

## SPEEDING UP MIXING WITH MOVING WALLS

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*Summary* Mixing in viscous fluids is challenging, but chaotic advection in principle allows efficient mixing. In the best possible scenario, the decay rate of the concentration profile of a passive scalar should be exponential in time. In practice, several authors have found that the no-slip boundary condition at the walls of a vessel can slow down mixing considerably, turning an exponential decay into a power law. This slowdown affects the whole mixing region, and not just the vicinity of the wall. The reason is that when the ergodic mixing region extends to the wall, a separatrix connects to it. The approach to the wall along that separatrix is polynomial in time and dominates the long-time decay. However, if we move the walls closed orbits appear, separated from the bulk by a hyperbolic fixed point with a homoclinic orbit. The long-time approach to the fixed point is exponential, so we recover an overall exponential decay, albeit with a thin unmixed region near the wall.

### THE FIGURE-EIGHT PROTOCOL

Figure (a) shows the result of a mixing experiment where a circular rod is moved slowly in a viscous flow following a ‘figure-eight’ trajectory. Three-dimensional effects are negligible, and the fluid (sugar syrup) can be treated as a Stokes flow. The passive scalar (darker fluid) is India ink. The rod stretches and folds an initial blob of ink until it filaments and diffusion takes over. A numerically-computed Poincaré section for this flow is shown in Fig. (b). It is evident that the phase space of trajectories consists of one large chaotic region, with no apparent islands. Furthermore, the chaotic region extends all the way to the wall. Two separatrices are also apparent, connected to the wall at the red dots. The upper separatrix corresponds to a separation point, and the lower one to a re-attachment point. Since the entire mixing domain consists of one ergodic component, we might expect that the decay rate for the concentration intensity would be exponential. This is not the case: recent work [1, 2, 3] has demonstrated that the slow dynamics associated with the no-slip boundary condition at the wall limit the rate of decay. Furthermore, we have shown experimentally and in a map [3] that this slow decay affects the entire mixing region, and is not limited to the vicinity of the wall. We will show why this is so by a simple model, and demonstrate that mixing can be accelerated by moving the walls.

### FLOW NEAR THE WALL

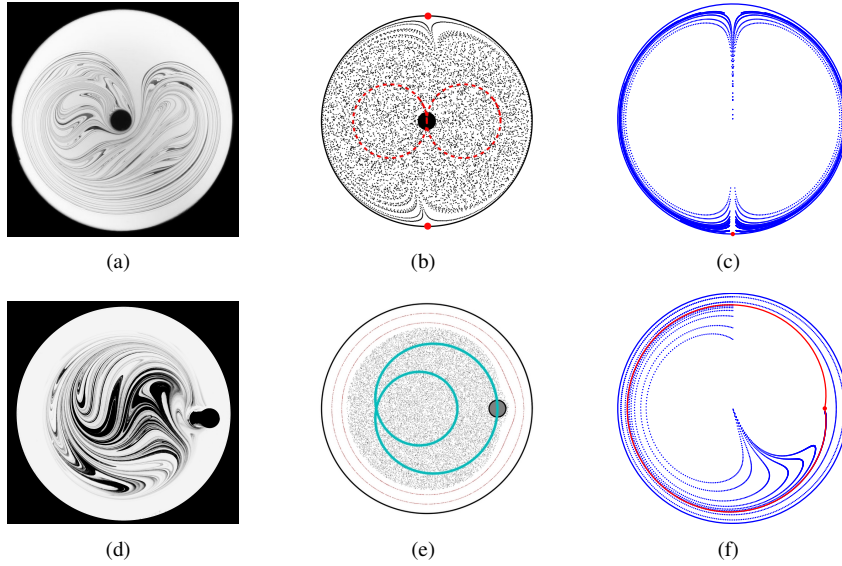
As can easily be seen in the Poincaré section (Fig. (b)), the ‘effective’ velocity field near the wall only reverses sign in two places, at separatrices connected to the wall. Near the wall, incompressibility and the no-slip boundary condition dictate

$$u = A(x)y + O(y^2), \quad v = -\frac{1}{2}A'(x)y^2 + O(y^3). \quad (1)$$

Here  $x$  is a periodic variable along the wall,  $0 \leq y \ll 1$  measures the distance from the wall, and  $u$  and  $v$  are the corresponding velocity components. The streamfunction for (1) is  $\psi(x, y) = \frac{1}{2}A(x)y^2 + O(y^3)$ , with  $(u, v) = (\partial_y\psi, -\partial_x\psi)$ . The streamfunction tells us the obvious fact that the wall  $y = 0$  is a streamline with  $\psi = 0$ . If two separatrices are to be connected to the wall, as is evident in Fig. (b), then we need  $\psi = 0$  for  $y > 0$ ; this can only happen at points  $x$  with  $A(x) = 0$ . We conclude that  $A(x)$  must have two zeros corresponding to the separatrices in Fig. (b). We choose  $x = 0$  to be the point where the lower separatrix is attached, and measure  $x$  counterclockwise around the wall. Every point  $(x, y) = (x_0, 0)$  on the wall is a parabolic fixed point. Near most of these points the dynamics are boring: particles just stream along following  $\dot{x} = A(x_0)y$ , and approach or recede from the wall depending on the sign of  $A'(x_0)$ . Eventually all trajectories leave the neighbourhood of  $(x_0, 0)$ . However, for the two values of  $x$  for which  $A(x)$  vanishes, we get separatrices. We focus on the separatrix at  $x = 0$ , where  $A(0) = 0$ , and Eq. (1) become

