PRE TEST - Answers

A careful algebraic computation consists of a string of identities each of which is justfied as a special case of a geberal principle in which one has confidence. By arranging the work systematically we can find errors more easily. I have redone the problems in this spirit indicating in the column on the right[1] which general principle I am applying.

(i)

step	reason
$\dfrac{x^2+(-1)}{x+1} = \dfrac{x^2-1}{x+1}$	$a+(-b)=a-b$
$= \dfrac{(x+1)(x-1)}{(x+1)}$	$a^2-b^2=(a+b)(a-b)$
$= x-1$	$ab/a=b$
$= \dfrac{x^2}{x}+\dfrac{-1}{1}$	$a=a^2/a, \quad -1=(-1)/1$

It is interesting that all the expressions in the original computation appear here. Was the original incorrect? It contained the line

$$(*) \qquad \frac{x^2+(-1)}{x+1} = \frac{x^2}{x}+\frac{-1}{1}$$

which appears to be an application of the incorrect rule $(a+b)/(c+d) = a/b+c/d$. Any instructor who saw equation $(*)$ in a student's paper would be convinced the student was confused, even though the equation is valid. After all, $64/16 = 4$ but usually you can't cancel the 6's.

(ii)

step	reason
$(x+y)^2-(x-y)^2$	
$= (x^2+2xy+y^2)-(x^2-2xy+y^2)$	$(a\pm b)^2 = a^2 \pm 2ab + b^2$
$= x^2+2xy+y^2-x^2+2xy-y^2$	$A-(a+b+c)=A-a-b-c$
$= 4xy.$	

[1]We don't expect you to do algebra in this much detail on an exam, but you should do it carefully.

In the last line we have used all the laws off addition: $(a+b)+c = a+(b+c)$, $a+b = b+a$, $a+(-b) = a-b$, $a-a = 0$, and $a+0 = a$.

(iii)

$$\frac{9(x-4)^2}{3x-12} = \frac{3^2(x-4)^2}{3x-12} = \frac{(3x-12)^2}{3x-12} = 3x-12.$$

Reasons: $9 = 3^2$, $a^2b^2 = (ab)^2$, $a^2/a = a$.

(iv)

$$\frac{x^2y^5}{2x^{-3}} = \frac{x^2y^5x^3}{2} = \frac{x^5y^5}{2}.$$

Reasons: $a/(bc^{-p}) = ac^p/b$ and $a^pba^q = a^{p+q}b$.

(v)

$$\begin{aligned} &\frac{(2x^3+7x^2+6)-(2x^3-3x^2-17x+3)}{(x+8)+(x-8)} \\ &\quad = \frac{(2-2)x^3+(7+3)x^2+(0+17)+(6-3)}{2x+(8-8)} \\ &\quad = \frac{10x^2+17x+3}{2x}. \end{aligned}$$

The key step in (v) involves collecting powers of x. I broke it up into two steps to emphasize the idea.

(vi)

step	reason
$\dfrac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$	
$= \dfrac{(x^{-1}+y^{-1})xy}{(x^{-1}-y^{-1})xy}$	$a/b = (ac)/(bc)$
$= \dfrac{y+x}{y-x}$	$(a+b)c = ac+bc.$

In the last step we also used the laws $ab = ba$, $a(bc) = (ab)c$, $a^{-1}a = 1$, and $1 \cdot a = a$.

How many of you have noticed the analogy between the laws of addition and the laws of multiplication?

$a + b = b + a$	$ab = ba$
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
$a + 0 = a$	$a \cdot 1 = a$
$a + (-a) = 0$	$a \cdot a^{-1} = 1$
$a - b = a + (-b)$	$a/b = a \cdot b^{-1}$
$a - b = (a + c) - (b + c)$	$a/b = (ac)/(bc)$
$(a - b) + (c - d) = (a + c) - (b + d)$	$(a/b) \cdot (c/d) = (ac)/(bd)$
$(a - b) - (c - d) = (a + d) - (b + c)$	$(a/b)/(c/d) = (ad)/(bc)$

The last line explains why *we invert and multiply to divide fractions.*