Math 221 – 1st Semester Calculus

Lecture Notes for Fall 2006.

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All references to “Thomas” or “the textbook” in these notes refer to

Thomas’ CALCULUS 11th edition

published by Pearson Addison Wesley in 2005.

These notes may be downloaded from

`http://www.math.wisc.edu/~robbin/221dir/lecs-221.pdf`.

Some portions of these notes are adapted from

`http://www.math.wisc.edu/~angenent/Free-Lecture-Notes/`.

Some problems come from a list compiled by Arun Ram, others come from the WES program, and others come from the aforementioned Thomas text or from Stewart, *Calculus (Early Transcendentals)*, 3rd Edition.

Contents

Chapter I

What you need to know to take Calculus 221

In this chapter we will review material from high school mathematics. We also teach some of this material at UW in Math 112, Math 113, and Math 114, and some of it will be reviewed again as we need it. This material should be familiar to you. If it is not, you may not be ready for calculus. Pay special attention to the definitions. Important terms are shown in **boldface** *when they are first defined.*

1 Algebra

This section contains some things which should be easy for you. (If they are not, you may not be ready for calculus.)

§1.1. Answer these questions.

1. Factor $x^2 - 6x + 8$.
2. Find the values of x which satisfy $x^2 - 7x + 9 = 0$. (Quadratic formula.)
3. $x^2 - y^2 =$? Does $x^2 + y^2$ factor?
4. True or False: $\sqrt{x^2 + 4} = x + 2$?
5. True or False: $(9x)^{1/2} = 3\sqrt{x}$?
6. True or False: $\dfrac{x^2 x^8}{x^3} = x^{2+8-3} = x^7$?
7. Find x if $3 = \log_2(x)$.
8. What is $\log_7(7^x)$?
9. True or False: $\log(x + y) = \log(x) + \log(y)$?
10. True or False: $\sin(x + y) = \sin(x) + \sin(y)$?

§1.2. There are conventions about the order of operations. For example,

$$
\begin{aligned}
ab + c \qquad & \text{means } (ab) + c \text{ and not } a(b + c),\\
\frac{a}{\frac{b}{c}} \qquad & \text{means } a/(b/c) \text{ and not } (a/b)/c,\\
\frac{\frac{a}{b}}{c} \qquad & \text{means } (a/b)/c \text{ and not } a/(b/c),\\
\log a + b \qquad & \text{means } (\log a) + b \text{ and not } \log(a + b).
\end{aligned}
$$

If necessary, we use parentheses to indicate the order of doing the operations.

§1.3. There is analogy between the laws of addition and the laws of multiplication:

$a + b = b + a$	$ab = ba$
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
$a + 0 = a$	$a \cdot 1 = a$
$a + (-a) = 0$	$a \cdot a^{-1} = 1$
$a - b = a + (-b)$	$a/b = a \cdot b^{-1}$
$a - b = (a + c) - (b + c)$	$a/b = (ac)/(bc)$
$(a - b) + (c - d) = (a + c) - (b + d)$	$(a/b) \cdot (c/d) = (ac)/(bd)$
$(a - b) - (c - d) = (a + d) - (b + c)$	$(a/b)/(c/d) = (ad)/(bc)$

The last line explains why *we invert and multiply to divide fractions.* The only other law of arithmetic is the distributive law

$$(a + b)c = ac + bc, \qquad c(a + b) = ca + cb.$$

Note that

$$(a + b)/c = (a/c) + (b/c), \quad \text{but}^1 \quad c/(a + b) \neq (c/a) + (c/b).$$

Exercises

Exercise 1.4. Answer the questions in §1.1

Exercise 1.5. Here is a list of some algebraic expressions that have been "simplified." Some steps in the simplification processes are correct and some of them are WRONG! For each problem:

1. Determine if the simplified result is correct.
2. Determine if there are any mistakes made in the simplication process. (NOTE: just because the result is correct does not mean there are no mistakes).
3. If there are mistakes, redo the problem correctly. If there are no mistakes, redo the problem with another correct method.

(i)
$$\frac{x^2 - 1}{x + 1} = \frac{x^2 + (-1)}{x + 1} = \frac{x^2}{x} + \frac{-1}{1} = x - 1$$

(ii)
$$(x + y)^2 - (x - y)^2 = x^2 + y^2 - x^2 - y^2 = 0$$

[1] usually

(*iii*) $$\frac{9(x-4)^2}{3x-12} = \frac{3^2(x-4)^2}{3x-12} = \frac{(3x-12)^2}{3x-12} = 3x-12$$

(*iv*) $$\frac{x^2y^5}{2x^{-3}} = x^2y^5 \cdot 2x^3 = 2x^6y^5$$

(*v*) $$\frac{(2x^3+7x^2+6)-(2x^3-3x^2-17x+3)}{(x+8)+(x-8)} = \frac{4x^2-17x+9}{2x} = 2x-17+\frac{9}{2x}$$

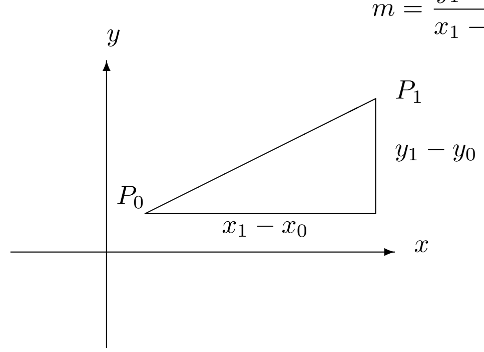
(*vi*) $$\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}} = \frac{(x+y)^{-1}}{(x-y)^{-1}} = \left(\frac{x+y}{x-y}\right)^{-1} = -\frac{x+y}{x-y} = \frac{x+y}{y-x}.$$

2 Coordinate Geometry

The material in this section is crucial for understanding calculus. It is reviewed in Thomas pages 9-18, but if you have not seen it before you may not be ready for calculus.

§2.1. Points and Slope. The notation $P(x, y)$ is used as an abbreviation for the more cumbersome phrase "the point P whose coordinates are (x, y)." The **slope** of the line through the distinct points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ is

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$



§2.2. Point-Slope Equation of a Line. If we use a different pair of points on the line to compute the slope, we get the same answer. Hence, a point $P(x, y)$ lies on the line P_0P_1 if and only if we get the same answer for the slope when we use (x, y) in place of (x_1, y_1):

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = m.$$

This equation has one minor flaw; it doesn't work when $(x, y) = (x_0, y_0)$ (never divide by zero). To remedy this multiply by $(x - x_0)$ and add y_0 to both sides:

$$y = y_0 + m(x - x_0).$$

This is the equation for the *line through $P_0(x_0, y_0)$ with slope m*; this form makes it obvious that the point $P_0(x_0, y_0)$ lies on the line. For example, the equation for the line through $P_0(2, 3)$ and $P_1(4, 11)$ is

$$\frac{y-3}{x-2} = \frac{11-3}{4-2} = 4, \qquad \text{or} \qquad y = 3 + 4(x-2),$$

See Thomas page 11.

§2.3. If a line has slope m, then any line perpendicular to it has slope $-1/m$. (Proof: The slope of a line is the tangent of the angle it makes with the x-axis and $\tan(\phi + \frac{\pi}{2}) = -1/\tan(\phi)$.

§2.4. Sometimes we use letters other than x and y. For this reason, always label the axes when you draw a graph. For example the line $2u + 3v = 1$ can be written as $v = -\frac{2}{3}u + \frac{1}{3}$ and as $u = -\frac{3}{2}v + \frac{1}{2}$. If the u-axis is horizontal, the slope is $-\frac{2}{3}$. If the v-axis is horizontal, the slope is $-\frac{3}{2}$. The slope is always the "rise over the run", i.e. to find the slope, take two points on the line, subtract the vertical coordinates (this is the "rise"), subtract the horizontal coordinates (this is the "run"), and divide the rise by the run.

§2.5. Distance and Circles. By the Pythagorean theorem the **distance** from $P_0(x_0, y_0)$ to $P_1(x_1, y_1)$ is

$$|P_0P_1| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

The distance from a point $P(x, y)$ to the point $P_0(x_0, y_0)$ is exactly a when $|P_0P|^2 = a^2$, i.e.

$$(x - x_0)^2 + (y - y_0)^2 = a^2.$$

This is the equation for the **circle** *centered at P_0 with radius a.* For example, the equation of the circle of radius 5 centered at $P_0(2, 3)$ is

$$(x-2)^2 + (y-3)^2 = 5^2, \qquad \text{or} \qquad x^2 + y^2 - 4x - 6y - 12 = 0.$$

The latter equation has the form of a general 2nd degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(with $A = C = 1$, $B = 0$, $D = -4$, $E = -6$, $F = -12$). Ellipses (including circles), parabolas, and hyperbolas all have equations of this form.

Exercises

Exercise 2.6. Let L_1 and L_2 be two lines. An equation for L_1 is $6x + 4y = 7$. The line L_2 is perpendicular to L_1 and goes through the point $P_0(-1, 0)$. Find an equation for the line L_2 and the point where the two lines intersect.

Exercise 2.7. Repeat the previous exercise replacing $6x+4y = 7$ by $ax+by = c$ and $P_0(-1, 0)$ by $P_0(x_0, y_0)$.

Exercise 2.8. Find the center $P_0(x_0, y_0)$ and radius a of the circle $x^2 + y^2 - 2x + 8y - 20 = 0$.

Exercise 2.9. Find the center $P_0(x_0, y_0)$ and radius a of the circle $x^2 + y^2 + Ax + By + C = 0$.

Exercise 2.10. Alice and Bob each graph the same line but Alice chooses one of the variables to label the horizontal axes and Bob the other. What is the relation between their values for the slope?

3 Functions

> *This material will be reviewed as we need it, but you should have seen it before. It is explained in Thomas pages 19-27 .*

Definition 3.1. A **function** is a rule which produces an output $f(x)$ from an input x. The set of inputs x for which the function is defined is called the **domain** and $f(x)$ (pronounced "f of x") is the **value** of f at x. The set of all possible outputs $f(x)$ as x runs over the domain is called the **range** of the function.

§3.2. If a function $f(x)$ is given by an expression in the variable x and the domain is not explicitly specified the domain is understood to be the set of all x for which the expression is meaningful. For example, for the function $f(x) = 1/x^2$ the domain is the set of all nonzero real numbers x (the value $f(0)$ is not defined because we don't divide by zero) and the range is the set of all positive real numbers (the square of any nonzero number is positive). The domain and range of the square root function $\sqrt{x}$ is the set of all nonnegative numbers x. The domain of the function $y = \sqrt{1 - x^2}$ is the interval $[-1, 1]$ and the range is the interval $[0, 1]$, i.e. $\sqrt{1 - x^2}$ is meaningful only if $-1 \le x \le 1$ (otherwise the input to the square root function is negative) and $0 \le \sqrt{1 - x^2} \le 1$.

Definition 3.3. The **graph** of a function

$$y = f(x)$$

is the set of all points $P(x, y)$ whose coordinates (x, y) satsfy the equation $y = f(x)$. More generally, an equation of form

$$F(x, y) = 0$$

determines a set (graph) in the (x, y)-plane consisting of all points $P(x, y)$ whose coordinates (x, y) satisfy the equation. The graph of a function $y = f(x)$ is a special case: take $F(x, y) = y - f(x)$. To decide if a set is the graph of a function we apply the

Vertical Line Test. *A set in the (x, y)-plane is the graph of a function if and only if every vertical line $x =$ constant intersects the graph in at most one point. [If the number a is in the domain, the vertical line $x = a$ intersects the graph $y = f(x)$ in the point $P(a, f(a))$.]*

§3.4. For example, the graph of the equation $x^2 + y^2 = 1$ is a circle; it is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. This graph is however the union of two different graphs each of which is the graph of a function:[2]

$$x^2 + y^2 = 1 \iff y = \sqrt{1 - x^2} \quad \text{or} \quad y = -\sqrt{1 - x^2}.$$

§3.5. Usually a function is defined by giving a formula as in $f(x) = \sqrt{1 - x^2}$. Sometimes several formulas are used as in the definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

of the **absolute value function**.

Remark 3.6. In calculus, we learn to reason about a function even when we cannot find an explicit formula for it. For example, in theory the equation $x = y^5 + y$ (which has the form $x = g(y)$) can be rewritten in the form $y = f(x)$ but there is no formula for $f(x)$ involving the mathematical operations studied in high school. (Your TA will learn this in Math 742.)

Definition 3.7. When two functions are related by the condition[3]

$$y = f(x) \iff x = g(y)$$

(for all appropriate x and y) we say that the functions are **inverse** to one another and write $g = f^{-1}$. The range of f is the domain of g and the range of g is the domain of f. Often a function f has an inverse function g only after we modify f by artificially restricting its domain. The following table lists some

[2]See Thomas Figure 1.28 page 4.

[3]The notation $\iff$ is an abbreviation for "if and only if".

common functions (suitably restricted) and their inverses.

$y = f(x)$		$x = f^{-1}(y)$	
$y = y_0 + m(x - x_0)$		$x = x_0 + (y - y_0)/m$	
$y = x^2$	$(x \geq 0)$	$x = \sqrt{y}$	$(y \geq 0)$
$y = a^x$		$x = \log_a(y)$	$(y > 0)$
$y = \sin\theta$	$(-\pi/2 \leq \theta \leq \pi/2)$	$\theta = \sin^{-1}(y)$	$(-1 \leq y \leq 1)$
$x = \cos\theta$	$(0 \leq \theta \leq \pi)$	$\theta = \cos^{-1}(x)$	$(-1 \leq x \leq 1)$
$u = \tan\theta$	$(-\pi/2 < \theta < \pi/2)$	$\theta = \tan^{-1}(u)$	

(The notations $\arcsin y = \sin^{-1} y$, $\arccos x = \cos^{-1} x$, and $\arctan u = \tan^{-1} u$ are also commonly used for the inverse trigonometric functions. The exponential function $y = a^x$ is only defined for a positive.) To decide if a function has an inverse we apply the

Horizontal Line Test. *A function has an inverse if and only if every horizontal line* $y =$ constant *intersects the graph in at most one point. [The horizontal line $y = b$ intersects the graph $y = f(x)$ in the point $P(f^{-1}(b), b)$.]*

Remark 3.8. To find an expression for $f^{-1}(y)$ solve the equation $y = f(x)$ for x in terms of y. (It is not always possible to find a nice formula.)

Remark 3.9. Two kinds of notation are used in calculus which I call **functional notation** and **variable notation**. In functional notation the function has a name, usually f or g, and is defined by an equation of form

$$f(x) = \text{ some formula in } x.$$

Changing the variable in this notation does not change the function, e.g. the function g defined by the formula

$$g(t) = t^2$$

is the same as the function g defined by the formula

$$g(x) = x^2.$$

(Both formulas give $g(3) = 9$.) In variable notation we write

$$y = x^2$$

and say that y is a function of x. I like to write expressions like

$$y\Big|_{x=3} = 9$$

to indicate that the value of y is 9 when the value of x is 3. The advantage of functional notation is that it is makes clear where a function is being evaluated. The disadvantage is that the notation tends to be cumbersome since we have to introduce a name for each function we consider. I like functional notation for proofs and variable notation for problems.

Exercises

Exercise 3.10. Let $f(x) = \dfrac{x-2}{x+4}$.

(i) Find the domain and range of f.

(ii) Find the domain and range of f^{-1}.

(iii) Find a formula for $f^{-1}(y)$.

Exercise 3.11. Repeat the previous exercise with $f(x) = \dfrac{ax+b}{cx+d}$ where a, b, c, d are constants with $ad - bc \neq 0$.

Exercise 3.12. What is $\sqrt{x^2}$? $(\sqrt{x})^2$?

Exercise 3.13. Draw the graph $y = f(x)$ where $f(x)$ is the function defined by

$$f(x) = \begin{cases} 3x & \text{for } 0 \leq x < 1; \\ 4x - 1 & \text{for } 1 \leq x. \end{cases}$$

Give a formula (like the formula for $f(x)$) for the inverse function $x = f^{-1}(y)$.

4 Trigonometry

There is a review of trigonometry in Thomas pages 48-58 but you should already be familiar with this material. You should also have seen the inverse trigonometric functions before. These are reviewed in Thomas section 7.7 (page 517) and we'll review it again later in the course.

§4.1. In calculus we always measure angles in radians rather than degrees. The radian measure of an angle is the arclength of a circle of radius one (centered at the vertex of the angle) cut out by the angle. Since the total length of the circumference of a circle is 2π we get

$$2\pi \text{ radians} = 360^o.$$

(Since $2\pi = 6.283\ldots$ is about 6, this means that one radian is a little less than 60 degrees.) Let $O(0,0)$ denote the origin, $P(x,y)$ be a point of the (x,y) plane distinct from the origin, $r = |OP|$ be the distance from O to P, and θ denote the angle between the positive x-axis and the ray OP. Then

$$\sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x} \qquad \sec\theta = \frac{r}{x}$$
$$\cos\theta = \frac{x}{r} \qquad \cot\theta = \frac{x}{y} \qquad \csc\theta = \frac{r}{y}$$

Since r is positive ($\sqrt{\cdots}$ always means the positive square root) these formulas make it easy to remember the symmetries

$$\sin(-\theta) = -\sin(\theta), \qquad \cos(-\theta) = \cos\theta,$$

(which say that the sine is an odd function and the cosine is an even function) and the sign reversals

$$\sin(\theta+\pi) = -\sin\theta, \qquad \cos(\theta+\pi) = -\cos\theta.$$

All the trigonometric functions have period 2π:

$$f(\theta+2\pi) = f(\theta), \qquad f = \sin, \cos, \tan, \cot, \sec, \csc\,.$$

Because of the above sign reversal formulas for the sine and cosine and the equations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

the tangent and cotangent have period π:

$$\tan(\theta+\pi) = \tan\theta, \qquad \cot(\theta+\pi) = \cot\theta.$$

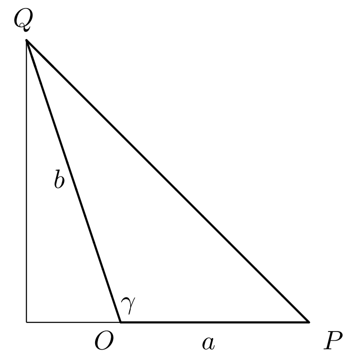
The **co**function of an angle is the function of its **co**mplement:

$$\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right), \qquad \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right), \qquad \csc\theta = \sec\left(\frac{\pi}{2}-\theta\right).$$

The trigonometric addition formulas are

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Here is a proof of the addition formula for the sine function. The area of a triangle $\triangle POQ$ is

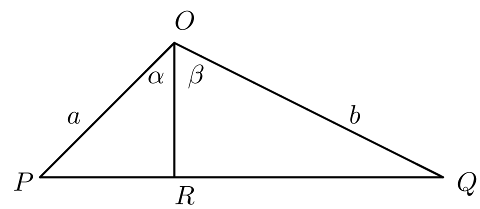


$$\text{Area}\,\triangle POQ = \frac{ab}{2}\sin\gamma$$

where $\gamma = \angle POQ$, $a = |OP|$ is the length of the base, and $b = |OQ|$ so $b\sin\gamma$ is the altitude. In the picture at the right the angle γ is obtuse but this makes no difference since

$$\sin\gamma = \sin(\pi - \gamma)$$

Turn the picture on its side and drop the altitude OR from the vertex O to the (new) base PQ. Define



$$\alpha = \angle POR, \qquad \beta = \angle ROQ,$$

so

$$\alpha + \beta = \angle POQ = \gamma$$

The (new) altitude is

$$h = |OR| = a\cos\alpha = b\cos\beta$$

The areas of the right triangles are

$$\text{Area}\,\triangle POR = \frac{1}{2}(a\sin\alpha)h = \frac{ab}{2}\sin\alpha\cos\beta,$$

and

$$\text{Area}\,\triangle ROQ = \frac{1}{2}(b\sin\beta)h = \frac{ab}{2}\cos\alpha\sin\beta.$$

From above

$$\text{Area}\,\triangle POQ = \frac{ab}{2}\sin\gamma = \frac{ab}{2}\sin(\alpha+\beta)$$

But

$$\text{Area}\,\triangle POQ = \text{Area}\,\triangle POR + \text{Area}\,\triangle ROQ.$$

Substituting and dividing by $ab/2$ gives the trigonometric addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

for the sine. The trigonometric addition formula for the cosine follows easily from the formula for the sine and the principle that the cosine of an angle is the sine of its complement:

$$\begin{aligned}
\cos(\alpha+\beta) &= \sin\bigl(\tfrac{\pi}{2}-(\alpha+\beta)\bigr) \\
&= \sin\bigl((\tfrac{\pi}{2}-\alpha)+(-\beta)\bigr) \\
&= \sin(\tfrac{\pi}{2}-\alpha)\cos(-\beta)+\cos(\tfrac{\pi}{2}-\alpha)\sin(-\beta) \\
&= \cos\alpha\cos\beta-\sin\alpha\sin\beta.
\end{aligned}$$

Exercises

Exercise 4.2. Let $u = \tan\alpha$ be the tangent of an acute angle. Express $\sin\alpha$, $\cos\alpha$, $\cot\alpha$, $\sec\alpha$, and $\csc\alpha$ as functions of u. (Hint: Draw a right triangle whose legs are 1 and u.)