$$\dot{X} = A'(0)XY + O(X^2Y, Y^2), \quad \dot{Y} = -\frac{1}{2}A'(0)Y^2 + O(XY^2, Y^3), \quad (2)$$

with  $(x, y) = (0 + X, 0 + Y)$  and  $(X, Y)$  small expansion variables. Now the set  $\{X = 0, Y > 0\}$  is invariant for small  $Y$  and corresponds to the separatrix, which is the stable manifold of the fixed point  $(0, 0)$ . The evolution along the stable manifold is obtained by solving  $\dot{Y} = -\frac{1}{2}A'(0)Y^2$ , which gives  $Y(t) = Y_0/(1 + \frac{1}{2}A'(0)Y_0t)$ . For this to represent the stable manifold, we require  $A'(0) > 0$ . (The other separatrix exhibits finite-time escape to infinity, which takes particles away from the wall and into the bulk.) For long times, the rate of approach is  $Y(t) \sim (2/A'(0))t^{-1}$ . The asymptotic form



for  $Y(t)$  is algebraic and is independent of  $Y_0$ . The consequence of this independence is visible at the bottom of Fig. (a): material lines ‘bunch-up’ against each other faster than they approach the wall, thereby forgetting their initial position. Since  $A(x)$  must be periodic in  $x$  and have only two zeros, a simple model is to take  $A(x) = \sin x$ . In that case  $x_2 = 3\pi/2$ ,  $A(x_2) = -1$ , and  $A''(x_2) = 1$ . Figure (c) shows trajectories for this mock flow. Once a particle leaves the vicinity of the wall, its trajectory becomes meaningless, since our expansion (1) is only valid for small  $y$ . However, the cusp structure for  $U = 0$  in Fig. (c) is evident and is remarkably similar to Fig. (b).

### MOVING WALL

Now consider the case of a ‘rotating wall’, where we add a constant speed  $U > 0$  to the velocity  $u$  in (1). Again we look for fixed points: all the parabolic fixed points on the wall have disappeared, as well as the two separatrices. Since  $A(x)$  is continuous, has two zeros, and  $A'(0) > 0$ ,  $A(x)$  must have a maximum at  $x_1$  and a minimum at  $x_2$ , where  $A'(x_{1,2}) = 0$  and hence  $v(x_{1,2}, y) = 0$  for all  $y$ . Enforcing that the along-wall velocity also vanish, there will be fixed points at  $y_{1,2} = -U/A(x_{1,2})$ . Since  $A(x_1) > 0$  (maximum) and  $A(x_2) < 0$  (minimum), only  $x_2$  has  $y_2 \geq 0$ . The other fixed point lies outside our domain. Hence, we focus on the unique fixed point  $(x_2, -U/A(x_2))$ . Now we look at the linearised dynamics near the fixed point. Let  $(x, y) = (x_2 + X, -U/A(x_2) + Y)$ ; then

$$\dot{X} = A(x_2)Y + O(X^2, Y^2, XY), \quad \dot{Y} = -\frac{1}{2}A''(x_2)y_2^2 X + O(X^2, Y^2, XY). \quad (3)$$

The linearised motion thus has eigenvalues  $\lambda_{\pm} = \pm\lambda = \pm\sqrt{-A''(x_2)/2A(x_2)}U$  where the argument in the square root is nonnegative since  $A(x_2) < 0$  and  $A''(x_2) \geq 0$ . For  $A''(x_2) > 0$  and  $U > 0$ , this is a hyperbolic fixed point, and the approach along its stable manifold is given by  $Y(t) \sim Y_0 \exp(-\lambda t)$  for  $(X_0, Y_0)$  initially on the stable manifold. Compare this to the  $1/t$  approach for a fixed wall: the approach to the fixed point is now exponential, at a rate proportional to the speed of rotation of the wall. One expects that this exponential decay will dominate if it is slower than the mixing rate in the bulk. Otherwise, if  $\lambda$  is large enough, then the rate of mixing in the bulk dominates.

We do not have experimental results for a moving wall, but Fig. (d) shows a stirring protocol where the rod undergoes a ‘loop’ (epitochroid) motion. This creates closed trajectories near the wall, as is evident in the Poincaré section Fig. (e). We have verified experimentally the the decay of the passive scalar in this case is *exponential*, for the same reason as the moving wall. Indeed, Fig. (f) shows trajectories for a moving wall with  $U = 0.2$  and  $A(x) = \sin x$ . Compare this to Fig. (d): in both cases closed trajectories are evident near the wall. A separatrix, consisting of a homoclinic orbit connecting the hyperbolic fixed point to itself, isolates the wall region from the bulk. It is the approach to this separatrix that will limit the decay rate, and it is exponential in time. In a forthcoming publication, we will analyse the case of closed orbits with a fixed wall (as in Fig. (d)) in greater detail.

### References

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