topological entropy, which guarantees exponential stretching and folding of material lines. The theory is particularly attractive because it requires only continuity, and so the rapid stretching and folding is independent of the dynamical equations satisfied by the fluid.

Topology of a fluid flow may be encoded in several ways, but the most common and succinct description uses the language of braid groups. A braid encodes how the fluid stirrers pass around each other, and, in a spatially-periodic domain, how stirrers tour the periodic directions. A description of flow topology using braids has been presented before in a fluids context [1, 2], and more generally by Birman [3], so for brevity we omit one here. The key result is that the topological entropy of a braid embedded in a flow provides a rigorous lower bound on the topological entropy of the flow itself. In the fluid mixing context it can be desirable to study relatively long braids corresponding to large collections of periodic flow structures. This is a significant computational challenge, and so some of the literature in the area of topological fluid kinematics has been devoted to developing fast techniques for computing braid entropies [2, 4].

The fastest technique for a bounded domain is due to Moussafir and uses a dynamical system to compute the action of braids on a Dynnikov-coordinate encoding of topologically non-trivial material loops [4]. The exponential growth rate of such loops converges to the braid entropy. Finn and Thiffeault derived a more general dynamical system, using a triangulation method, to describe the action of braiding in domains with one or two spatially-periodic directions [2]. In the present work we present a canonical method for examining flows on surfaces of arbitrary genus given the flow topology encoded as a braid. The method may be used to study fluid mixing driven by an arbitrary number of stirrers in either bounded or spatially-periodic fluid domains. Additionally, and unlike previous techniques, the current work may also be applied to flows on manifolds of higher genus.

## PANTS DECOMPOSITION, DEHN–THURSTON COORDINATES AND BRAIDS

It is not possible to present full details of the technique in this short paper, so instead we give an outline and present an illustrative example. Full details will be prepared for a subsequent article. We consider a surface of genus $g$ with $n$ punctures. In a fluids context a puncture is a stirrer. The genus $g$ is the number of holes, with $g = 0$ for bounded flows or $g = 1$ for spatially-periodic flows. We also consider $g > 1$ for flows on surfaces with a more complicated topology that may be produced by some exotic physical constraint. Figure 1(a) illustrates a domain with three holes and four stirrers.

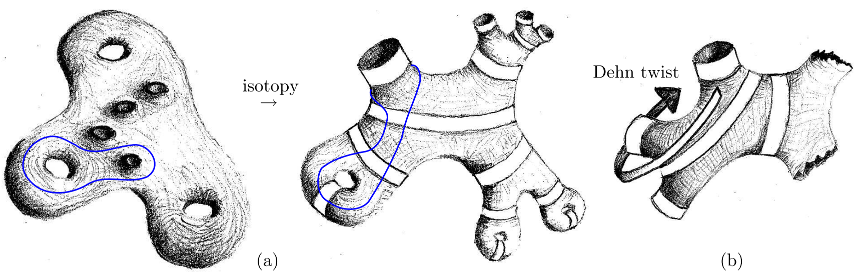


**Figure 1.** (a) A surface of genus three with four punctures continuously deformed to standard form and decomposed into pairs of pants; (b) The action of braid letters $\sigma_i$, $\tau_i$ and $\rho_i$ may be expressed in terms of fundamental transformations and ‘half’ and full Dehn twists.