Exercise 4.3. Prove that

$$\sin(\theta+\pi) = -\sin\theta, \qquad \cos(\theta+\pi) = -\cos\theta, \qquad \tan(\theta+\pi) = \tan\theta.$$

Exercise 4.4. Prove the trigonometric addition formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

for the tangent function. (Hint: the tangent is the sine divided by the cosine.)

Exercise 4.5. True or false: $\sin^2(x) = (\sin x)^2$? $\sin^{-1}(x) = (\sin x)^{-1}$?

5 Additional Exercises

Exercise 5.1. Find an equation for the set of points in the plane which are equidistant from the points $P_1(1,3)$ and $P_2(2,4)$. What is the geometric shape of this set?

Exercise 5.2. Find an equation for the set of points in the plane which are equidistant from the point $P_0(3,1)$ and the line $y = 2$. What is the geometric shape of this set?

Exercise 5.3. Find an equation for the set of points in the plane whose distance to $P_1(1,0)$ is twice their distance to $P_2(3,0)$. What is the geometric shape of this set?

Exercise 5.4. Find the distance from the point $P_0(4,3)$ to the line $y = 3x+1$.

Exercise 5.5. Find the distance from the point $P_0(x_0,y_0)$ to the line $y = mx+b$.

Exercise 5.6. **(i)** Draw the graph of $y = \sin\theta$ with the θ-axis horizontal and the y-axis vertical.

(ii) Thicken that part of the graph where $-\frac{\pi}{2} \le \theta\frac{\pi}{2}$. This is the graph of $\theta = \sin^{-1}(y)$.

(iii) Of course, $\sin^{-1}(\sin\theta) = \theta$ for $-\frac{\pi}{2} \leq \theta\frac{\pi}{2}$ but what is $\sin^{-1}(\sin 10)$? Hint: $\sin(\theta + 2\pi) = \sin(\pi - \theta) = \sin\theta$ and $3.1 < \pi < 3.2$.

Exercise 5.7. A revolving beacon from a lighthouse shines on the straight shore, and the closest point on the shore is a pier one half mile from the lighthouse. Let θ denote the positive acute angle between the shore and the beam of light. Write the distance from the pier to the point where the light shines on the shore as a function of θ.

Exercise 5.8. Your town is threatened by flooding and must build a levy. The cost C (in millions of dollars) of building a levy is a linear function $C = f(H)$ of the height H (in feet) of the levy. (This means that the graph of $C = f(H)$ is a straight line.) A one foot high levy costs \$2.5 million to build; a half foot high levy costs \$1.5 million to build. The state government and the federal government will subsidize the project in different ways. The state will pay \$400,000 (\$0.4 million) independent of the height, whereas the feds will pay \$1.5 million per foot. Denote by $f_1(H)$ the net cost if the state subsidy is used, $f_2(H)$ the net cost if the federal subsidy is used, and $f_3(H)$ the net cost if both subsidies are used. Find expressions for all three functions, $f_1(H)$, $f_2(H)$, $f_3(H)$. Draw and label all their graphs (on the same axes). Show where the graphs meet the C axis. (The horizontal axis should be labeled as the H-axis and the vertical axis should be labeled as the C-axis.)

Chapter II

Limits

6 Tangent and Velocity

This section is a warmup for calculus. It roughly corresponds to Thomas pages 73-83 and 134-140.

§6.1. Why is division by zero undefined? If $a/a = 1$ why doesn't $0/0 = 1$? *First answer: the laws of algebra would fail.* For example,

$$\frac{a \times b}{a} = b$$

if $a \neq 0$. If this were to work with $a = 0$ we could prove $2 = 3$ as follows: $0 \times 2 = 0 = 0 \times 3$ so

$$2 = \frac{0 \times 2}{0} = \frac{0}{0} = \frac{0 \times 3}{0} = 3!??!?$$

Second answer: Division would not be continuous. If $a \neq 0$ then $y/x \approx b/a$ when $y \approx b$ and $x \approx a$ as in

$$y = 8.001 \approx 8 \text{ and } x = 1.995 \approx 2 \qquad \Longrightarrow \qquad \frac{y}{x} = \frac{8.001}{1.995} \approx \frac{8}{2} = 4.$$

(The notation $x \approx a$ means x is 'approximately equal to a.') But there is no number which is approximately equal to y/x when $y \approx 0$ and $x \approx 0$. For example,

$$y = 0.01 \approx 0 \text{ and } x = 0.0001 \approx 0 \qquad \Longrightarrow \qquad \frac{y}{x} = \frac{0.0001}{0.01} = 0.01,$$

but

$$y = 0.0001 \approx 0 \text{ and } x = 0.01 \approx 0 \qquad \Longrightarrow \qquad \frac{y}{x} = \frac{0.01}{0.0001} = 100.$$

§6.2. The points $P(a, a^2)$ and $Q(x, x^2)$ lie on the parabola $y = x^2$. Imagine that these points are distinct but very close to each other, say $(a, a^2) = (3, 9)$ and $(x, x^2) = (3.01, 9.0601)$. The slope of the "secant" line joining P and Q is

$$m_{PQ} = \frac{\Delta y}{\Delta x}$$

where

$$\Delta y = x^2 - a^2$$

is the difference between the vertical coordinate x^2 of Q and the vertical coordinate a^2 of P and

$$\Delta x = x - a$$

is the difference of the horizontal coordinates of P and Q. Since $\Delta y = x^2 - a^2 = (x-a)(x+a)$ we get

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{x^2 - a^2}{x - a} = \frac{(x-a)(x+a)}{x-a} = x + a.$$

As x gets closer and closer to a, the point Q gets closer and closer to the point P and the secant line PQ gets closer and closer to the tangent line to the graph $y = x^2$ at the point P. The slope of the tangent line is the limiting value

$$m_P = \lim_{Q \to P} m_{PQ}$$

of the slope of the secant line as the point Q gets closer and closer to the point P. By the above formula for m_{PQ}, this limit is

$$m_P = \lim_{x \to a} (x + a) = a + a = 2a.$$

We will learn to write

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{x \to a} \frac{\Delta y}{\Delta x}$$

for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ on that curve.

And now for something completely different.

(But it's not.)

§6.3. What is the difference between average velocity and instantaneous velocity? Consider a car moving on a highway. When we go from Madison to Milwaukee along highway I-94 the mile posts go from 240 to 300. If we complete the trip in one hour, our average velocity is 60 miles per hour, but the speedometer gives the instantaneous velocity and this will vary over the course of the trip. Suppose we leave Madison at one PM and arrive at Milwaukee at two PM and the quantity $s = f(t)$ is our position at time t as determined by the mileposts. (Imagine that there is a milepost every few feet.) Thus $240 \le f(t) \le 300$ for $1 \le t \le 2$. In a short time Δt we move a distance $\Delta s = f(t + \Delta t) - f(t)$. Our **average velocity** over the tiny time interval from t to $t + \Delta t$ is

$$v_{\mathrm{av}}(t, t + \Delta t) = \frac{\Delta s}{\Delta t}.$$

If Δt is very small (say Δt = one second) this is the speedometer reading at time t. This is (almost exactly the same as) the **instantaneous velocity**

$$v_{\text{inst}}(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

7 Limits

The material in this section of these notes is treated in Thomas pages 84-90, 102-114, and 115-123. The precise definition of the notation $\lim_{x\to a} F(x) = L$ is given in Thomas section 2.3 page 91, but we will deemphasize it. If you want to learn it, you are probably a very good student. Come to my office hour. In the lectures we will use the informal definitions of this section.

§7.1. The notation

$$\lim_{x\to a} F(x) = L$$

is read "the limit of $F(x)$ as x approaches a is L." It means $F(x)$ gets closer and closer to L as x gets closer and closer to a. Sometimes the textbook writes this as

$$F(x) \to L \qquad \text{as} \qquad x \to a;$$

this is read as "$F(x)$ approaches L as x approaches a" or "$F(x)$ goes to L as x goes to a". I like to explain limits by writing

$$F(x) \approx L \quad \textit{when} \quad x \approx a \qquad (\text{but } x \neq a).$$

The notation $A \approx B$ means A is approximately equal to B. In Chapter III we will study a limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

called the *derivative*. The derivative is a limit, but the concept of limit is more general. Note that in the definition of limit we consider values of x close to a but never plug in $x = a$. In the interesting examples (like the derivative) setting $x = a$ in $F(x)$ leads to something undefined like $0/0$.

§7.2. In the notation

$$\lim_{x\to a} F(x) = L$$

either a or L or both can be either ∞ or $-\infty$. The notation $x \approx \infty$ means x is large and positive (like $x = 1000000$), and the notation $x \approx -\infty$ means x is large and negative (like $x = -1000000$). We also use one sided limits: The notation

$$\lim_{x\to a+} F(x) = L$$

means $F(x) \approx L$ when $x \approx a$ but $x > a$. The notation

$$\lim_{x\to a-} F(x) = L$$

means $F(x) \approx L$ when $x \approx a$ but $x < a$. (Think $a+$ means $a + 0.0000000001$ and $a-$ means $a - 0.0000000001$.)

§7.3. (**Limit Laws**)[4] If c is a constant, then

$$(I) \qquad \lim_{x\to a} c = c,$$

and

$$(II) \qquad \lim_{x\to a} x = a.$$

These laws are obvious. Assume that the limits

$$\lim_{x\to a} F_1(x) = L_1, \qquad \lim_{x\to a} F_2(x) = L_2,$$

exist and are finite. Then

$$(III) \qquad \lim_{x\to a}\big(F_1(x) + F_2(x)\big) = \lim_{x\to a} F_1(x) + \lim_{x\to a} F_2(x),$$

$$(IV) \qquad \lim_{x\to a}\big(F_1(x) - F_2(x)\big) = \lim_{x\to a} F_1(x) - \lim_{x\to a} F_2(x),$$

$$(V) \qquad \lim_{x\to a}\big(F_1(x) \cdot F_2(x)\big) = \Big(\lim_{x\to a} F_1(x)\Big) \cdot \Big(\lim_{x\to a} F_2(x)\Big),$$

and, if $\lim_{x\to a} F_2(x) \neq 0$,

$$(VI) \qquad \lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{\lim_{x\to a} F_1(x)}{\lim_{x\to a} F_2(x)}.$$

In other words the limit of the sum is the sum of the limits, etc. These Laws are true because if $F_1(x) \approx L_1$ and $F_2(x) \approx L_2$, then

$$F_1(x)+F_2(x) \approx L_1+L_2, \qquad F_1(x)-F_2(x) \approx L_1-L_2, \qquad F_1(x)\cdot F_2(x) \approx L_1\cdot L_2,$$

and (if $L_2 \neq 0$)

$$\frac{F_1(x)}{F_2(x)} \approx \frac{L_1}{L_2}.$$

§7.4. The Limit Laws say that you can evaluate limits by 'just plugging in' *so long as 'just plugging in' doesn't give you something undefined like* $0/0$ *or* ∞/∞. For example,

$$\lim_{x\to 2} \frac{x^3-1}{x^2-1} = \frac{2^3-1}{2^2-1} = \frac{8-1}{4-1} = \frac{7}{3} \qquad (\heartsuit)$$

but if $x \to 2$ is replaced by $x \to 1$ we can't just plug in $x = 1$ because we will get $0/0$. We have to do some algebra first as in

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{(x^2+x+1)}{(x+1)} = \frac{3}{2}.$$

[4] See Thomas section 2.2 page 84.

The point is that

$$\frac{x^3-1}{x^2-1} = \frac{(x^2+x+1)}{(x+1)}$$

for $x \neq 1$ and the right hand side (but not the left) is meaningful even when $x = 1$.

§7.5. When there is no single L such that $F(x) \approx L$ for $x \approx a$ we write

$$\lim_{x \to a} F(x) \quad \text{D.N.E.}$$

and say that the *limit does not exist*. This happens because there are two distinct quantities L_1 and L_2 with $F(x) \approx L_1$ for some values of x arbitrarily near a and $F(x) \approx L_2$ for some other values of x near a. Then if there were a quantity L such that $F(x) \approx L$ whenever $x \approx a$ we would have that $L = L_1$ *and* $L = L_2$ which is absurd: $L_1 \neq L_2$. For example, $x/|x| = 1$ if $x > 0$ and $x/|x| = -1$ if $x < 0$ so

$$\lim_{x \to 0+} \frac{x}{|x|} = 1, \qquad \lim_{x \to 0-} \frac{x}{|x|} = -1,$$

but

$$\lim_{x \to 0} \frac{x}{|x|} \quad \text{D.N.E.}$$

as there are values of x arbitrarily near 0 (like $x = 0.001$) where $x/|x| = 1$ and other values of x arbitrarily near 0 (like $x = -0.001$) where $x/|x| = -1$. The same thing can happen with infinite limits:

$$\lim_{x \to 0+} \frac{1}{x} = \infty, \qquad \lim_{x \to 0-} \frac{1}{x} = -\infty,$$

but

$$\lim_{x \to 0} \frac{1}{x} \quad \text{D.N.E.}$$

Notice that we distinguish between an infinite limit and one which does not exist:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Example 7.6. Sometimes even the one sided limits don't exist. For example

$$\lim_{x \to \infty} \cos(x) \quad \text{D.N.E.}$$

as there are large values of x (like $x = 1000\pi$) where $\cos(x) = 1$ and other large values of x (like $x = 1001\pi$) where $\cos(x) = -1$. The function $\cos(x)$ bounces back and forth between these two values as x gets bigger and bigger. Similarly

$$\lim_{x \to 0+} \cos\left(\frac{1}{x}\right) \quad \text{D.N.E.}$$

as there are small positive values of x where $\cos(1/x) = 1$ (like $x = (1000\pi)^{-1}$) and other small positive values of x where $\cos(1/x) = -1$ (like $x = (1001\pi)^{-1}$). The function $\cos(1/x)$ bounces back and forth between these two values as x gets closer and closer to zero.

§7.7. Dummy Variables. Beginners sometimes use the notation $\lim_{x\to a} F(x)$ incorrectly. The formula (see equation (♡) §7.4)

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \frac{2}{3}$$

says that if we take a number which is close to 1 (like 1.001), call it x, and evaluate $(x^2-1)/(x^3-1)$ we get an answer which is close to 2/3:

$$\frac{x^2-1}{x^3-1} = \frac{1.001^2-1}{1.001^3-1} \approx \frac{2}{3}.$$

The formula

$$\lim_{u\to 1} \frac{u^2-1}{u^3-1} = \frac{2}{3}$$

says that if we take a number which is close to 1 (like 1.001), call it u, and evaluate $(u^2-1)/(u^3-1)$ we get an answer which is close to 2/3:

$$\frac{u^2-1}{u^3-1} = \frac{1.001^2-1}{1.001^3-1} \approx \frac{2}{3}.$$

Clearly these formulas have the same meaning:

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{u\to 1} \frac{u^2-1}{u^3-1}.$$

We express this formula by saying that the variable x on the left and the variable u on the right are **dummy variables**. *If a dummy variable in an expression is systematically replaced by a completely different variable, the meaning of the expression is unchanged.*

§7.8. Free Variables. The method used in computing equation (♡) §7.4 shows more generally that

$$\lim_{x\to a} \frac{x^2-a^2}{x^3-a^3} = \frac{2}{3a}.$$

In this formula the variable x on the left is a dummy variable but the variable a which appears on both sides of the formula is a **free variable**; the formula asserts a fact that is true for *all* values of the free variable. Thus the formula

$$\lim_{x\to b} \frac{x^2-b^2}{x^3-b^3} = \frac{2}{3b}$$

conveys the same information whereas the formula

$$\lim_{x\to b} \frac{x^2-b^2}{x^3-b^3} = \frac{2}{3a}$$

is incorrect (unless $a=b$). If we substitute a number for the free variable a we get the valid formula as in

$$\lim_{x\to 7} \frac{x^2-7^2}{x^3-7^3} = \frac{2}{21}$$

whereas it is meaningless to replace the dummy variable x by a number.

§7.9. Another kind of change of dummy variable is

$$\lim_{x \to a} F(x) = \lim_{h \to 0} F(a + h).$$

Here the dummy variable x on the left is effectively replaced by $a + h$ on the right. If $x = a + h$ the condition $x \approx a$ and $h \approx 0$ are the same. Sometimes these substitution makes the algebra more straight forward as in

$$\lim_{x \to a} \frac{x^2 - a^2}{x^3 - a^3} = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{(a + h)^3 - a^3} = \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{2ah + h^2} = \frac{2}{3a}. \qquad (\diamondsuit)$$

Compare this with equation ($\heartsuit$) of §7.4.

§7.10. A **rational function** is a ratio of two polynomials, i.e. one which can be written in the form

$$F(x) = \frac{P(x)}{Q(x)}.$$

where $P(x)$ and $Q(x)$ are polynomials. Evaluate

$$\lim_{x \to a} \frac{P(x)}{Q(x)}$$

as follows:

Case I. If a is finite and $Q(a) \neq 0$, then $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ by the limit laws.

For example,

$$\lim_{x \to 3} \frac{x^2 - 1}{x^3 - 1} = \frac{9 - 1}{27 - 1} = \frac{8}{26}$$

and

$$\lim_{x \to 0} \frac{x^2 - 1}{x^3 - 1} = \frac{0 - 1}{0 - 1} = 1.$$

Case II. The most common case is where a is finite and $Q(a) = P(a) = 0$. In this case we make the change of dummy variable as in §7.9.

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{h \to 0} \frac{P(a + h)}{Q(a + h)}$$

There will be some cancellation and we can evaluate the limit. For example,

$$\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1} = \lim_{h \to 0} \frac{(1 + h)^2 - 1}{(1 + h)^3 - 1} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{(1 + 3h + 3h^2 + h^3) - 1} =$$
$$= \lim_{h \to 0} \frac{2h + h^2}{3h + 3h^2 + h^3} = \lim_{h \to 0} \frac{2 + h}{3 + 3h + h^2} = \frac{2}{3}$$

Case III. If $P(a) \neq 0$ and $Q(a) = 0$ then we can factor out a a power of h from the denominator $Q(a+h)$, i.e.

$$Q(a+h) = h^n M(a+h)$$

where $M(a) \neq 0$. If n is odd, the one sided limits exist and are infinite and opposite in sign so the two sided limit does not exist. For example,

$$\lim_{x\to 1+} \frac{1}{x-1} = \lim_{h\to 0+} \frac{1}{h} = \infty, \qquad \lim_{x\to 1-} \frac{1}{x-1} == \lim_{h\to 0-} \frac{1}{h} = -\infty,$$

and thus the two sided limit does not exist:

$$\lim_{x\to 1} \frac{1}{x-1} \quad \text{D.N.E.}$$

If n is even, the one sided limits are infinite and have the same sign so the two sided limit is infinite. For example,

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \lim_{h\to 0} \frac{1}{h^2} = \infty.$$

Case IV. If a is infinite, multiply top and bottom by the appropriate power of x so that the limit on the bottom is nonzero and finite. Then use the law that the limit of the quotient is the quotient of the limits. For example,

$$\lim_{x\to\infty} \frac{x^2-1}{x^3-1} = \lim_{x\to\infty} \frac{x^2-1}{x^3-1} \cdot \frac{x^{-3}}{x^{-3}} = \lim_{x\to\infty} \frac{x^{-1}-x^{-3}}{1-x^{-3}} = \frac{0-0}{1-0} = 0$$

Theorem 7.11. Sandwich Theorem.[5] *Suppose that*

$$f(x) \leq g(x) \leq h(x)$$

(for all x) and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

Proof: If $f(x) \approx L$ and $h(x) \approx L$ then $g(x) \approx L$ since $g(x)$ is between $f(x)$ and $h(x)$.

Example 7.12. The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \leq x\cos\left(\frac{1}{x}\right) \leq |x|$$

[5] See Thomas section 2.2 page 88. Some books call this the *Squeeze Theorem*.

since the cosine is always between -1 and 1. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$$

the Sandwich Theorem tells us that

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

Recall that in Example 7.6 we saw that the limit $\lim_{x\to 0}\cos(1/x)$ does not exist.

Exercises

Exercise 7.13. Find the following limits. (Write D.N.E. if the limit does not exist.)

(a) $\lim_{x\to -7}(2x+5)$ (b) $\lim_{x\to 7-}(2x+5)$ (c) $\lim_{x\to -\infty}(2x+5)$

(d) $\lim_{x\to -4}(x+3)^{2006}$ (e) $\lim_{x\to -4}(x+3)^{2007}$ (f) $\lim_{x\to -\infty}(x+3)^{2007}$

(g) $\lim_{t\to 1}\dfrac{t^2+t-2}{t^2-1}$ (h) $\lim_{t\to 1-}\dfrac{t^2+t-2}{t^2-1}$ (i) $\lim_{t\to -1}\dfrac{t^2+t-2}{t^2-1}$

(j) $\lim_{x\to\infty}\dfrac{x^2+3}{x^2+4}$ (k) $\lim_{x\to\infty}\dfrac{x^5+3}{x^2+4}$ (l) $\lim_{x\to\infty}\dfrac{x^2+1}{x^5+2}$

(m) $\lim_{x\to\infty}\dfrac{(2x+1)^4}{(3x^2+1)^2}$ (n) $\lim_{u\to\infty}\dfrac{(2u+1)^4}{(3u^2+1)^2}$ (o) $\lim_{t\to 0}\dfrac{(2t+1)^4}{(3t^2+1)^2}$

(p) $\lim_{x\to 2}\dfrac{x^2-4}{x^3-8}$ (q) $\lim_{x\to\infty} x\sin(x)$ (q) $\lim_{x\to\infty}\dfrac{x^2-\sin x}{x+\sin x}$

Exercise 7.14. Evaluate $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$. Hint: Multiply top and bottom by $\sqrt{x}+3$.

Exercise 7.15. Evaluate $\lim_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$. Hint: The function is rational.

Exercise 7.16. Evaluate $\lim_{x\to 2}\dfrac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{2}}}{x-2}$.

Exercise 7.17. Compute the one-sided limits $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ for $c=1$ and $c=-1$ where the function f is defined by

$$f(x) = \begin{cases} x^3 & for \quad x < -1 \\ ax+b & for \quad -1 \le x < 1 \\ x^2+2 & for \quad x \ge 1. \end{cases}$$

where a and b are constants.

8 Two Limits in Trigonometry

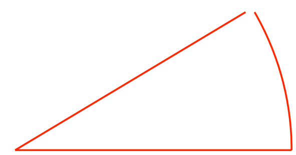
In this section we prove two important limit formulas which we will need in Section 15. The first is proved as Thomas page 105 and the second is an easy consequence.

§ 8.1. The length of a circular arc is given by

$$L = r\theta$$

where r is the radius of the circle and θ is the central angle of the circle. Convince yourself of this by trying $\theta = 2\pi$, $\theta = \pi$, $\theta = \pi/2$, $\theta = \pi/4$ etc. Similarly, the area of a circular sector is given by

$$A = r^2 \frac{\theta}{2}.$$



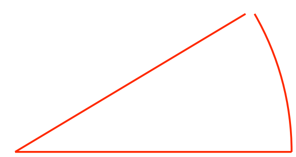
Theorem 8.2.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1, \qquad \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Proof: Comparing the area of the circular sector with the areas of two right triangles gives

$$\frac{\sin\theta\cos\theta}{2} < \frac{\theta}{2} < \frac{\sin\theta}{2\cos\theta},$$

from which we get the limit formula by algebra and the Sandwich Theorem. The second formula is proved by multiplying top and bottom by $1 + \cos\theta$.



Exercises

Exercise 8.3. Evaluate Find the limit or show that it does not exist. Distinguish between a limit which is infinite from one which does not exist.

(a) $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}$. (b) $\lim_{x\to 0}\dfrac{1-\cos x}{x\sin x}$. (c) $\lim_{x\to\infty}\dfrac{2x^3+3x^2\cos x}{(x+2)^3}$.

(d) $\lim_{x\to 0}\dfrac{\sin(x^2)}{x^2}$. (e) $\lim_{x\to 0}\dfrac{x(1-\cos x)}{\tan^3 x}$. (f) $\lim_{x\to 0}\dfrac{\sin(x^2)}{1-\cos x}$.

(g) $\lim_{x\to 0}\dfrac{\cos x}{x^2+9}$. (h) $\lim_{x\to \pi}\dfrac{\sin x}{x-\pi}$. (i) $\lim_{x\to 0}\dfrac{\sin x}{x+\sin x}$.

9 Continuity

This material in this section comes from Thomas pages 124-133 and 147-149.

Definition 9.1. A real valued function f is **continuous**at a iff

$$\lim_{x\to a} f(x) = f(a) \tag{C}$$

A function is continuous iff it is continuous at every a in its domain. A function is continuous on a set (e.g. interval) I iff it is defined and continuous at every point a of I. Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval. For example, the function $f(x) = 1/x^2$ is continuous on the interval $1 < x < 5$ but is *not* continuous on the interval $-1 < x < 1$.

Definition 9.2. The **derivative** of the function f is the function f' whose value at the point a is given by

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}. \tag{D}$$

A function f is said to be **differentiable** on an interval I iff the limit $f'(a)$ exists for every point a in I, i.e. iff the domain of the derivative f' contains the interval I.

Theorem 9.3. *A differentiable function is continuous.*

Proof: Assume that the limit (D) exists. Then

$$\begin{aligned}
\Big(\lim_{x\to a} f(x)\Big) - f(a) &= \Big(\lim_{x\to a} f(x)\Big) - \Big(\lim_{x\to a} f(a)\Big) && \text{(I)} \\
&= \lim_{x\to a}\big(f(x) - f(a)\big) && \text{(III)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a) && \text{(hsa)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot \lim_{x\to a}(x-a) && \text{(V)} \\
&= \left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)\cdot 0 && \text{(I, IV, II)} \\
&= f'(a)\cdot 0 && \text{(definition)} \\
&= 0 && \text{(hsa)}
\end{aligned}$$

so Equation (C) holds. In this proof the Roman numerals refer to the corresponding Limit Law from §7.3, *(definition)* means *by definition*, and *(hsa)* means *high school algebra*.)

Remark 9.4. The converse of Theorem 9.3 is false. The function

$$f(x) = |x|$$

is continuous but not differentiable at $x = 0$ since

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x}$$

and $\lim_{x\to 0} \frac{|x|}{x}$ does not exist by §7.5.

Example 9.5. A stupid way to make an example of a discontinuous function is the following:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 47 & \text{if } x = 3. \end{cases}$$

Then

$$\lim_{x\to 3} f(x) = 9 \neq 47 = f(3).$$

The reason that the limit is 9 is that $\lim_{x\to a} f(x) = L$ means that $f(x) \approx L$ when $x \approx a$ but $x \neq a$; i.e. in the definition of limit (see §7.1) the actual value of $f(a)$ is explicitly excluded.

Example 9.6. The function

$$f(x) = \begin{cases} x\cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is continuous by the Sandwich Theorem (see Example 7.12) but the function

$$g(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) & \text{fot } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is not continuous at 0 (see Example 7.6).

Theorem 9.7 (Intermediate Value Theorem). *A continuous function $f(s)$ defined on the closed interval $a \leq x \leq b$ takes every value between $f(a)$ and $f(b)$. In other words, if $f(a) < v < f(b)$ or $f(b) < v < f(a)$ there is a c such that $a < c < b$ and $f(c) = v$.*

§ 9.8. Informally, the Intermediate Value Theorem says that the graph of a continuous function is connected, i.e. you can draw the graph without lifting your pencil. For example, the equation $23 = x^5 + x$ has a solution x satisfying $1 < x < 2$ because the function $f(x) = x^5 + x$ is continuous and

$$f(1) = 2 < 23 < 34 = f(2).$$

As you draw the graph of $y = x^5 + x$ starting at the point $(1, 2)$ and ending at the point $(2, 34)$ your pencil crosses over some point $(x, 23)$. This is not true for example for the discontinuous function

$$h(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0, \\ \\ 0 & \text{if } x = 0. \end{cases}$$

Since $h(x) = -1$ for $x < 0$ and $h(x) = 1$ for $x > 0$ you cannot draw the graph $y = h(x)$ without lifting your pencil.

Remark 9.9. The previous paragraph notwithstanding do not say that "a function is continuous iff you can draw its graph without lifting your pencil". A pencil is not a (formal) mathematical concept.

§ 9.10. We say that f **attains** its maximum at c and its minimum at d and that $f(c)$ is the **maximum value** of f and $f(d)$ is the **minimum value** of f on the interval. Often the maximum is attained at an endpoint, i.e. $c = a$ or $c = b$ and similarly for the minimum. For example, on the interval $2 \leq x \leq 3$ the function $f(x) = x^2$ attains its minimum value $4 = f(2)$ at the left endpoint and its maximum value $9 = f(3)$ at the right endpoint. When $a < c < b$ we say that c is an **interior minimum**; when $a < d < b$ we say that d is an **interior**

maximum. For example, on the interval $-2 \le x \le 1$ the function $f(x) = x^2$ attains its maximum $f(-2) = 4$ at the left endpoint and its minimum value $f(0) = 0$ at the interior point 0. On an interval which is not closed a continuous function need not assume its maximum or minimum. For example, the function $g(x) = 1/x$ is continuous on the interval $0 < x \le 1$ but there is no number c satisfying $0 < c \le 1$ and $g(x) \le g(c)$ for all x. This is because no matter what c we pick in the interval $0 < x \le 1$ we have that $f(c') > f(c)$ when c' is nearer 0 then c, e.g. when $c' = c/2$.

Theorem 9.11 (The Extreme Value Theorem). *Let the function $f(x)$ be continuous on the closed finite interval $a \le x \le b$. Then there is at least one number c such that $a \le c \le b$ and $f(x) \le f(c)$ for all x in the interval $a \le x \le b$ and there is at least one number d such that $a \le d \le b$ and $f(d) \le f(x)$ for all x in the interval $a \le x \le b$.*

Remark 9.12. The Intermediate Value Theorem and the Extreme Value Theorem are stated but not proved in the textbook (see Thomas Theorem 11 page 130 and Theorem 1 page 246). They are normally proved in more advanced courses like Math 521. Both theorems tell us that certain problems have a solution, but the theorems don't tell us how to find it. They will provide the theoretical justification for the reasoning we employ in Chapter III.

Exercises

Exercise 9.13. Why doesn't the proof of Theorem 9.3 shows that every function is continuous?

Exercise 9.14. Find a constant k such that the function

$$f(x) = \begin{cases} 3x + 2 & \text{for } x < 2 \\ x^2 + k & \text{for } x \ge 2. \end{cases}$$

is continuous. Hint: Compute the one-sided limits.

Exercise 9.15. Find constants a and b such that the function

$$f(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax + b & \text{for } -1 \le x < 1 \\ x^2 + 2 & \text{for } x \ge 1. \end{cases}$$

is continuous for all x.

Exercise 9.16. Is there a constant k such that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \ne 0 \\ k & \text{for } x = 0. \end{cases}$$

is continuous? If so, find it; if not, say why.

Exercise 9.17. Let $g(x)$ be continuous on the interval $-1 \le x \le 1$ and suppose that $g(-1) = 3$ and $g(1) = 5$. Can we necessarily find an x between -1 and 1 with $g(x) = 4$? How about $g(x) = 6$? Explain.

Exercise 9.18. Prove that the equation $x^5 - 4x + 1 = \sin x$ has a solution x in the interval $1 \le x \le 2$.

Chapter III
Differentiation

10 Derivatives Defined

Derivatives and the differentiation laws are explained in Thomas pages 147-170.

Definition 10.1. The **derivative** of the function f is the function f' whose value at the point a is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

A function f is said to be **differentiable** at a iff this limit exists and **differentiable** on an interval I iff it is differentiable every point a in I, i.e. iff the domain of the derivative f' contains the interval I; f is said to be **continuously differentiable** on an interval I iff it is differentiable on I and its derivative f' is continuous on I.

§10.2. Here are two ways of writing the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{2}$$

To use equation (1) to estimate $f'(4)$ we might take $x = 4$ and $h = 0.001 \approx 0$, so $x + h = 4.001$ and

$$f'(4) = f'(x) \approx \frac{f(x+h) - f(x)}{h} = \frac{f(4.001) - f(4)}{0.001}.$$

To use equation (2) to estimate $f'(4)$ we might take $a = 4$ and $x = 4.001 \approx 4$, so $x - a = 0.001$ and

$$f'(4) = f'(a) \approx \frac{f(x) - f(a)}{x - a} = \frac{f(4.001) - f(4)}{0.001}.$$

This illustrates that the two formulas are different ways of expressing the same thing. In equation (1) the variable h is a **dummy variable** whereas the variable

x is a **free variable**. If all occurrences of a dummy variable in an expression are changed to another letter, the meaning of the expression is unchanged:

$$\lim_{k\to 0} \frac{f(x+k)-f(x)}{k} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

A free variable is used to assert that an equation is valid for a range of values; if the free variable is changed on one side of an equation it must be changed on the other side as well. Thus (1) could be written

$$f'(y) = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h}.$$

One can substitute a number for a free variable as in

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h},$$

but substituting a number for a dummy variable yields nonsense. In equation (2), x is the dummy variable and a is the free variable.

§ 10.3. We can calculate some derivatives using the method of equation ($\diamondsuit$) of §7.9. For example,

- the derivative of the function

  $$f(x) = x^2$$

  is the function

  $$f'(x) = 2x.$$

  Here is the proof:

  $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

- the derivative of the identity function

  $$g(x) = x$$

  is the constant function

  $$g'(x) = 1.$$

  Here is the proof:

  $$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

- the derivative of the constant function

  $$k(x) = c$$

is the zero constant function

$$k'(x) = 0.$$

Here is the proof:

$$k'(x) = \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

§10.4. The Differentiation Rules. These are used to differentiate expressions in functions u and v when you know how to differentiate u and v. In the following Let c and n are constants, u and v are functions, and $'$ denotes differentiation. The Differentiation Rules are

(Constant Rule)	$c' = 0,$
(Sum Rule)	$(u \pm v)' = u' \pm v',$
(Product Rule)	$(u \cdot v)' = u' \cdot v + u \cdot v',$
(Quotient Rule)	$\left(\frac{u}{v}\right)' = \frac{u/ \cdot v - u \cdot v/}{v^2},$
(Power Rule)	$(u^n)' = nu^{n-1} \cdot u'.$

Note that we already proved the Constant Rule above.

§10.5. Proof of the Sum Rule. Suppose that $f(x) = u(x) + v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)}\\
&= \lim_{x\to a} \frac{\big(u(x)+v(x)\big)-\big(u(a)+v(a)\big)}{x-a} && \text{(hypothesis)}\\
&= \lim_{x\to a} \left(\frac{u(x)-u(a)}{x-a} + \frac{v(x)-v(a)}{x-a}\right) && \text{(hsa)}\\
&= \lim_{x\to a} \frac{u(x)-u(a)}{x-a} + \lim_{x\to a} \frac{v(x)-v(a)}{x-a} && \text{(limit law)}\\
&= u'(a) + v'(a) && \text{(definition)}
\end{aligned}$$

§10.6. Proof of the Product Rule. Suppose that $f(x) = u(x)v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)}\\
&= \lim_{x\to a} \frac{u(x)\cdot v(x) - u(a)\cdot v(a)}{x-a} && \text{(hypothesis)}\\
&= \lim_{x\to a} \left(\left(\frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(x) \cdot \left(\frac{v(x)-v(a)}{x-a} \right) \right) && \text{(hsa)}\\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(a) \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(limit laws)}\\
&= u'(a)v(a) + v'(a)u(a) && \text{(definition)}
\end{aligned}$$

(In the fourth step the theorem that a differentiable function is continuous is also used.)

§10.7. Proof of the Quotient Rule. Suppose that $f(x) = u(x)/v(x)$ for all x where u and v are differentiable and $v(a) \neq 0$. Then

$$\begin{aligned}
&f'(a) =\\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)}\\
&= \lim_{x\to a} \frac{\big(u(x)/v(x)\big) - \big(u(a)/v(a)\big)}{x-a} && \text{(hypothesis)}\\
&= \lim_{x\to a} \frac{u(x)v(a) - u(a)v(x)}{v(x)v(a)(x-a)} && \text{(hsa)}\\
&= \lim_{x\to a} \left(\frac{u(x)-u(a)}{x-a} \cdot \frac{v(a)}{v(x)v(a)} - \frac{u(a)}{v(x)v(a)} \cdot \frac{v(x)-v(a)}{x-a} \right) && \text{(hsa)}\\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(lim law)}\\
&= u'(a) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot v'(a) && \text{(definition)}\\
&= \frac{u'(a)v(a) - u(a)v'(a)}{v(a)^2} && \text{(hsa)}
\end{aligned}$$

(In the fifth step the theorem that a differentiable function is continuous is also used.)