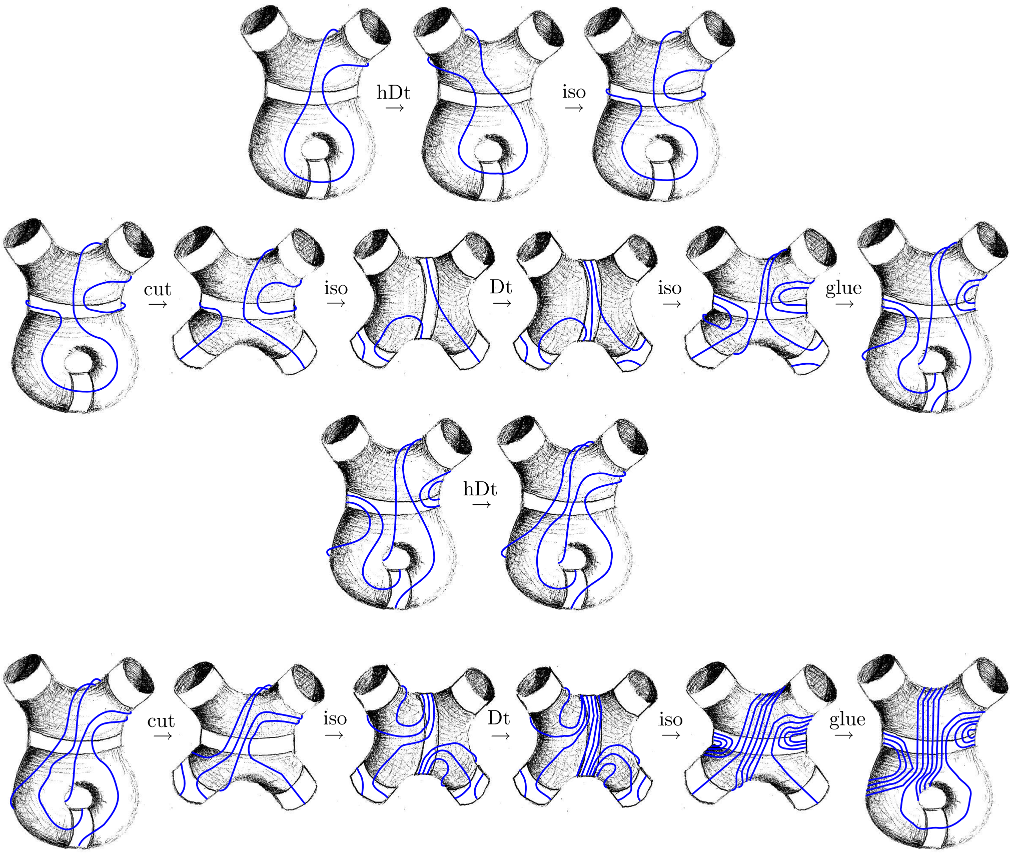


**Figure 2.** Deformation of a closed loop under the action of the silver braid $\sigma_1\rho_1^{-1}\sigma_1^{-1}\rho_1$ on a surface of genus one with two punctures. Each row corresponds to one braid letter, beginning with $\sigma_1$. The actions of $\sigma_1$ and $\sigma_1^{-1}$ are achieved by half Dehn twists (hDt). The motions around the hole, $\rho_1$ and $\rho_1^{-1}$, are achieved by making cuts, full Dehn twists (Dt), and gluing. To apply these Dehn twists it is necessary to perform an isotopy (iso) between the various steps to maintain the correct twisting conventions needed in the pants coordinate system [5].

Following the notation of Birman [3], the braiding motions on the surface are labelled $\sigma_i$, corresponding to interchanging the position of neighbouring punctures, and $\tau_i$ or $\rho_i$, corresponding to a puncture touring each hole (which can be done in two ways). To determine the action of braids on material loops we employ a pair-of-pants decomposition of the surface, and use Dehn–Thurston coordinates to encode loops, following the conventions used by Penner [5]. By Euler's formula exactly $p = 2g + n - 2$ pants are required for such a decomposition. To give a systematic way of computing braid actions it is convenient to perform first a continuous deformation of the surface to a standard form with a tree of punctures and a tree of holes. This is shown for our example surface in Figure 1(a).

Our main contribution is how to perform the action of the braid letters $\sigma_i$, $\tau_i$ and $\rho_i$. These actions are achieved through a sequence of Dehn twists (full twists around the boundary between two pants—see Figure 1(b)) and a special manoeuvre which may only be applied to pairs of punctures which amounts to a *half* Dehn twist. To compute these actions in Dehn–Thurston coordinates also involves Penner's two fundamental transformations [5], and some cutting and rejoining of the surface. The details are non-trivial, but fortunately the action of certain braids can be simplified by appealing to the braid group presentation [3].

To illustrate the process we show in Figure 2 the action of the braid $\sigma_1\rho_1^{-1}\sigma_1^{-1}\rho_1$ on a loop in a surface with genus $g = 1$ and $n = 2$ punctures. This corresponds to a flow on a domain with periodicity in one direction driven by two stirrers. The braid is readily realised with rotating gearing in a batch mixer and produces a large topological entropy of $\log(3 + 2\sqrt{2}) \approx 1.7627$ [6]. An entropy estimate based on iteration of our dynamical system converges to this quantity.

## References


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