Remark 10.8. The Quotient Rule can be derived from the Product Rule as follows: if $w = u/v$ then $w \cdot v = u$ so $w' \cdot v + w \cdot v' = u'$ so

$$w' = \frac{u' - w \cdot v'}{v} = \frac{u' - (u/v) \cdot v'}{v} = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

Unlike the argument above, this argument does not prove that w is differentiable if u and v are. Note that the rule

$$(cu)' = cu'$$

is a trivial consequence of the Constant Rule and the Product Rule though of course it can be quite easily proved directly. A special case of the Quotient Rule is when $u = 1$. Then $u' = 0$ and the Quotient Rule reduces to

$$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}.$$

§10.9. Proof of the Power Rule for Positive Integer Exponents. First we prove the Power Rule when the exponent n is a nonnegative integer. When $n = 0$ we have $u^0 = 1$ and the Power Rule is the Constant Rule. When $n = 1$ the formula says that $u' = u'$. Taking $u = v$ in the Product Rule gives

$$(u^2)' = u' \cdot u + u \cdot u' = 2u \cdot u'$$

which is the Power Rule with $n = 2$. Using this and taking $v = u^2$ in the Product Rule gives

$$(u^3)' = (u \cdot u^2)' = u' \cdot u^2 + u \cdot (u^2)' = u' \cdot u^2 + u \cdot (2u \cdot u') = 3u^2 \cdot u'.$$

Proceeding in this way we see that once we have proved the Power Rule for some exponent n we can take $v = u^n$ in the Product Rule and get

$$(u^{n+1})' = (u \cdot u^n)' = u' \cdot u^n + u \cdot (u^n)' = u' \cdot u^n + u \cdot (nu^{n-1} \cdot u') = (n+1)u^n \cdot u'$$

which proves the Power Rule for exponent $n+1$. Hence the Power Rule holds for any nonnegative integer exponent n. (This style of proof is called *Mathematical Induction.*)

§10.10. Proof of the Power Rule for Negative Integer Exponents. We can now prove the Power Rule for negative exponents using the the Quotient Rule with $u = 1$ and $v = u^n$:

$$(u^{-n})' = \left(\frac{1}{u^n}\right)' = -\frac{nu^n \cdot u'}{u^{2n}} = -nu^{-n-1} \cdot u'.$$

§10.11. Proof of the Power Rule for Rational Powers. We prove the Power Rule when the exponent n is a rational number, i.e. a number of form $n = p/q$ where p and q are integers. For thus assume that

$$w = u^{p/q}.$$

Raising both sides to the qth power gives

$$w^q = u^p.$$

Applying the Power Rule to both sides gives

$$qw^{q-1} \cdot w' = pu^{p-1} \cdot u'.$$

Dividing both sides by qw^{q-1} and substituting $u^{p/q}$ for w gives

$$w' = \frac{pu^{p-1} \cdot u'}{qw^{q-1}} = \frac{pu^{p-1} \cdot u'}{qu^{p(q-1)/q}} = \frac{pu^{p-1} \cdot u'}{qu^{p-(p/q)}} = \frac{p}{q} \cdot u^{(p/q)-1} \cdot u'$$

which is the Power Rule for $n = p/q$. (In §38.4 we will prove the Power Rule for any real exponent n.)

Example 10.12. Using the Differentiation Rules you can easily differentiate any polynomial and hence any rational function. For example, using the Sum Rule, the Power Rule with $u(x) = x$, the rule $(cu)' = cu'$, the derivative of the polynomial

$$f(x) = 2x^4 - x^3 + 7$$

is

$$f'(x) = 8x^3 - 3x^2.$$

By the Quotient Rule the derivative of rational function

$$g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}$$

is

$$\begin{aligned} g'(x) &= \frac{(8x^3 - 3x^2)(1 + x^2) - (2x^4 - x^3 + 7)2x}{(1 + x^2)^2} \\ &= \frac{6x^5 - 5x^4 + 8x^3 - 3x^2 + 14x}{(1 + x^2)^2}. \end{aligned}$$

Example 10.13. The derivative of $f(x) = \sqrt{x} = x^{1/2}$ is

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

where we used the power rule with $n = 1/2$ and $u(x) = x$.

Exercises

Exercise 10.14. Using the methods of Chapter II compute the limit

$$\lim_{\Delta x \to 0} \frac{\dfrac{x + \Delta x}{x + 2 + \Delta x} - \dfrac{x}{x + 2}}{\Delta x}$$

Exercise 10.15. Using the definition of the derivative as a limit (only method allowed), compute the derivative of $f(x) = \dfrac{x}{x+2}$. Hint: See previous problem.

Exercise 10.16. Find $f'(x)$ and $g'(x)$ if $f(x) = x^4$ and $g(x) = (1+x^2)^4$. Hint: Use the power rule with $u = x$ to find $f'(x)$ and with $u = 1 + x^2$ to find $g'(x)$.

Exercise 10.17. Let $f(x) = (x^2 + 1)(x^3 + 3)$. Find $f'(x)$ in two ways, first by multiplying and then differentiating, and then using the product rule. Are the answers the same?

Exercise 10.18. Let $f(x) = (1+x^2)^4$. Find dy/dx in two ways, first by expanding to get an expression for $f(x)$ as a polynomial in x and then differentiating, and then by using the power rule. Are the answers the same?

Exercise 10.19. The equation

$$\frac{2x}{x^2 - 1} = \frac{1}{x+1} + \frac{1}{x-1}$$

holds for all values of x so you should get the same answer if you differentiate both sides. Check this.

Exercise 10.20. Find $f'(x)$ and $g'(x)$ if

$$f(x) = \frac{1 + x^2}{2x^4 - x^3 + 7}, \qquad g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}.$$

Note that $f(x) = 1/g(x)$. Is it true that $f'(x) = 1/g'(x)$? What is the relation between $f'(x)$ and $g'(x)$? Hint: $f(x) = g(x)^{-1}$. What is the derivative of $f(x) \cdot g(x)$?

Exercise 10.21. Find $f'(x)$ if $f(x) = 1 + x + x^2/2 + x^3/3 + x^4/4$.

11 Higher Derivatives and Differential Notation

Higher derivatives and Differential Notation are explained in Thomas page 168 and pages 221-231 respectively.

§11.1. If $y = f(x)$ we often use the notation, called "**Leibniz notation**",[6]

$$\frac{dy}{dx} = f'(x)$$

for the derivative. This notation is very suggestive: changing the dummy variable h to Δx (see §10.2) gives

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

[6] I called it **variable notation** in Remark 3.9

where

$$\Delta y = f(x + \Delta x) - f(x).$$

Thus

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \qquad \text{when} \qquad \Delta x \approx 0.$$

When $y = f(x)$, the equations

$$m = f'(x_0) \qquad \text{and} \qquad m = \left.\frac{dy}{dx}\right|_{x=x_0}$$

are synonymous: both indicate that the derivative is to be evaluated at $x = x_0$. We use the former in functional notation and the latter in Leibniz notation.

Example 11.2. When we use **functional notation** as in

$$f(x) = x^3 - x$$

we may write things like $f(2) = 2^3 - 2 = 6$, $f(t) = t^3 - t$, $f'(x) = 3x^2 - 1$, and $f'(2) = 3(2)^2 - 1 = 11$. When we use **variable (Leibniz) notation** as in

$$y = x^3 - x$$

we may say things like "the point $(x, y) = (2, 6)$ lies on the curve $y = x^3 - x$", and write things like

$$\frac{dy}{dx} = 3x^2 - 1, \qquad \left.\frac{dy}{dx}\right|_{x=2} = 11.$$

§11.3. The second derivative of a function f is the derivative of the derivative of f. It is denoted f''. The third derivative of a function f is the derivative of the second derivative of f. It is denoted f'''. The nth derivative of f is sometimes denoted $f^{(n)}$. Thus

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \qquad f^{(3)} = f''', \ldots.$$

Another notation for the nth derivative of $y = f(x)$ is

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

§11.4. For reasons which will become apparent when we study integration, we sometimes also use **differential notation**

$$dy = (3x^2 - 1)\, dx$$

instead of $\dfrac{dy}{dx} = 3x^2 - 1$. This handy notation reminds us that a derivative is roughly a quotient of two "infinitely small" quantities.

Warning: Never write something like $dy = 3x^2 - 1$, i.e. don't forget the dx. The quantity dy is "infinitely small" whereas $3x^2 - 1$ is not.

§11.5. Another common notation is **operator notation** as in

$$\frac{d}{dx}(x^3 - x) = 3x^2 - 1.$$

This allows us to avoiding introducing a name for $x^3 - x$. It also explains why we write

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dx}\right)^2 y$$

for the second derivative of y with respect to x. Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2.$$

Definition 11.6. (**Slopes and Tangents**). The **tangent line** to a curve at a $P_0(x_0, y_0)$ on a curve is the limit of the secant line connecting $P_0(x_0, y_0)$ to a nearby point $P(x, y)$ on the graph as the point P approaches the point P_0; the **slope of the curve** at P_0 is the slope of tangent line at P_0. The slope of the secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

so the slope of the curve at P_0 is

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)}$$

(see Section 6). More precisely:

When a curve is the graph of a function $y = f(x)$, the slope of the curve at the point $P_0(x_0, f(x_0))$ on the curve is the derivative

$$m = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

evaluated at the point $x = x_0$ so, by the point slope formula, an equation for the tangent line at P_0 is $y = y_0 + m(x - x_0)$, i.e.

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Remark 11.7. The **normal line** to a curve at a point P_0 on the curve is the line through P_0 which is perpendicular to the tangent line. Hence an equation for the normal line to the curve $y = f(x_0)$ at the point $P_0(x_0, f(x_0)$ is

$$y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0).$$

(Because $\tan(\phi + \frac{\pi}{2}) = -1/\tan\phi$, the slope of a line which is perpendicular to a line of slope m is $-1/m$. Hence an equation for the line through $P_0(x_0 y_0)$ which is perpendicular to the line $y = y_0 + m(x - x_0)$ is $y = y_0 - (1/m)(x - x_0)$.)

Example 11.8. We find equations for the tangent and normal lines to the curve $y = x^2$ at the point $(x, y) = (3, 9)$. The slope of the curve at the point $(3, 9)$ is

$$m = \left.\frac{dy}{dx}\right|_{x=3} = 2x\Big|_{x=3} = 6$$

so the tangent line is $y = 9 + 6(x - 3)$ and the normal line is $y = 9 - \frac{1}{6}(x - 3)$. **Warning:** A common mistake is to forget to evaluate the derivative. The equation $y = 9 + 2x(x - 3)$ is not an equation for the tangent line, it is not even an equation for a line. The correct answer is $y = 9+6(x-3)$ not $y = 9+2x(x-3)$.

Exercises

Exercise 11.9. Find the second derivative of x^7 with respect to x.

Exercise 11.10. Find the first two derivatives $f'(x)$ and $f''(x)$ of the function $f(x) = \dfrac{x}{x+2}$.

Exercise 11.11. Find dy/dx and d^2y/dx^2 if $y = x/(x + 2)$. Hint: See previous problem.

Exercise 11.12. Find du/dt and d^2u/dt^2 if $u = t/(t + 2)$. Hint: See previous problem.

Exercise 11.13. Find $\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)$ and $\dfrac{d^2}{dx^2}\left(\dfrac{x}{x+2}\right)$. Hint: See previous problem.

Exercise 11.14. Find $\left.\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)\right|_{x=1}$ and $\dfrac{d}{dx}\left(\dfrac{1}{1+2}\right)$.

Exercise 11.15. Find $g(x)$ such that $dy = g(x)\,dx$ if $y = x/(x + 2)$.

Exercise 11.16. Let $x = (1 - t^2)/(1 + t^2)$, $y = 2t/(1 + t^2)$ and $u = y/x$. Find dx/dt, dy/dt, and du/dt.

Exercise 11.17. Find d^2y/dx^2 and $(dy/dx)^2$ if $y = x^3$.

Exercise 11.18. Find equations for the tangent and normal lines to the curve $y = 4x/(1 + x^2)$ at origin and at the point $(1, 2)$.

Exercise 11.19. Find equations for the tangent and normal lines to the curve $y = 8/(4 + x^2)$ at the point $(2, 1)$ and at the point $(0, 2)$.

Exercise 11.20. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where? Hint: A line is horizontal when its slope is zero.

Exercise 11.21. Let $f(x) = 5+3x+\frac{9}{2}x^2+\frac{5}{6}x^3$. Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(k)}(0)$ for $k \geq 4$.

Exercise 11.22. Let $f(x) = 5 + 3(x-7) + \frac{9}{2}(x-7)^2 + \frac{5}{6}(x-7)^3$. Find $f(7)$, $f'(7)$, $f''(7)$, $f'''(7)$ and $f^{(k)}(7)$ for $k \geq 4$.

Exercise 11.23. Let $f(x) = c_0 + c_1x + \dfrac{c_2}{2}x^2 + \dfrac{c_3}{6}x^3$ where c_0, c_1, c_2, c_3 are constants. Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ and $f^{(k)}(0)$ for $k \geq 4$.

Exercise 11.24. Let $f(x) = c_0 + c_1(x-a) + \dfrac{c_2}{2}(x-a)^2 + \dfrac{c_3}{6}(x-a)^3$ where a, c_0, c_1, c_2, c_3 are constants. Find $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$ and $f^{(n)}(a)$ for $n \geq 4$.

Exercise 11.25. For each non-negative integer k, the notation $k!$ is pronounced k-**factorial** and is defined to be the product of the first k positive integers, i.e.

$$k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k.$$

Thus $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc. We also define $0! = 1$. Find $f^{(k)}(x)$ and $f^{(k)}(0)$ for all $k = 0, 1, 2, \ldots$ if $f(x) = x^n$ and n is a positive integer.

Exercise 11.26. Find $f^{(k)}(x)$ and $f^{(k)}(a)$ for all $k = 0, 1, 2, \ldots$ if

$$f(x) = \frac{c_0}{0!} + \frac{c_1}{1!}(x-a) + \frac{c_2}{2!}(x-a)^2 + \cdots + \frac{c_n}{n!}(x-a)^n$$

where $a, c_0, c_1, c_2, \ldots, c_n$ are constants.

12 Implicit Functions

Implicit differentiation is explained in Thomas pages 205-213.

§12.1. When we say that the function $y = f(x)$ is **implicitly defined** by an equation in x and y we mean that if we substitute $f(x)$ for y in that equation, we get an equation (in x) that holds for all values of x. In this case, we can find the derivative by differentiating the equation and solving for the derivative.

Example 12.2. The function $y = \sqrt{1-x^2}$ is implicitly defined by the equation $x^2+y^2 = 1$ (with the additional condition that $y \geq 0$). We can find the derivative explicitly via

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

but it is easier to view $x^2 + y^2$ as a (constant) function of x, differentiate to get

$$0 = \frac{d}{dx}(x^2+y^2) = 2x + 2y\frac{dy}{dx},$$

and then solve to get

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}.$$

Example 12.3. Here is a more complicated example. A differentiable function $y = f(x)$ is implicitly defined by the equation

$$y^2 + 3xy + 7x^2 - 17 = 0. \tag{†}$$

and satisfies $f(1) = 2$. To find $f'(1)$ we can differentiate (†) and solve:

$$2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + 14x = 0$$

(we used the product rule when we differentiated $3xy$) so

$$\frac{dy}{dx} = -\frac{3y + 14x}{2y + 3x}. \tag{†′}$$

Then

$$f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = -\left.\frac{3y + 14x}{2y + 3x}\right|_{(x,y)=(1,2)} = -\frac{6 + 14}{4 + 3}.$$

Another (harder) way is to find an explicit formula for y by using the quadratic formula:

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-3x \pm \sqrt{9x^2 - 4(7x^2 - 17)}}{2}$$

where $A = 1$, $B = 3x$, and $C = 7x^2 - 17$. Because $f(1) = 2$ we must take the plus sign on the right and we see that $y = f(x)$ is explicitly defined by

$$y = \frac{-3x + \sqrt{9x^2 - 4(7x^2 - 17)}}{2} = \frac{-3x + \sqrt{68 - 19x^2}}{2} \tag{‡}$$

We can find $f'(x)$ by differentiating (‡):

$$\frac{dy}{dx} = -\frac{3}{2} - \frac{19x}{2\sqrt{68 - 19x^2}}. \tag{‡′}$$

In even more complicated examples, it will be impossible (not merely difficult) to find a formula for the implicitly defined function. Nonetheless we can still compute the derivative.

Example 12.4. A typical problem asks you to find an equation for the tangent line to a curve at a point on the curve. For example, to find the equation for the tangent line to the graph of (†) at the point $(x, y) = (1, 2)$ we calculate the slope by implicit differentiation as before:

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\left.\frac{3y + 14x}{2y + 3x}\right|_{(x,y)=(1,2)} = -\frac{20}{7}.$$

Then, since the value of the derivative at a point is the slope of the tangent line at that point, the equation of the tangent line is

$$y = 2 - \frac{20}{7}(x - 1)$$

where we have used the point slope equation

$$y = y_0 + m(x - x_0)$$

(see §2.2) for the equation of the line of slope m through the point (x_0, y_0).

Exercises

Exercise 12.5. Check that the two formulas (†′) and (‡′) for dy/dx in Example 12.3 are actually equal.

Exercise 12.6. Find an equation for the tangent line to the curve

$$x^2 + xy - y^2 = 1$$

at the point $P_0(2, 3)$. Answer: $y - 3 = \frac{7}{4}(x - 2)$

Exercise 12.7. The point $P(1, 2)$ lies on the curve

$$y^5 + 3xy + 7x^5 - 45 = 0.$$

Find equations for the tangent line at P via the method of Example 12.4. In this case you must use implicit differentiation: there is no analog of Equation (‡). (Your TA will learn this in Math 742.)

Exercise 12.8. Find equations for the tangent line and the normal line to the curve $x^3 + y^3 = 9xy$ at the point $(x, y) = (2, 4)$. Hint: The slope of the normal line is the negative reciprocal of the slope of the tangent line.

13 The Chain Rule

The Chain Rule is explained in Thomas pages 190-194.

Definition 13.1. The **composition** of two functions f and g is the function is $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x)).$$

Theorem 13.2 (Chain Rule). *If f and g are differentiable, so is the composition $f \circ g$ and its derivative at the point $x = a$ is*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof: The idea is that

$$x \approx a \implies \frac{g(x) - g(a)}{x - a} \approx g'(a).$$

$$u \approx g(a) \implies \frac{f(u) - f(g(a))}{u - g(a)} \approx f'(g(a)),$$

and (because a differentiable function is continuous)

$$x \approx a \implies g(x) \approx g(a).$$

Hence

$$\frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \approx f'(g(a))g'(a)$$

when $x \approx a$. This proof isn't quite a correct proof because if $g(x) = g(a)$ the expression in the middle is meaningless. In general we might have $g(x) = g(a)$ for some values of x which are arbitarily close to a and $g(x) \neq g(a)$ for some other values of x which are arbitrarily close to a. But if this happens we must have $g'(a) = 0$ since $(g(x) - g(a))/(x - a) \approx g'(a)$ for $x \approx a$. As before the difference quotient $\big(f(g(x)) - f(g(a))\big)/(x-a) \approx f'(g(a))g'(a) = 0$ for $x \approx a$ and $g(x) \neq g(a)$ whereas the difference quotient exactly equal to zero if $g(x) = g(a)$.

Example 13.3. Take $f(u) = 5 + u^2$ and $g(x) = 7 + x^3$ so $f \circ g$ is given by

$$(f \circ g)(x) = f(g(x)) = 5 + g(x)^2 = 5 + (7 + x^3)^2 = 54 + 14x^3 + x^6.$$

Thus $(f \circ g)'(x) = 42x^2 + 6x^5$ (without the chain rule) while the chain rule give the same answer:

$$(f \circ g)'(x) = f'(g(x))g'(x) = 2(7 + x^3)3x^2 = 42x^2 + 6x^5.$$

If we modify the example by taking $f(u) = 5 + \sqrt{u}$, the direct method doesn't apply and we have to use the Chain Rule.

§13.4. In differential notation the chain rule looks like the cancellation rule for multiplying fractions. Thus if $y = f(u)$ and $u = g(x)$ the chain rule is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \qquad (*)$$

For example, if $y = 5 + u^2$ and $u = 7 + x^3$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 3x^2 = 2(7 + x^3)3x^2$$

as in Example 13.3. The formula $(*)$ is hardly surprising since the Chain Rule is proved by taking the limit as Δx tend to zero in the formula

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \tag{$**$}$$

and the quantities in $(**)$ really are fractions, not just limits of fractions.

Remark 13.5. In most problems you are given a complicated formula of y in terms of x and asked to find dy/dx. In such a case you must decide what to take for u. For example, if you are told that $y = 4 + \sqrt{7 + x^3}$ and asked to find dy/dx you might take $u = 7 + x^3$ so $y = 4 + \sqrt{u}$ and then compute

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{7 + x^3}} \cdot 3x^2.$$

The last step (where you replace u by its definition in terms of x) is important because the problem was presented to you with only x and y as variables and u was a variable you introduced yourself to do the problem. After awhile you will be able to apply the Chain Rule without introducing new letters, and you will simply think "the derivative is the derivative of the outside with respect to the inside times the derivative of the inside" and write

$$\frac{d}{dx}(4 + \sqrt{7 + x^3}) = \frac{1}{2\sqrt{7 + x^3}} \cdot 3x^2.$$

Remark 13.6. When $f(u) = u^n$, the Chain Rule becomes the Power Rule, i.e.

$$\frac{d}{dx}u^n = nu^{n-1} \cdot \frac{du}{dx}.$$

Thus, for the functions we've encountered so far (rational functions and fractional powers), the Chain Rule gives nothing new.

§13.7. Usually we have to apply the Chain Rule more than once to compute a derivative. Thus if $y = f(u)$, $u = g(v)$, and $v = h(x)$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

In functional notation this is

$$(f \circ g \circ h)'(x) = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x).$$

Note that each of the three derivatives on the right is evaluated at a different point. Thus if $b = h(a)$ and $c = g(b)$ the Chain Rule is

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=c} \cdot \left.\frac{du}{dv}\right|_{v=b} \cdot \left.\frac{dv}{dx}\right|_{x=a}.$$

For example, if $y = \dfrac{1}{1+\sqrt{9+x^2}}$, then $y = 1/(1+u)$ where $u = 1+\sqrt{v}$ and $v = 9+x^2$ so

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = -\frac{1}{(1+u)^2}\cdot\frac{1}{2\sqrt{v}}\cdot 2x.$$

so

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{dy}{du}\right|_{u=6}\cdot\left.\frac{du}{dv}\right|_{v=25}\cdot\left.\frac{dv}{dx}\right|_{x=4} = -\frac{1}{7}\cdot\frac{1}{10}\cdot 8.$$

Exercises

Exercise 13.8. Let $y = \sqrt{1+x^3}$ and find dy/dx using the Chain Rule. Say what plays the role of $y = f(u)$ and $u = g(x)$.

Exercise 13.9. Repeat the previous exercise with $y = (1+\sqrt{1+x})^3$.

Exercise 13.10. Alice and Bob differentiated $y = \sqrt{1+x^3}$ with respect to x differently. Alice wrote $y = \sqrt{u}$ and $u = 1+x^3$ while Bob wrote $y = \sqrt{1+v}$ and $v = x^3$. Assuming neither one made a mistake, did they get the same answer?

Exercise 13.11. Let $y = u^3+1$ and $u = 3x+7$. Find $\dfrac{dy}{dx}$ and $\dfrac{dy}{du}$. Express the former in terms of x and the latter in terms of u.

Exercise 13.12. Suppose that $f(x) = \sqrt{x}$, $g(x) = 1+x^2$, $v(x) = f\circ g(x)$, $w(x) = g\circ f(x)$. Find formulas for $v(x)$, $w(x)$, $v'(x)$, and $w'(x)$.

Exercise 13.13. Suppose that $f(x) = x^2+1$, $g(x) = x+5$, and

$$v = f\circ g,\qquad w = g\circ f,\qquad p = f\cdot g,\qquad q = g\cdot f.$$

Find $v(x)$, $w(x)$, $p(x)$, and $q(x)$.

Exercise 13.14. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	f'(x)	g'(x)
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

Define

$$v(x) = f(g(x)),\qquad w(x) = g(f(x)),\qquad p(x) = f(x)g(x),\qquad q(x) = g(x)f(x).$$

Evaluate $v(0)$, $w(0)$, $p(0)$, $q(0)$, $v'(0)$ and $w'(0)$, $p'(0)$, $q'(0)$. If there is insufficient information to answer the question, so indicate.

Exercise 13.15. A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$ and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x,y) = (3,7)$.

14 Inverse Functions

Inverse functions are explained in Thomas pages 466-475.

§14.1. A graph of an equation of form $y = f(x)$ satisfies the **Vertical Line Test**: every vertical line $x = a$ intersects the graph in at most one point. If a is in the domain of f, then the vertical line $x = a$ intersects the graph $y = f(x)$ in the point $P(a, f(a))$; if a is not in the domain of f, then the vertical line $x = a$ does not intersect the graph $y = f(x)$ at all. A graph of an equation of form $x = g(y)$ satisfies the **Horizontal Line Test**: every horizontal line $y = b$ intersects the graph in at most one point. If b is in the domain of g, then the horizontal line $y = b$ intersects the graph in the point $P(g(b), b)$; if b is not in the domain of g, then the line $y = b$ does not intersect the graph $x = g(y)$ at all.

Definition 14.2. When the graphs $y = f(x)$ and $x = g(y)$ are the same, i.e. when

$$y = f(x) \iff x = g(y)$$

we say that f and g are **inverse functions** and write $g = f^{-1}$. Thus

$$\text{domain}(f^{-1}) = \text{range}(f), \qquad \text{range}(f^{-1}) = \text{domain}(f),$$

and

$$y = f(x) \iff x = f^{-1}(y) \tag{\#}$$

for x in the domain of f and y in the range of f.

Example 14.3. The graph $y = x^2$ does not satisfy the horizontal line test since the horizontal line $y = 9$ intersects the graph in the two points $(-3, 9)$ and $(3, 9)$. Therefore this graph cannot be written in the form $x = g(y)$. However, if we restrict the the domain to $x \geq 0$ the resulting graph does have the form $x = g(y)$:

$$\text{For } x \geq 0: \qquad y = x^2 \iff x = \sqrt{y}.$$

Let $f(x) = x^2$ (with the domain artificially restricted to $x \geq 0$); then $f^{-1}(y) = \sqrt{y}$. Thus $f(3) = 9$ so by (#) $f^{-1}(9) = 3$. Hence $f(f^{-1}(9)) = 9$ and $f^{-1}(f(3)) = 3$. In general:

$$\sqrt{x^2} = x, \qquad (\sqrt{y})^2 = y$$

for $x \geq 0$. But there is nothing special about this example:

§14.4. Cancellation Equations. If the graph $y = f(x)$ satisfies the horizontal line test (so that the inverse function f^{-1} is defined) then

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y$$

for x in domain(f) = range(f^{-1}) and y in range(f) = domain(f^{-1}).

To see this choose x and let $y = f(x)$. Then $x = f^{-1}(y)$ by Equation (#) in Definition 14.2. Substituting the former in the latter gives $x = f^{-1}(f(x))$. Reversing the roles of f and f^{-1} proves the other cancellation equation.

Theorem 14.5 (Inverse Function Theorem). *Suppose that f and g are inverse functions, that f is differentiable, and that $f'(x) \neq 0$ for all x. Then g is differentiable and*

$$g'(y) = \frac{1}{f'(g(y))}.$$

Proof: The fact that g is differentiable is normally proved in more advanced courses like Math 521. Assuming this we prove the formula for $g'(y)$ as follows. By the Cancellation Equations of 14.4 we have

$$f(g(y)) = y.$$

Differentiate with respect to y and use the Chain Rule to get

$$f'(g(y))g'(y) = 1.$$

Now divide both sides by $f'(g(y))$.

Remark 14.6. We can also write the Inverse Function Theorem as

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

If we use this notation, we don't need a name for the inverse function.

Remark 14.7. A handy way to summarize the formula $(f^{-1})'(y) = 1/f'(f^{-1}(y))$ from Theorem 14.5 is with Leibniz notation:

$$\boxed{\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.}$$

For example, for $x > 0$ and $y = x^2$ we have $x = \sqrt{y} = y^{\frac{1}{2}}$ so

$$\frac{dy}{dx} = 2x, \qquad \frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}},$$

in agreement with the power rule

$$\frac{d}{dy}y^{\frac{1}{2}} = \frac{dx}{dy} = \frac{1}{2\sqrt{y}} = \frac{1}{2}y^{\frac{1}{2}-1}.$$

Example 14.8. We find the inverse function $g = f^{-1}$ of the function

$$f(x) = x^3 + 1$$

and its derivative. Since $y = x^3 + 1 \iff x = (y-1)^{1/3}$, the inverse function is $g(y) = (y-1)^{1/3}$ so

$$g'(y) = \frac{(y-1)^{-2/3}}{3}.$$

The following calculation confirms that $g'(y) = 1/f'(g(y))$:

$$\frac{1}{f'(g(y))} = \frac{1}{3g(y)^2} = \frac{1}{3\big((y-1)^{1/3}\big)^2} = \frac{1}{3(y-1)^{2/3}} = \frac{(y-1)^{-2/3}}{3}.$$

Exercises

Exercise 14.9. Find the inverse function to $f(x) = 3x + 6$.

Exercise 14.10. Find the inverse function to $f(x) = 7 + 5x^3$. Then find its derivative.

Exercise 14.11. Does the function $f(x) = x^3 - x$ have an inverse? (i.e. does it satisfy the horizontal line test?) Hint: Factor $x^3 - x$ and draw the graph.)

Exercise 14.12. Find the inverse function to $f(x) = \sqrt{1-x^2}$ where the domain is artificially restricted to the interval $0 \le x \le 1$. Draw a graph.

Exercise 14.13. Let $f(x) = x^5 + x$ and $g(y) = f^{-1}(y)$. What is $f(1)$? $g(2)$? $f(2)$? $g(34)$? Find $f'(1)$, $g'(2)$, $f'(2)$, and $g'(34)$. Warning: Don't try to find a formula for $g(y)$.

Exercise 14.14. Assume that $y = f(x)$. With the information given below you can find dx/dy for some values of y. Which values of y and what are the corresponding values of dx/dy?

$$f(3) = 4, \quad f(5) = 6, \quad f'(3) = 1, \quad f'(4) = 2, \quad f'(5) = 3, \quad f'(6) = 4.$$

Exercise 14.15. (i) For which constants c does is the function defined by

$$f(x) = \begin{cases} 3x & \text{for } 0 \le x < 1; \\ 4x - c & \text{for } 1 \le x, \end{cases}$$

have an inverse function? (Hint: Horizontal Line Test.) (ii) For which value of c is $f(x)$ continuous? (iii) Draw a graph of $y = f(x)$ for this value of c. (iv) Find a formula (like the formula for $f(x)$) for the inverse function $x = f^{-1}(y)$.

15 Differentiating Trig Functions

Differentiation of trig functions and of inverse trig functions is covered in Thomas pages 183-190 and 517-527. Ignore for the moment the "Integration Formulas" on page 528. These formulas involve the integral sign $\int$ which you don't need to understand till later in the course.

§15.1. We calculate

$$\sin'(\theta) = \lim_{h \to 0} \frac{\sin(\theta + h) - \sin(\theta)}{h}.$$

By the trigonometric addition formula

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

with $\alpha = \theta$ and $\beta = h$ the difference quotient is

$$\begin{aligned} \frac{\sin(\theta + h) - \sin(\theta)}{h} &= \frac{\sin(\theta)\cos(h) + \cos(\theta)\sin(h) - \sin(\theta)}{h} \\ &= \cos(\theta)\frac{\sin(h)}{h} + \sin(\theta)\frac{\cos(h) - 1}{h} \end{aligned}$$

Hence by the formulas

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

from Section 8 we have

$$\boxed{\frac{d}{d\theta} \sin\theta = \cos\theta.}$$

By the principle that the **co**sine of an angle is the sine of its **co**mplement we have

$$\cos(\theta) = \sin(\frac{\pi}{2} - \theta),$$

Differentiating this gives

$$\cos'(\theta) = -\sin'(\frac{\pi}{2} - \theta) = -\cos(\frac{\pi}{2} - \theta) = -\sin(\theta).$$

(we used the Chain Rule and $\frac{d}{d\theta}(\frac{\pi}{2} - \theta) = -1$), i.e.

$$\boxed{\frac{d}{d\theta}\cos\theta = -\sin\theta.}$$

Differentiating the formula

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

using the quotient rule and the formulas just proved for the derivatives of the sine and cosine gives

$$\tan'\theta = \frac{\frac{dy}{d\theta}\cdot x - y\cdot\frac{dx}{d\theta}}{x^2} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

so

$$\boxed{\frac{d}{d\theta}\tan\theta = \sec^2\theta.}$$

§ 15.2. Inverse Trig Functions. The trig functions sine, cosine, tangent etc. do not satisfy the horizontal line test: they are periodic. The inverse trig functions are defined by artificially restricting the domain of the corresponding trig function. When we do this (see §3.7) we get

- If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ then $y = \sin\theta \iff \theta = \sin^{-1}(y)$
- If $0 \le \theta \le \pi$ then $x = \cos\theta \iff \theta = \cos^{-1}(y)$
- If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ then $u = \tan\theta \iff \theta = \tan^{-1}(y)$.

Using these and the formula from Remark 14.7 we can differentiate the inverse trig functions.

$$\boxed{\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}}.}$$

Proof: Let $y = \sin\theta$ so $\theta = \sin^{-1}(y)$. Then

$$\frac{d\theta}{dy} = \left(\frac{dy}{d\theta}\right)^{-1} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-y^2}}$$

where we used the Pythogorean Theorem $\cos^2\theta + \sin^2\theta = 1$.

$$\boxed{\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}.}$$

Proof: Let $x = \cos\theta$ so $\theta = \cos^{-1}(y)$. Then

$$\frac{d\theta}{dx} = \left(\frac{dx}{d\theta}\right)^{-1} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\cos^2\theta}} = -\frac{1}{\sqrt{1-x^2}}$$

where we used the Pythogorean Theorem as before.

$$\boxed{\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2}.}$$

Proof: Let $u = \tan\theta$ so $\theta = \tan^{-1}(u)$. Then

$$\frac{d\theta}{du} = \left(\frac{du}{d\theta}\right)^{-1} = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+u^2}$$

where we used the Pythogorean Theorem in the form $1 + \tan^2\theta = \sec^2\theta$. This follows from $\cos^2\theta + \sin^2\theta = 1$ by dividing by $\cos^2\theta$. (Recall that the secant function is the reciprocal of the cosine.)

Example 15.3. We calculate the derivative of $y = \sin\sqrt{1+x^2}$ using the Chain Rule:

$$\frac{dy}{dx} = \left(\cos\sqrt{1+x^2}\right)\cdot\left(\frac{1}{2\sqrt{1+x^2}}\right)\cdot 2x.$$

Exercises

Exercise 15.4. Find $\dfrac{d}{d\theta}\cot\theta$ and $\dfrac{d}{dv}\cot^{-1}(v)$.

Exercise 15.5. Find $\dfrac{d}{d\theta}\sec\theta$ and $\dfrac{d}{dw}\sec^{-1}(w)$.

Exercise 15.6. Find the second derivative of $\tan\theta$ with respect to θ.

Exercise 15.7. In each of the following, find dy/dx.

(a) $y = \sin x$. (b) $y = (\sin x)^{-1}$. (c) $y = \sin(x^{-1})$. (d) $y = \sin^{-1}(x)$.

Exercise 15.8. Consider the following functions

$$f_1(x) = \sin(x^2), \quad f_2(x) = (\sin x)^2, \quad f_3(x) = (\sin x)x,$$

$$f_4(x) = \sin^2 x, \quad f_5(x) = \sin(\sin x).$$

Which (if any) of these functions are the same? Evaluate the derivative of each of them. Use parentheses to make absolutely certain the order of evaluation is unambiguous. When you use the Chain Rule to differentiate a composition $f \circ g$ say which function plays the role of g and which plays the role of f.

Exercise 15.9. Find the limit. Distinguish between an infinite limit and one which doesn't exist. (Give reasons!)

(**a**) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$. (**b**) $\displaystyle\lim_{x\to\infty} \frac{\sin 3x}{x}$. (**c**) $\displaystyle\lim_{x\to 0+} \frac{\sin 3}{x}$.

(**d**) $\displaystyle\lim_{h\to 0} \frac{\sin(3+h) - \sin 3}{h}$. (**e**) $\displaystyle\lim_{x\to 3} \frac{\sin x - \sin 3}{x - 3}$.

16 Exponentials and Logarithms

The material in this section of the notes corresponds roughly to Sections 7.3 and 7.4 of Thomas pages 486-500 but Section 7.3 depends on Section 7.2 which uses integration. Thomas postpones exponentials and logarithms till late in the semester, but other books[7] do it here. The theorems which are stated but not proved here will be proved in Section 38 of these notes and in Section 7.3 of Thomas.

Theorem 16.1. *For each positive number a there is a unique function called the* **exponential function base** *a satisfying the following conditions:*

(i) *The domain of $\exp_a$ is the set of all real numbers.*

(ii) *The range of $\exp_a$ is the set of positive real numbers.*

(iii) *The function $\exp_a$ is continuous.*

(iv) *The function $\exp_a$ converts addition into multiplication, i.e.*

$$\exp_a(x + y) = \exp_a(x) \cdot \exp_a(y).$$

(v) *The value of $\exp_a(x)$ when $x = 1$ is*

$$\exp_a(1) = a.$$

[7] My favorite is Stewart: *Calculus: Early Transcendentals.*

§16.2. The more familiar notation for the exponential function is

$$\boxed{\exp_a(x) = a^x}$$

so parts (iv) and (v) of Theorem 16.1 take the more familiar form

$$a^{x+y} = a^x \cdot a^y, \qquad a^1 = a. \tag{$\heartsuit$}$$

This implies that, for integers, exponentiation is repeated multiplication,[8] e.g. $a^3 = a^{1+1+1} = a^1 \cdot a^1 \cdot a^1 = a \cdot a \cdot a$. Using ($\heartsuit$) repeatedly gives

$$a^{nx} = a^{\overbrace{x+x+\cdots+x}^{n}} = \underbrace{a^x \cdot a^x \cdot \cdots \cdot a^x}_{n} = (a^x)^n$$

so taking $x = 1/n$ and using ($\heartsuit$) again proves that for any positive integer n

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

the nth root of a (i.e. the functions $b = a^n$ and $a = b^{\frac{1}{n}}$ are inverse functions). It follows easily that parts (iv) and (v) of Theorem 16.1 determines the value a^x uniquely when x is a rational number[9] and, as every real number is a limit of rational numbers, this shows that the conditions of Theorem 16.1 uniquely determine the exponential function. In Section 38 we give a formula for $\exp_a$ and show that it satisfies the conditions of Theorem 16.1.

§16.3. The following familiar laws of algebra all follow easily from equation ($\heartsuit$).

$$\boxed{\begin{array}{lll} a^0 = 1, & a^{x+y} = a^x \cdot a^y, & a^{-x} = \dfrac{1}{a^x}, \\ a^1 = a, & (ab)^x = a^x \cdot b^x, & (a^p)^x = a^{p \cdot x}. \end{array}}$$

For example, the reason why $(ab)^x = a^x \cdot b^x$ is that both sides satisfy the conditions of Theorem 16.1 (reading ab for a). Similarly, to prove $(a^p)^x = a^{p \cdot x}$ note that both sides (as functions of x and reading a^p for a) satisfy the conditions of Theorem 16.1 (at least if $p \neq 0$; if $p = 0$ both sides equal 1).

[8] Just as multiplication is repeated addition.
[9] A **rational number** is a ratio of integers, i.e. a fraction.

Theorem 16.4. *For $a > 1$, the exponential function is differentiable, strictly increasing, and satisfies so*

$$\lim_{x\to\infty} a^x = \infty, \qquad \textit{and} \qquad \lim_{x\to-\infty} a^x = 0.$$

These will be proved in section 38. That the exponential function $y = a^x$ is strictly increasing means that $a^{x_1} < a^{x_2}$ for $x_1 < x_2$. (For example, $2^3 < 2^4$.) This, combined with the Intermediate Value Theorem 9.7 means that the range of the exponential function is the set of all positive numbers and the graph of the exponential function passes the vertical line test so the exponential function has an inverse function.

Definition 16.5. The inverse function to the exponential function $\exp_a(x) = a^x$ is called the **logarithm function base** a:

$$y = a^x \iff x = \log_a(y).$$

The range of $\log_a$ is all real numbers, the domain is all positive rea numbers, and

$$\log_a(a^x) = x, \qquad a^{\log_a(y)} = y$$

by the Cancellation Law for inverse functions.

Remark 16.6. Since $(1/a)^x = 1/a^x = a^{-x}$, similar statements hold if $a < 1$. Thus for $b < 1$

$$\lim_{x\to\infty} b^x = 0 \qquad \text{and} \qquad \lim_{x\to\infty} b^{-x} = \infty$$

and the function b^x is strictly decreasing.

§ 16.7. Because the exponential and the logarithm are inverse functions the properties the latter may be expressed in terms of the former. For example, the exponential function converts addition into multiplication so the logarithm function coverts multiplication into addition. (This is why the logarithm function was invented.) The following table summarizes this.

$$\begin{array}{ll}
a^0 = 1, & \log_a(1) = 0, \\
a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}, & \log_a(y_1 \cdot y_2) = \log_a(y_1) + \log_a(y_2), \\
a^{-x} = 1/a^x, & \log_a(1/y) = -\log_a(y), \\
a^1 = a, & \log_a(a) = 1, \\
(a^p)^x = a^{p\cdot x}, & \log_a(y^p) = p\log_a(y).
\end{array}$$

For example, to prove $\log_a(y_1 \cdot y_2) = \log_a(y_1) + \log_a(y_2)$ let $x_1 = \log_a(y_1)$ and $x_2 = \log_a(y_2)$. Then $y_1 \cdot y_2 = a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$ so

$$\log_a(y_1 \cdot y_2) = \log_a(a^{x_1+x_2}) = x_1 + x_1 = \log_a(y_1) + \log_a(y_2).$$

To prove that $\log_a(y^p) = p\log_a(y)$ let $x = \log_a(y)$. Then $y = a^x$ so

$$y^p = (a^x)^p = a^{x\cdot p}$$

so

$$\log_a(y^p) = \log_a(a^{x\cdot p}) = x \cdot p = p \cdot x = p\log_a(y).$$

§16.8. We compute the derivative of $\exp_a(x)$ with respect to x:

$$\exp_a'(x) = \lim_{h\to 0} \frac{\exp_a(x+h) - \exp_a(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}.$$

This shows that

$$\exp_a'(x) = a^x \exp_a'(0),$$

i.e.

$$\frac{d}{dx}a^x = c \cdot a^x, \qquad c = \exp_a'(0), \tag{$\diamondsuit$}$$

i.e. the exponential function and its derivative are proportional.

§16.9. For the function $y = a^x = \exp_a(x)$ the formula ($\diamondsuit$) is

$$\frac{dy}{dx} = cy, \qquad c = \exp_a'(0) \tag{$\dagger$}$$

so the inverse function $x = \log_a(y)$ satisfies

$$\frac{dx}{dy} = \frac{1}{cy}. \tag{$\ddagger$}$$

In Theorem 38.3 of Section 38 we will prove that there is a number

$$e = 2.71827182845904523536\ldots$$

such that

$$\exp_e'(0) = 1.$$

We abbreviate $\exp_e$ by exp so by definition

$$\exp(x) = \exp_e(x) = e^x.$$

When $a = e$ the constant c in (†) is 1 so (†) simplifies to

$$\frac{d}{dx} e^x = e^x \tag{†'}$$

i.e. the function $y = e^x$ is its own derivative. The inverse function $\log_e(y)$ is called the **natural logarithm** and is usually written

$$\boxed{\ln(y) = \log_e(y).}$$

By (†′) its derivative is given by

$$\frac{d}{dy} \ln y = \frac{1}{e^x} = \frac{1}{y}.$$

To summarize:

> The derivatives of the exponential function $y = e^x$ and its inverse function $x = \ln(y)$ are given by
>
> $$\frac{d}{dx} e^x = e^x, \qquad \frac{d}{dy} \ln(y) = \frac{1}{y}.$$

§16.10. In science it is customary to express all exponentials in base e. Substituting a for y in the cancellation law $e^{\ln y} = e^{\log_e(y)} = y$ gives

$$a = e^{\ln a}$$

so

$$\boxed{a^x = e^{(\ln a)x}}$$

so by the Chain Rule

$$\boxed{\frac{d}{dx}a^x = e^{(\ln a)x}\ln a = a^x \ln a,}$$

i.e. the constant c in Equation (†) of §16.9 is $c = \ln a$. We express $\log_a(y)$ in terms of the natural logarithm as follows. Let

$$y = a^x \tag{$*$}$$

so by $a^x = e^{(\ln a)x}$ we have

$$y = e^{(\ln a)x}. \tag{$**$}$$

By ($*$) we have $x = \log_a(y)$ and by ($**$) we have $\ln y = x \ln a$. Hence

$$\boxed{\log_a(y) = \frac{\ln y}{\ln a}}$$

as both side equal x.

Example 16.11. We find $\frac{d}{dx}\sqrt{\ln x}$ using the Chain Rule as follows:

$$\frac{d}{dx}\sqrt{\ln x} = \frac{1}{2\sqrt{\ln x}} \cdot \frac{d}{dx}\ln x = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}.$$

Example 16.12. If $y = e^{x^2}$, then, by the Chain Rule, $\frac{dy}{dx} = e^{x^2} \cdot (2x)$. Note: e^{x^2} means $e^{(x^2)}$ not $(e^x)^2$. The latter is e^{2x}.

Exercises

Exercise 16.13. If x is large, which is bigger: 2^x or x^2? Hint: Try $x = 1, 2, 3, \ldots, 10$.

Exercise 16.14. Find $\lim_{x\to\infty} \frac{e^x - 1}{e^x + 1}$.

Exercise 16.15. Draw the graph of $y = 2^x$. What is $\lim_{x\to\infty} 2^x$? $\lim_{x\to-\infty} 2^x$?

Exercise 16.16. Find the second derivative of $\ln x$ with respect to x.

Exercise 16.17. Find the second derivative of 5^x with respect to x.

Exercise 16.18. Find the second derivative of x^2e^{3x} with respect to x.

Exercise 16.19. Find $\frac{d}{dx}x^2$ and $\frac{d}{dx}2^x$.

Exercise 16.20. Find $\frac{d}{dx}\sin(e^x)$ and $\frac{d}{dx}e^{\sin x}$.

Exercise 16.21. Find $\frac{d}{dx}\ln(\sin x)$ and $\frac{d}{dx}\sin(\ln x)$.

Exercise 16.22. Let $y = e^{-t}\cos t$. Show that $\frac{d^2y}{dt^2} = 2e^{-t}\sin t$.

Exercise 16.23. Let $y = xe^x$. Find $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y$.

Exercise 16.24. In Exercise 11.17 you were asked to compute d^2y/dx^2 and $(dy/dx)^2$ to reinforce the warning that these are usually not equal. Are they ever equal? Hint: Try $y = -\ln x$.

Exercise 16.25. Find $\frac{d}{dx}x^x$, $\frac{d}{dx}x^{x^x}$, and $\frac{d}{dx}(x^x)^x$. Hint: $x^x = e^?$.

Exercise 16.26. Let $y = (x+1)^2(x+3)^4(x+5)^6$ and $u = \ln y$. Find du/dx. Hint: Use the fact that ln converts multiplication to addition before you differentiate. It will simplify the calculation.

17 Parametric Equations

Parametric Equations are explained in Thomas pages 195-200.

Definition 17.1. A pair of equations

$$x = f(t), \qquad y = g(t)$$

assigns to each value of t a corresponding point $P(x, y)$. The set of these points is called a **parametric curve** and the equations are called **parametric equations** for the curve. The variable t is called the **parameter** and we say that the equations "**trace out**" or **parameterize** the curve. Often t has the interpretation of time and the parametric equations describe the position of a moving particle at time t, i.e. the point corresponding to to the parameter value t is the position of the particle at time t. Parameters other than time are also used. The following examples show that sometimes (but not always) we can eliminate the parameter and find an equation of the form

$$F(x, y) = 0$$

which describes the curve.

Example 17.2. Rectilinear Motion. Here's a parametric curve:

$$x = 1 + t, \qquad y = 2 + 3t.$$

Both x and y are linear functions of time (i.e. the parameter t), so every time t increases by an amount Δt (every time Δt seconds go by) the first component

x increases by Δt, and the second component y increases by $3\Delta t$. The point at $P(x, y)$ moves horizontally to the left with speed 1, and it moves vertically upwards with speed 3.

Which curve is traced out by these equations? In this example we can find out by eliminating the parameter, i.e. solving one of the two equations for t, and substituting the value of t you find in the other equation. Here you can solve $x = 1 + t$ for t, with result $t = x - 1$. From there you find that

$$y = 2 + 3t = 2 + 3(x - 1) = 3x - 1.$$

So for any t the point $P(x, y)$ is on the line $y = 3x-1$. This particular parametric curve traces out a straight line with equation $y = 3x-1$, going from left to right.

Example 17.3. Rectilinear Motion (More Generally). Any constants x_0, y_0, a, b such that either $a \neq 0$ or $b \neq 0$ give parametric equations

$$x = x_0 + a(t - t_0), \qquad y = y_0 + b(t - t_0) \tag{$*$}$$

which trace out the line

$$a(y - y_0) = b(x - x_0). \tag{$**$}$$

(Both sides equal $ab(t - t_0)$.) At time $t = t_0$ the point $P(x, y)$ is at $P_0(x_0, y_0)$. The values corresponding to Example 17.2 are $t_0 = 0$, $x_0 = 1$, $y_0 = 2$, $a = 1$, $b = 3$.

Example 17.4. Going back and forth on a straight line. Consider

$$x = x_0 + a \sin t, \qquad y = y_0 + b \sin t.$$

At each moment in time the point whose motion is described by this parametric curve is on the straight line with equation ($*$) as in Example 17.3. However, instead of moving along the line from one end to the other, the point at $P(x, y)$ keeps moving back and forth along the line ($**$) between the point P_1 corresponding to time $t = \pi/2$ and the point P_2 corresponding to time $t = 3\pi/2$.

Example 17.5. Motion along a graph. Let $y = f(x)$ be some function of one variable (defined for x in some interval) and consider the parametric curve given by

$$x = t, \qquad y = f(t).$$

At any moment in time the point $P(x, y)$ has x coordinate equal to t, and $y = f(t) = f(x)$, since $x = t$. So this parametric curve describes motion on the graph of $y = f(x)$ in which the horizontal coordinate increases at a constant rate.

Example 17.6. The standard parametrization of a circle. The parametric equations

$$x = \cos\theta, \quad y = \sin\theta$$

satisfy

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1,$$

so that $P(x, y)$ always lies on the unit circle. As θ increases from $-\infty$ to $+\infty$ the point will move around the circle, going around infinitely often. The point runs around the circle in the *counterclockwise direction,* which is the mathematician's favorite way of running around in circles. The more general equations

$$x = a\cos t, \quad y = b\sin t.$$

parameterize the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example 17.7. Another parametrization of a circle. The equations

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

also parameterize the unit circle. To see this divide both sides of the identity[10]

$$(1-t^2)^2 + (2t)^2 = (1+t^2)^2$$

by $(1+t^2)^2$ to get $x^2 + y^2 = 1$. However the point $Q(-1, 0)$ is left out since $y = 0$ only when $t = 0$ and $x = 1 \neq -1$ when $t = 0$.

Example 17.8. A parametrization of a hyperbola. The functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}$$

are called the **hyperbolic sine** and **hyperbolic cosine** respectively. This is because the equations

$$x = \cosh(t), \qquad y = \sinh(t),$$

parameterise the part of the hyperbola

$$x^2 - y^2 = 1$$

which to the right of the y-axis.

§17.9. For parametric equations as in Definition 17.1 the chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

so dividing gives the formula

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.}$$

[10] $(1-t^2)^2 + (2t)^2 = (1-2t^2+t^4) + 4t^2 = 1 + 2t^2 + t^4 = (1+t^2)^2$

We can use this formula to find the slope of the tangent line at a point on the curve. The following example illustrates this.

Example 17.10. The point $P_0\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ lies on the unit circle $x^2+y^2 = 1$. This point corresponds to the parameter value $\theta = \pi/6$ in the standard parameterization $x = \cos\theta$, $y = \sin\theta$ of Example 17.6. Since

$$\frac{dx}{d\theta} = -\sin\theta, \qquad \frac{dy}{d\theta} = \cos\theta$$

we get

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -\frac{\sqrt{3}}{2}, \qquad \left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = \frac{1}{2},$$

and so the slope of the tangent line at P_0 is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/6} = -\frac{1}{\sqrt{3}}.$$

The point slope equation $y = y_0 + m(x - x_0)$ for the tangent line is

$$y = \frac{1}{2} - \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right).$$

Remark 17.11. Let $P_0(x_0, y_0)$ be a point on a parametric curve corresponding to a parameter value $t = t_0$ and let

$$a = \left.\frac{dx}{dt}\right|_{t=t_0} \qquad \text{and} \qquad b = \left.\frac{dy}{dt}\right|_{t=t_0}.$$

Then

$$x = x_0 + a(t - t_0), \qquad y = y_0 + b(t - t_0),$$

are parametric equations for the tangent line to the curve at P_0. This is because the point slope equation for the tangent line is

$$y = y_0 + m(x - x_0), \qquad \text{where} \qquad m = \left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)}.$$

and the slope is $m = b/a$. (See Example 17.3.)

Exercises

Exercise 17.12. Confirm Example 17.8 by showing that

$$\big(\cosh(t)\big)^2 - \big(\sinh(t)\big)^2 = 1.$$

This is analogous to the Pythogorean Theorem

$$\big(\cos(t)\big)^2 + \big(\sin(t)\big)^2 = 1.$$

Also show that the hyperbolic sine and hyperbolic cosine are derivatives of each other. Thus we have the analogous equations

$$\frac{d}{dt}\sinh(t) = \cosh(t), \qquad \frac{d}{dt}\cosh(t) = \sinh(t),$$

$$\frac{d}{dt}\sin(t) = \cos(t), \qquad \frac{d}{dt}\cos(t) = -\sin(t).$$

Note the signs!

Exercise 17.13. The point $P_0(-\frac{3}{5}, \frac{4}{5})$ lies on the unit circle $x^2+y^2 = 1$. In the parameterization of Example 17.7 it corresponds to the parameter value $t = 2$. Use this parameterization to find the equation of the tangent line at this point. Then find the (same) equation using $y = \sqrt{1-x^2}$.

Exercise 17.14. Consider the parameterization

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

of the unit circlefrom 17.7. For which value of t is $(x, y) = (1, 0)$? $(0, 1)$? $(0, -1)$? $(\frac{3}{5}, \frac{4}{5})$? $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$? Is there a value of t for which $(x, y) = (-1, 0)$?

18 Approximation*

Approximation is explained in Thomas pages 221-231 and 807-810. In the latter reference Taylor Approximation is treated at a higher level than here. This section is included as a warmup for infinite series which we study in Math 222. Before reading this section you should do Exercises 11.21-11.26.

Definition 18.1. Let $f(x)$ be a differentiable function and a a point in its domain. The **linear approximation** to $f(x)$ at $x = a$ is the linear function $L(x)$ whose graph is the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$, i.e.

$$L(x) = f(a) + f'(a)(x-a)$$

Note that $L(a) = f(a)$ and $L'(a) = f'(a)$.

Theorem 18.2 (Linear Approximation Theorem). *The linear approximation $L(x)$ is the linear function which best approximates $f(x)$ near $x = a$ in the sense that*

$$\lim_{x\to a}\frac{f(x)-L(x)}{x-a} = 0.$$

Proof: $\displaystyle\lim_{x\to a}\frac{f(x)-L(x)}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} - f'(a)\right) = 0.$

Remark 18.3. The idea is that $L(x)$ is a good approximation to $f(x)$ when x is close to a, i.e. the "error" $f(x) - L(x)$ is small. Theorem 18.2 says that not only is the error small it is so small that even when it is divided by the small number $x - a$, the result is still small.

Definition 18.4. Given a number a in the domain of f and an integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \qquad (\#)$$

is called the **degree n Taylor polynomial of f at the point** a.

§18.5. The letter $\sum$ is the Greek S (for *sum*) and is pronounced *sigma* so the notation used in (#) is called **sigma notation**. It is a handy notation but if you don't like it you can indicate the summation with dots:

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_{n-1} + a_n.$$

Hence the first few Taylor polynomials are

$$P_0(x) = f(a),$$

$$P_1(x) = f(a) + f'(a)(x-a),$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2},$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}.$$

The linear approximation to f at a is the degree one Taylor Polynomial

$$L(x) = P_1(x) = f(a) + f'(a)(x-a).$$

The degree two Taylor Polynomial

$$Q(x) = P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2}$$

is also called the **quadratic approximation** to f at a.

Theorem 18.6. *The Taylor polynomial $P_n(x)$ is the unique polynomial of degree n which has the same derivatives as f at a up to order n:*

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad\qquad \textit{for } k = 0, 1, 2, \ldots, n.$$

(Of course, $P_n^{(k)}(x) = 0$ for $k > n$.)

§18.7. *The Taylor polynomial $P_n(x)$ for $f(x)$ at a is the polynomial of degree n which best approximates $f(x)$ for x near a.* To make this precise let

$$R_n(x) = f(x) - P_n(x)$$

denote the nth **Taylor Error** of f at a. When $R_n(x)$ is small $P_n(x)$ is a good approximation for $f(x)$. How small is small? The answer is given by

Theorem 18.8 (Taylor's Formula). *Suppose that f is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous. Let a be a point in the domain of f. $P_n(x)$ be the Taylor Polynomial for f at a, and $R_n(x)$ be Taylor Error for f at a. Then*

(i) $f(x) = P_n(x) + R_n(x)$

(ii) $\displaystyle\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0.$

(iii) *$P_n(x)$ is polynomial of degree $\le n$ whch satisfies (iii).*

We'll study this more in Math 222 but for now we'll be satisfied to show how it gives accurate approximations to complicated functions. (Your calculator uses this method.)

Example 18.9. Take $f(x) = x^{1/3}$ and $a = 8$. Then

$$f'(x) = x^{-2/3}/3, \qquad f''(x) = -2x^{-5/3}/9,$$

so

$$f(a) = 2, \qquad f'(a) = 1/12, \qquad f''(a) = -1/144,$$

and hence

$$L(x) = 2 + \frac{1}{12}(x-8), \qquad Q(x) = 2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{288}.$$

Notice that

$$L(9) = 2 + \frac{1}{12} = 2.083333333\ldots$$

is close to

$$f(9) = 9^{1/3} = 2.080083823\ldots$$

and

$$Q(9) = L(9) - \frac{1}{288} = 2.079861111\ldots$$

is even closer.

Exercises

Exercise 18.10. Evaluate $\sum_{k=1}^{5} \frac{1}{k}$.

Exercise 18.11. Let $f(x) = x^{1/3}$. Find the polynomial $P(x)$ of degree three which best approximates $f(x)$ near $x = 8$. Calculate $P(9)$ and compare it with the value of $9^{1/3}$ given by your calculator.

Exercise 18.12. Let $f(x) = \sqrt{x}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(4) = f^{(k)}(4)$ for $k = 0, 1, 2, 3$. Calculate $P(3)$, $P(5)$, $P(3.5)$, $P(4.5)$, $P(3.9)$, $P(4.1)$ and compare the results with the values $\sqrt{3}$, $\sqrt{5}$, $\sqrt{3.5}$, $\sqrt{4.5}$, $\sqrt{3.9}$, $\sqrt{4.1}$

Exercise 18.13. If $f(x) = e^x$ what is $f^{(k)}(x)$? $f^{(k)}(0)$? Find the first five Taylor polynomials $P_k(x)$ $(k = 1, 2, \ldots, 5)$ at $a = 0$ for e^x. Evaluate $P_k(1)$ and compare the result with $e = f(1)$.

Exercise 18.14. If $f(x) = \ln x$ what is $f^{(k)}(x)$? $f^{(k)}(1)$? Find the first five Taylor polynomials $P_k(x)$ $(k = 1, 2, \ldots, 5)$ at $a = 1$ for $\ln x$. Evaluate $P_k(2)$ and compare the result with $\ln 2$.

Exercise 18.15. If $f(x) = \sin x$ what is $f^{(k)}(x)$? $f^{(k)}(0)$? Hint: What is $f^{(4)}(x)$? Find the first five Taylor polynomials $P_k(x)$ $(k = 1, 2, \ldots, 5)$ at $a = 0$ for $\sin x$. Evaluate $P_k(0.5)$ and compare the result with $\sin 0.5 = f(0.5)$. Warning: Make sure your calculator is computing in radians.

Exercise 18.16. Is the linear approximation the only linear function satisfying the conclusion of Theorem 18.2. Why? Is the Taylor polynomial of degree n the only polynomial of degree n to satisfy the conclusion of Theorem 18.8?

Exercise 18.17. Denote the linear approximations to f near a and to g near b respectively by

$$L(x) = f(a) + f'(a)(x-a), \qquad M(y) = g(b) + g'(b)(y-b),$$

and assume $b = f(a)$. Show that

$$M \circ L(x) = (g \circ f)(a) + (g \circ f)'(a)(x - a).$$

(This says that *The linear approximation to the composition is the composition of the linear approximations.*)

19 Additional Exercises

Exercise 19.1. Let $f(x) = \sqrt{(a + x)(b + x)}$ where a and b are constants. Show that

$$f''(x) = -\frac{(b-a)^2}{4f(x)^3}.$$

Exercise 19.2. Find all points on the parabola with the equation $y = x^2 - 1$ such that the normal line at the point goes through the origin.

Exercise 19.3. (i) Find c so that function

$$f(x) = \begin{cases} x + c & \text{for } x < 1 \\ 3^x & \text{for } x \geq 1 \end{cases}$$

is continuous. (ii) Draw a crude graph of the equation $y = f(x)$. (iii) Give a formula (like the above formula for $f(x)$) for the inverse function $x = f^{-1}(y)$ of the function $y = f(x)$.

%!TEX root = lecs-221.tex

Chapter IV

Applications of Derivatives

20 The Derivative as A Rate of Change

See Thomas pages 171-177.

§20.1. The area of a circle of radius r is

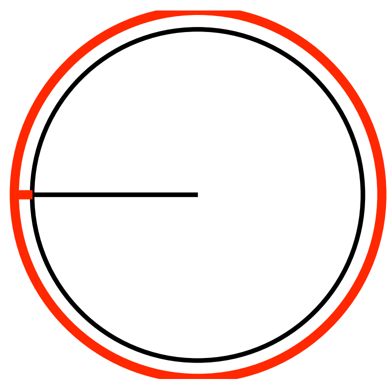
$$A = \pi r^2$$

and the circumference of a circle is

$$C = 2\pi r.$$

It is no coincidence that

$$\frac{dA}{dr} = 2\pi r = C.$$



The picture shows a circle of radius r and a slightly larger circle with the same center of radius $r + \Delta r$. The difference in the areas is

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r \Delta r + \pi(\Delta r)^2 \approx 2\pi r \Delta r.$$

The picture shows that ΔA is the area between the two circles and illustrates that it is roughly (and in the limit exactly) the circumference of the inner circle times the change Δr in the radius. (Imagine cutting up the area between the two circles in a bunch of small rectangles of height Δr and whose bases sum to C.)

§20.2. The volume of a sphere of radius r is

$$V = \frac{4}{3}\pi r^3$$

and the area is

$$S = 4\pi r^2.$$

It is no coincidence that

$$\frac{dV}{dr} = 4\pi r^2 = S$$

To see this rotate the two circles in the picture about a common diameter to make two concentric spheres. The spherical shell between the two spheres has volume

$$\Delta V = \frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3 \approx 4\pi r^2 \Delta r.$$

This volume is roughly (and in the limit exactly) the area of the sphere times the change Δr in the radius. (Imagine cutting up the volume between the two spheres in a bunch of small blocks of height Δr and whose bases have areas which sum to S.)

§20.3. The **average rate of change** of a function $y = f(x)$ as x changes from x_0 to $x_0 + \Delta x$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The **instantaneous rate of change** of a function $y = f(x)$ at $x = x_0$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0).$$

For example, suppose that the position of a particle at time t is

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and t is measured in meters. The distance travelled over a tiny time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t).$$

The **average velocity** over that time interval iss

$$v_{\text{av}} = \frac{\Delta s}{\Delta t}$$

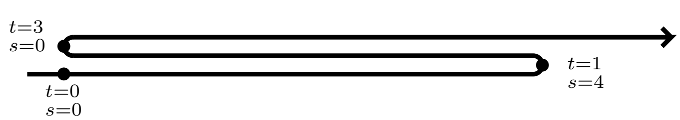
and the **instantaneous velocity** is the limit

$$v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

as the size Δt of the time interval shrinks to zero. Thus

$$v_{\text{inst}} = \frac{ds}{dt} = f'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3).$$

The particle is at rest when $ds/dt = 0$, i.e. when $t = 1$ and when $t = 3$. The particle is moving in the positive direction when ds/dt is positive, i.e. for $t < 1$ and for $t > 3$ and is moving in the negative direction when ds/dt is negative, i.e. for $1 < t < 3$. Here is a schematic diagram:



Exercises

Exercise 20.4. The position in meters of a particle is given by $s = t^3 - 4.5t^2 - 7t$ where t is the time in seconds. When does the particle reach a velocity of 5 meters per second?

Exercise 20.5. Find the average rate of change of the area of a circle with respect to its radius r as r changes from 2 to 3, from 2 to 2.5, and from 2 to 2.1. Then find the instantantaneous rate when $r = 2$.

Exercise 20.6. Find the average rate of change of the volume of a cube with respect to its edge length x as x changes from 5 to 6, from 5 to 5.1, and from 5 to 5.01. Then find the instantantaneous rate when $x = 5$.

Exercise 20.7. If a tank holds for 5000 gallons of water which drains from the bottom of the tank in 40 minutes then Toricelli's law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{t}{40}\right)^2, \qquad 0 \leq t \leq 40.$$

Find the rate at which water is draining from the tank after 5 minutes, after 10 minutes, and after 20 minutes.

Exercise 20.8. Water runs out of a cylindrical tank from a drain in the bottom. The water level in the tank t hours after the tank starts to drain is

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ meters}$$

and the tank drains completely after 12 hours. Find the rate (measured in meters per hour) at which the depth is decreasing after t hours. When is the depth decreasing the fastest? The slowest? Show that the rate at which the water level decreases is proportional to the square root of the water level.

Exercise 20.9. A heavy object is shot straight up from the Earth's surface at 200 feet per second. Elementary physics tells us that its height after t seconds is

$$y = 200t - 16t^2.$$

Find the velocity

$$v = \frac{dy}{dt}$$

and the acceleration

$$a = \frac{d^2y}{dt^2}$$

as functions of t. When is the object (momentarily) at rest? When does it move up? Down? When is it highest? When does it change direction? How high does it go? When it it moving fastest?

Exercise 20.10. The size of a poulation of bacteria in a nutrient broth after t hours is

$$N = 10^6 e^{0.06t}.$$

How large is the population and how fast is it growing at time $t = 0$? $t = 1$? $t = 2$?

Exercise 20.11. If f is the focal length of a lens and an object is placed at a distance p from the lens then the image will be at a distance q from the lens where

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

Find the rate of change of p with respect to q.

21 Related Rates

See Thomas pages 213-220.

§ 21.1. The first step (and usually the hardest step for students) in solving a word problem is to reformulate it in mathematical notation. This usually means expressing some of the quantities in the problem as functions of the other quantities. If the formulation of the problem does not explicitly assign letter names to these quantities, you will first have to name them yourself. In these problems it is very helpful to keep track of the units.

§21.2. The Balloon Problem. Air is pumped into a spherical balloon so that its volume increases at a rate of 100 cubic centimeters per second. How fast is the radius increasing at the instant when it is 25 centimeters?

SOLUTION. To solve this problem we first name everything in sight:

> *Let V denote the volume of the balloon, r denote the radius of the balloon, and t denote the time in seconds.*

The quantities are related by

$$V = \frac{4\pi}{3} r^3.$$

We write the given information in mathematical notation:

$$\frac{dV}{dt} = 100\text{cm}^3/\text{sec}.$$

We write what the problem asks us to find in mathematical notation:

$$\left.\frac{dr}{dt}\right|_{r=25} = ?$$

Now we use the chain rule.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

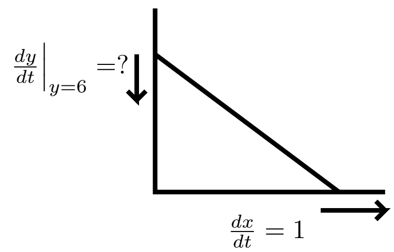
so when $r = 25$ we have

$$\left.\frac{dr}{dt}\right|_{r=25} = \left.\frac{dV/dt}{4\pi r^2}\right|_{r=25} = \frac{100}{4\pi 625}.$$

The answer has the units of

$$\frac{\text{cm}^3/\text{sec}}{\text{cm}^2} = \text{cm/sec}.$$

§21.3. The Ladder Problem. A ladder 10 feet long leans against a vertical wall. The bottom of the ladder moves away from the wall at one foot per second. How fast is the top sliding down the wall when it is 6 feet above the ground?



Solution. We name everything in sight: Let x be the distance of the bottom of the ladder from the wall, y be the height of the top of the ladder above the ground and t be the time in seconds. By the Pythogorean Theorem x and y are related by

$$x^2 + y^2 = 10^2.$$

Hence $x = \sqrt{10^2 - y^2}$ so $x = 8$ when $y = 6$. Differentiating gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

so

$$\left.\frac{dy}{dt}\right|_{y=6} = -\left.\frac{2x \cdot dx/dt}{2y}\right|_{y=6} = \frac{2 \cdot 8 \cdot 1}{2 \cdot 6}.$$

Exercises

Exercise 21.4. The radius r of a sphere at time t is given by the formula $r = \sqrt{t^2+1} - 1$.

1. Give a formula for the rate of change in the radius with respect to time.
2. Give a formula for the rate of change of the volume of the sphere with respect to time.
3. How fast is the volume changing when $t = 2$?
4. How fast is the volume changing when $r = 2$?

Exercise 21.5. Two cars, car A traveling west at 30 miles per hour and car B traveling south at 22.5 miles per hour, are heading toward an intersection I.

1. At what rate is the angle IAB changing at the instant when cars A and B are 300 feet and 400 feet, respectively, from the intersection?

2. At what rate is the angle IAB changing at the instant when cars A and B are 300 meters and 400 meters, respectively, from the intersection?

Exercise 21.6. A point is moving on the curve $y = x^2$. At the instant when it is passing through $(3, 9)$, y is changing so that $\dfrac{dy}{dt}$ is 7. How fast is x changing at that time?

Exercise 21.7. Sand is flowing from a pipe at the constant rate of s cubic meters per second, and is falling in a conical pile. The diameter of the base of this pile is always three times the altitude. At what rate is the altitude of the pile increasing when the altitude is h meters?[11]

Exercise 21.8. A highway patrol plane flies 3 miles above (and along) a level straight road at a steady 120 miles per hour. The pilot sees an oncoming car with radar and determines that at the moment that the distance from the plane to the car is 5 miles, this distance is decreasing at the rate of 160 miles per hour. Find the car's speed along the highway. This problem comes from Thomas page 221.

Exercise 21.9. If the highway patrol plane in the Exercise 21.8 were really designed to catch speeders, it might have a device which works as follows. It records the altitude h, the velocity u of the plane, the angle ϕ between the line of sight to the car and the vertical, and the rate of change $d\phi/dt$ of ϕ. (The values from the previous problem are $h = 3$, $u = 120$, and, at the moment the measurement is taken, $\phi = \cos^{-1}(3/5)$.) Then it computes the speed v of the car. Find a formula expressing v in terms of h, u, ϕ and $d\phi/dt$. As before assume that h and u are constant, but of course ϕ and $d\phi/dt$ are not.

Exercise 21.10. A light at the top of a pole which is h feet high. A ball is dropped from half the height of h at a point which is at a horizontal distance a in feet from the pole. Assume that the ball falls a distance $s = gt^2/2$ feet, where t is the time in seconds since it was dropped and g is a constant. Find how fast the tip of the shadow of the ball is moving along the ground t seconds after it is dropped. (Express the answer in terms of h, a, g, and t.)

Exercise 21.11. A lighthouse is located 1000 feet from the nearest point on shore and rotates three times per minute. How fast is the end of the beam of light it emits moving along along the shore when it passes the closest point on shore?

Exercise 21.12. You are videotaping a race from a stand 120 feet from the track, following a car that is moving at a constant velocity along a straight track.

[11] The volume of a right circular cone whose altitude is h and whose base has radious r is $V = \dfrac{\pi r^2 h}{3}$. We will prove this using calculus in Chapter VI.

When the car is directly in front of you the camera angle is changing at a rate of $\pi/3$ radians per second. How fast is the car going? How fast will the camera angle be changing a half second later? (The camera angle - race car problem is like the lighthouse problem in that both use the formula $x = b \tan \phi$ but in the former problem dx/dt is constant while in the latter $d\phi/dt$ is constant.)

22 Some Theorems about Derivatives

See Thomas pages 255-270.

Theorem 22.1 (First Derivative Test). *Suppose that a function $f(x)$ defined on an interval $a \le x \le b$ attains its minimum (or its maximum) at a point c in the interval. Then either*

(1) *c is an endpoint, i.e. $c = a$ or $c = b$; or*

(2) *f is not differentiable at c; or*

(3) *c is a* **critical point** *of f, i.e. $f'(c) = 0$.*

Proof. Assume that (1) and (2) fail, i.e. that $a < c < b$ and that $f'(c)$ exists. We will prove (3). Since $f'(c)$ exists we have that

$$\frac{f(x) - f(c)}{x - c} \approx f'(c) \qquad (*)$$

for $x \approx c$. If the ratio on the left is positive, then the numerator $f(x) - f(c)$ and the denominator $x - c$ have the same sign; if the ratio on the left is negative, then $f(x) - f(c)$ and $x - c$ have opposite signs. Since c is an interior point, there are numbers in the interval to the right of c and close to c and other numbers in the interval to the left of c and close to c. Thus

- If $f'(c)$ is positive, then $f(x) - f(c) > 0$ for x near and to the right of c, so c is not a maximum.
- If $f'(c)$ is positive, then $f(x) - f(c) < 0$ for x near and to the left of c, so c is not a minimum.
- If $f'(c)$ is negative, then $f(x) - f(c) < 0$ x near and to the right of c, so c is not a minimum.
- If $f'(c)$ is negative, then $f(x) - f(c) > 0$ for x near and to the left of c, so c is not a maximum.

Thus the only possibility is $f'(c) = 0$. □

Remark 22.2. A function $f(x)$ is said to have a

- **global minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f;
- **global maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f;
- **local minimum** at c if $f(c) \leq f(x)$ for all x some open interval about c;
- **local maximum** at c if $f(c) \geq f(x)$ for all x some open interval about c.

(Some books use the words "absolute" and "relative" for "global" and local".) The first derivative test says that the derivative of a differentiable function vanishes at a local minimum or maximum.

§22.3. Examples. (1) The function $L(x) = 2x+3$ satisfies $L'(x) = 2$ and so is differentiable and has no critical point. On any interval it attains its minimum at the left endpoint and its maximum at the right endpoint.

(2) The absolute value function $g(x) = |x|$ satisfies

$$g'(x) = \frac{x}{|x|}$$

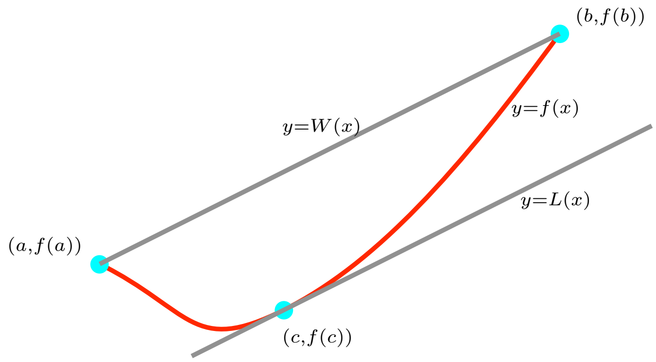
for $x \neq 0$ but $g'(0)$ does not exist. On any interval containing 0 it attains its minimum at 0 and its maximum at one of the two endpoints.

(3) The derivative of the function $f(x) = x^3 - 3x$ is $f'(x) = 3(x^2 - 1)$ and $f'(x) = 0$ for $x = \pm 1$. The point $x = -1$ does not lie in the interval $0 \leq x \leq 2$ and $f(0) = 0$, $f(1) = -2$, and $f(2) = -1$ so on the interval $0 \leq x \leq 2$ the function attains its minimum value at 1 and its maximum value at 0.

Theorem 22.4 (Mean Value Theorem). *Assume $f(x)$ is continuous on the closed interval $a \leq x \leq b$ and differentiable on $a < x < b$. Then there is a point c with $a < c < b$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

i.e. the tangent line to the graph at $(c, f(c)$ is parallel to the "secant line" joining $(a, f(a))$ and $(b, f(b))$.



Proof: Consider the linear function

$$W(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

The graph $y = W(x)$ is the line joining $(a, f(a))$ and $(b, f(b))$, i.e.

$$W(a) = f(a), \qquad W(b) = f(b).$$

The function

$$g(x) = f(x) - W(x)$$

satisfies

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \qquad g(a) = g(b) = 0.$$

By the Extreme Value Theorem §9.11 the function g attains its maximum and its minimum. Since $g(a) = g(b) = 0$ at least one of the maximum or the minimum must occur at an interior point c. By the First Derivative Test 22.1 we have

$$g'(c) = 0$$

as required. (The special case of the Mean Value Theorem where the values at the endpoints are the same is called **Rolle's Theorem**.)

Definition 22.5. A function $y = f(x)$ is said to be **increasing** on an interval iff

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for any two points x_1, x_2 of the interval. (The symbol $\implies$ means *implies*.) Similarly f is said to be **decreasing** on an interval iff

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

A function is **monotonic** on an interval iff either it is increasing on that interval or else it is decreasing on that interval.

Theorem 22.6 (Monotonicity Theorem). *If* $\left\{\begin{array}{l} f'(x) > 0 \\ f'(x) < 0 \\ f'(x) = 0 \end{array}\right\}$ *for all* x *in an interval* I*, then* f *is* $\left\{\begin{array}{l} \textit{increasing} \\ \textit{decreasing} \\ \textit{constant} \end{array}\right\}$ *on that interval.*

Proof. Choose x_1 and x_2 in the interval I with $x_1 < x_2$. By the Mean Value Theorem there is a c with $x_1 < c < x_2$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c). \tag{\#}$$

Since c is between x_1 and x_2 it lie in the interval I and hence $f'(c)$ has the sign (positive, negative, or zero) of the hypothesis. Hence the ratio on the left in equation (#) has this same sign. Since $x_1 < x_2$ the denominator $x_2 - x_1$ is positive and hence the numerator $f(x_2) - f(x_1)$ has this same sign. If the sign is positive, then $f'(c) > 0$ so $f(x_2) - f(x_1) > 0$ so $f(x_1) < f(x_2)$. If the sign is negative, then $f'(c) < 0$ so $f(x_2) - f(x_1) < 0$ so $f(x_1) > f(x_2)$. If the derivative is identically zero, then $f'(c) = 0$ so $f(x_2) - f(x_1) = 0$ so $f(x_1) = f(x_2)$.

Theorem 22.7 (Second Derivative Test). *If* $f''(x) > 0$ *for all* x *in some open interval* I*, and* $f'(c) = 0$ *at some* c *in* I*, then* $f(x)$ *assumes its minimum on* I *at* c*. Similarly, if* $f''(x) < 0$ *for all* x *in some open interval* I*, and* $f'(c) = 0$ *at some* c *in* I*, then* $f(x)$ *assumes its maximum on* I *at* c*.*

Proof: Let a and b be the endpoints of the interval so $a < c < b$. Assume $f''(x) > 0$ for $a < x < b$. Then $f'(x)$ is increasing on the interval. But $f'(c) = 0$ so $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, i.e. $f(x)$ is decreasing for $a < x < c$ and increasing for $c < c < a$. Hence $f(x)$ is smallest when $x = c$.

Remark 22.8. The proof shows more generally that if $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$ then $f(x)$ has a minimum at c on the interval $a < x < b$.

Definition 22.9. A function $y = f(x)$ is **concave up** on an interval iff the second derivative $f''(x)$ is positive at every point x in the interval. The derivative of f' is f'' so, by the Monotonicity Theorem, the derivative f' of a concave up function is increasing. Similarly, the function is **concave down** iff its second derivative is negative. For example, the function $f(x) = x^2$ is concave up on any interval and the function $g(x) = -x^2$ is concave down.

Theorem 22.10 (Secant Concavity Theorem). *Suppose that $f(x)$ is concave up on the interval $a \le x \le b$ and define*

$$W(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right) \cdot (x - a)$$

so that the graph $y = W(x)$ is the secant line joining the points $(a, f(a))$ and $(b, f(b))$. Then the graph $y = f(x)$ lies below the graph of the secant line, i.e.

$$f(x) < W(x).$$

for $a < x < b$. Similarly, if $f(x)$ is concave down, the graph of the function lies above the graph of the secant line.

Proof: Define $g(x) = f(x) - W(x)$ as in the proof of the Mean Value Theorem. Since $W(x)$ is a linear function, its second derivative is zero so $g''(x) = f''(x)$ and g is also concave up. By the Mean Value Theorem there is a point c with $a < c < b$ and $g'(c) = 0$. Since g' is increasing this means that $g'(x) < g'(c) = 0$ for $a < x < c$ and $g'(x) > 0$ for $c < x < b$. Hence g is decreasing on the interval $a < x < c$ and increasing on the interval $c < x < b$. Hence $g(a) > g(x)$ for $a < x < c$ and $g(x) < g(b)$ for $c < x < b$, i.e. $g(x) < 0$ for $a < x < b$. As $g = f - W$ we get $f(x) < W(x)$ for $a < x < b$ as required. If f is concave down, then $-f$ is concave up, so $-f < -W$, and hence $W < f$.

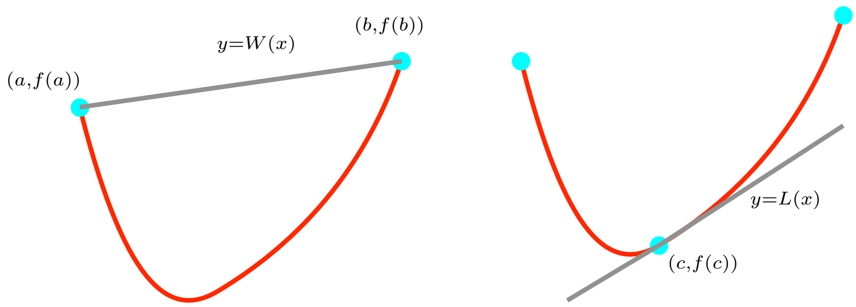


Figure 1: The Secant and Tangent Concavity Theorems

Theorem 22.11 (Tangent Concavity Theorem). *Suppose that $f(x)$ is concave up on the interval $a \le x \le b$ and let c be an interior point of that interval, i.e. $a < c < b$. Define*

$$L(x) = f(c) + f'(c)(x - c)$$

so that the graph $y = L(x)$ is the tangent line to the graph $y = f(x)$ at the point $(c, f(c))$. Then the graph $y = f(x)$ lies above the graph of the tangent line, i.e.

$$L(x) \le f(x)$$

for $a \le x \le b$. Similarly if $f(x)$ is concave down, the graph of the function lies below the graph of the tangent line.

Proof: Consider the function $g(x) = f(x) - L(x)$. As the function $L(x)$ is linear, its second derivative is zero, so $g''(x) = f''(x)$ and g is also concave up. Moreover $f'(c) = L'(c) =$ the slope of the tangent line at c, so $g'(c) = 0$. Since g' is increasing this means that $g'(x) < g'(c) = 0$ for $a < x < c$ and $g'(x) > 0$ for $c < x < b$. Hence g is decreasing on the interval $a < x < c$ and increasing on the interval $c < x < b$. Hence $g(x) > g(c) = 0$ for $a < x < c$ and $0 = g(c) < g(x)$ for $c < x < b$, i.e. $g(x) > 0$ for $a < x < b$. As $g = f - L$ we get $L(x) < f(x)$ for $a < x < b$ as required. If f is concave down, then $-f$ is concave up, so $-L < -f$, and hence $f < L$.

Remark 22.12. When I learned calculus the terms *convex* and *concave* were used; I could never remember which was which. Small wonder. In common parlance the concave side of a curve is a convex set. The present terminology is better because of the following dumb poem:

Concave up is like a cup $\smile$, Concave down is like a frown $\frown$.

Example 22.13. If $y = x^3$, then $d^2y/dx^2 > 0$ for $x > 0$ so by the Secant and Tangent Concavity Theorems it satisfies the following inequalities:

$$x^3 < a^3 + \frac{b^3 - a^3}{b - a}(x - a) \text{ for } 0 < a < x < b < \infty$$

and

$$c^3 + 3c^2(x - c) < x^3 \text{ for } 0 < c, x < \infty, \quad x \neq c.$$

Similarly, the function $d^2y/dx^2 < 0$ for $x < 0$ so

$$a^3 + \frac{b^3 - a^3}{b - a}(x - a) < x^3 \text{ for } -\infty < a < x < b < 0$$

and

$$x^3 < c^3 + 3c^2(x - c) \text{ for } -\infty < c, x < 0, \quad x \neq c.$$

To make sure you appreciate this, plug in a few particular values for a, b, c and x, say $a = 1$, $b = 4$, $c = 2$, $x = 3$, and evaluate both sides of the inequality.

Exercises

Exercise 22.14. Let $f(x) = x^3 + 3x^2 + 3x + 5$. Does f have an inverse? How do you know? If f has an inverse, determine $f^{-1}(5)$, $\left(f^{-1}\right)'(5)$, $f^{-1}(12)$, and $\left(f^{-1}\right)'(12)$. Hint: For what values of x is $f'(x) < 0$?

Exercise 22.15. True or false?

T F. An increasing function has an inverse.[12]

T F. A decreasing function has an inverse.

T F. If a continuous function has an inverse it must be monotonic. (Hint: Intermediate Value Theorem.)

T F. If a function has an inverse it must be monotonic.

Exercise 22.16. On what intervals is the function $y = x^3 - x$ increasing? Decreasing? Concave up? Concave down?

Exercise 22.17. Repeat the previous exercise for the following functions:

(a) $y = x^4 - x^2$ (b) $y = 1/(1 + x^2)$ (c) $y = (1 + x)/(1 - x)$

(d) $y = e^x$ (e) $y = \ln(x)$ (f) $y = 2^x$

(g) $y = \sin(x)$ (h) $y = \cos(x)$ (i) $y = \tan(x)$

[12] i.e. satisfies the horizontal line test.

Exercise 22.18. Write out the inequalities asserted by the Secant and Tangent Concavity Theorems for each of the functions $y = f(x)$ of the previous exercise and each interval on which the function does not change concavity. (See Example 22.13.)

Exercise 22.19. Find a point on the curve $y = x^3$ where the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$. What theorem does this illustrate?

23 Curve Plotting

See Thomas pages 270-277.

Definition 23.1. A point on a curve where the concavity changes is called a **point of inflection**. On one side of a point of inflection the tangent line is below the curve, on the other side it is above. At the point of inflection the tangent line crosses the curve.

Theorem 23.2. *If the curve is the graph $y = f(x)$ of a twice differentiable function, the second derivative $f''(x)$ vanishes at a point of inflection.*

Definition 23.3. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ iff either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to\infty} f(x) = b$ or both. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ iff either $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$ or both.

Example 23.4. **(i)** The line $y = 0$ is a horizontal asymptote for the function $f(x) = 1/(1 + x^2)$.

(ii) The lines $y = \pm\pi/2$ is are horizontal asymptotes for the function $f(x) = \tan^{-1}(x)$.

(iii) The line $x = 3$ is a vertical asymptote for the function $f(x) = 1/(x - 3)^2$.

(iv) The line $x = 2$ is a vertical asymptote for the function $f(x) = 1/(x - 3)$.

(v) The lines $x = (n + \frac{\pi}{2})$ are the vertical asymptotes for the function $f(x) = \tan x$.

§23.5. To graph $y = f(x)$ on $a < x < b$ proceed as follows:

(1) find the interesting values of x: critical points, inflection points, vertical asymptotes.

(2) find the value of y at each interesting point:

- At a critical point or inflection point x, find $f(x)$.
- At a vertical asymptote $x = c$ find both one sided limits

$$f(c-) = \lim_{x\to c-} f(x), \text{ and } f(c+) = \lim_{x\to c+} f(x).$$

(These values are $\pm\infty$.)

- At the endpoints a and b find

$$f(a+) = \lim_{x \to a+} f(x), \text{ and } f(b-) = \lim_{x \to b-} f(x).$$

(Usually $a = -\infty$ and $b = +\infty$.)

(3) Make a table. On each interval bounded by a pair of adjacent interesting points, find the sign of $f'(x)$ and the sign of $f''(x)$. (Now you know on which intervals f is increasing and on which intervals f is concave up.)

(4) Draw the graph.

Example 23.6. we graph $y = x^3 - 3x$. The formulas for the first derivative $y' = dy/dx$ and $y'' = d^2y/dx^2$ are

$$y' = 3x^2 - 3, \qquad y'' = 6x.$$

Here is a table showing the intervals on which f is increasing, decreasing, concave up, concave down:

x	$-\infty$		-1		0		1		∞
y	$-\infty$		2		-3		-2		∞
y'		$+++$	0	$---$	$-$	$---$	0	$+++$	
y''		$---$	$-$	$---$	0	$+++$	$+$	$+++$	

Figure 23 shows the graph and the tangent line at the point of inflection.

Example 23.7. We graph $y = x/(1-x^2)$. The formulas for the first derivative $y' = dy/dx$ and $y'' = d^2y/dx^2$ are

$$y' = \frac{1+x^2}{(1-x^2)^2}, \qquad y'' = \frac{2x(3+x^2)}{(1-x^2)^3}.$$

The x-axis is a horizontal asymptote at both ends as

$$\lim_{x \to -\infty} \frac{x}{1-x^2} = 0, \qquad \lim_{x \to \infty} \frac{x}{1-x^2} = 0,$$

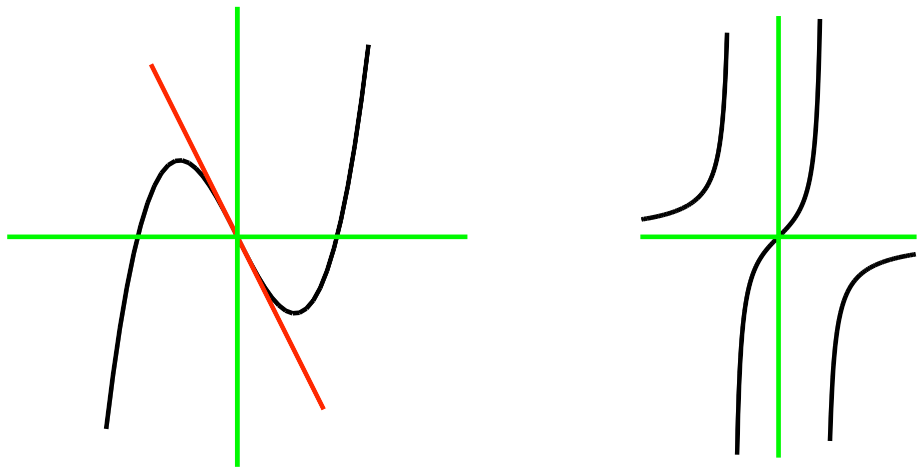
and the lines $x = -1$ and $x = 1$ are vertical asymptotes with

$$\lim_{x \to -1-} \frac{x}{1-x^2} = \infty, \qquad \lim_{x \to -1+} \frac{x}{1-x^2} = -\infty,$$
$$\lim_{x \to 1-} \frac{x}{1-x^2} = \infty, \qquad \lim_{x \to 1+} \frac{x}{1-x^2} = -\infty.$$

Here is a table showing the intervals on which f is increasing, decreasing, concave up, concave down:

x	$-\infty$		$-1-$	$-1+$		0		$1-$	$1+$		∞
y	0		∞	$-\infty$		0		∞	$-\infty$		0
y'		$+++$			$+++$	$+$	$+++$			$+++$	
y''		$+++$			$---$	0	$+++$			$---$	

Figure 23 shows the graph.



$y = x^3 - x$ $\qquad$ $y = 1/(1-x^2)$

Figure 2: Two Graphs

Exercises

Exercise 23.8. Draw the graph of a function $y = f(x)$ having all the following properties:

a) horizontal asymptote $y = -1/2$ (as $x \to -\infty$).

b) horizontal asymptote $y = 1$ (as $x \to +\infty$),

c) vertical asymptotes $x = -1$ and $x = 1/2$.

d) continuous and decreasing on the interval $(-\infty, -1)$.

e) continuous and increasing on the interval $(-1, 1/2)$.

f) continuous and increasing on the interval $(1/2, \infty)$.

Exercise 23.9. Determine where the curve $y = x^4 - 4x^3 + 1$ is increasing or decreasing, concave up or concave down. Where are its critical points and inflection points? Draw the graph.

Exercise 23.10. Describe the horizontal and vertical asymptotes (if any) of $y = \frac{x^2}{1-x^2}$.

Exercise 23.11. Find the intervals on which the curve $y = x^3 - 3x$ is increasing, decreasing, concave up, or concave down. Sketch the curve showing inflection points and local maxima and minima.

Exercise 23.12. Graph $y = x + 1/x$. The graph approaches what line as $x \to \pm\infty$?

Exercise 23.13. Graph $y = \dfrac{1}{x^2+1}$.

Exercise 23.14. Graph $y = \dfrac{x}{x^2+1}$.

Exercise 23.15. Graph $y = \dfrac{1}{x^2-1}$.

Exercise 23.16. Graph $y = \dfrac{x}{x^2-1}$.

24 Max Min Word Problems

See Thomas pages 278-292.

Example 24.1. We find the points on the curve $y = x^2$ which are closest to the point $(0, b)$ for various values of b. The distance is

$$r = \sqrt{(x-0)^2 + (y-b)^2} = \sqrt{x^2 + (x^2-b)^2} = \sqrt{y + (y-b)^2}.$$

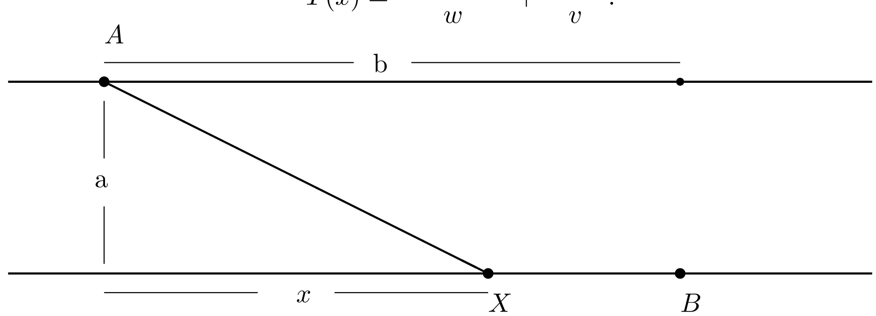
If r attains its minimum at some point, then r^2 also attains its minimum at that same point so we minimize r^2. The algebra is slightly simpler if we use y rather than x as a parameter but we must remember that, by symmetry, every positive value of y determines two points $(\sqrt{y}, y)$ and $(-\sqrt{y}, y)$ on the curve and that the domain of the function r^2 as a function of y has an endpoint at $y = 0$. The derivative of r^2 is

$$\frac{d(r^2)}{dy} = 1 + 2(y-b)$$

which vanishes when $y = b - 1/2$. If $b < 1/2$, then $b - 1/2 < 0$ so the derivative never vanishes and the minimum must occur at the endpoint $y = 0$ so the closest point is $(0, 0)$ Since the second derivative is positive there is a unique minimum if $b > 1/2$. In this case there are two closest points: $(\pm\sqrt{b-1/2}, b-1/2)$.

Example 24.2. Suppose the river is bounded by the horizontal lines $y = 0$ (the x-axis) and $y = a$. A man wants to go from $A(0, a)$ to $B(b, 0)$ as fast as possible by rowing to $P(x, 0)$ and running along the shore to B. He rows at speed w and runs at speed v so the total time is

$$T(x) = \frac{\sqrt{x^2+a^2}}{w} + \frac{b-x}{v}.$$



The function $T(x)$ is defined for all x but we seek its minimum on $0 \le x \le b$.

(Obviously $T(x) < T(0)$ for $x < 0$, and for $x > b$ the formula is not correct: one should replace $b - x$ by $|b - x|$.) The derivative is

$$T'(x) = \frac{x}{w\sqrt{x^2 + a^2}} - \frac{1}{v}$$

which vanishes at

$$\bar{x} = \frac{wa}{\sqrt{v^2 - w^2}}.$$

(This holds if $v \ge w$; otherwise $T'(x)$ doesn't vanish anywhere.) Now

$$T''(x) = \frac{a^2}{w(x^2 + a^2)^{3/2}} > 0$$

so $T(x)$ is concave up on $-\infty < x < \infty$ so has a unique minimum at $x = \bar{x}$. If $\bar{x} \ge b$ and the minimum occurs at the end point $x = b$; otherwise the minimum occurs at $x = \bar{x}$. The inequality $\bar{x} < b$ is $\bar{x}^2 < b^2$. From the formula for $\bar{x}$ this is

$$w^2(a^2 + b^2) < v^2 b^2.$$

If $a = 3$, $b = 8$, $w = 6$, $v = 8$, then $w^2(a^2 + b^2) = 36 \cdot 73 < 64 \cdot 64 = v^2 b^2$ and the min occurs at $x = \bar{x}$. If $a = 5$, $b = 5$, $w = 6$, $v = 8$, then $w^2(a^2 + b^2) = 36 \cdot 50 > 64 \cdot 25 = v^2 b^2$ and the min occurs at $x = b$.

Example 24.3. Here is a similar problem. A woman wants to cross a lake from point A to point C directly opposite. She rows in a straight line to a point B and then runs along the shore. Let $\theta = \angle CAB$ so $2\theta = \angle COB$ where O is the center. Now ABC is a right triangle (it is inscribed in a semicircle) so (assuming the radius of the lake is a) the time is

$$T(\theta) = \frac{2a\cos\theta}{w} + \frac{2a\theta}{v}.$$

for $\pi/2 \ge \theta \ge 0$. (Again w is the rowing speed and v is the running speed.) Now

$$T'(\theta) = -\frac{2a\sin\theta}{w} + \frac{2a}{v}$$

which vanishes at $\theta = \sin^{-1}(v/2w)$. But $T''(\theta) = -(2a/w)\cos\theta < 0$ so the interior critical point is a maximum, not a minimum. The minimum occurs at one of the two end points. At the end points

$$T(\pi/2) = \frac{2a\pi}{v}, \qquad T(0) = \frac{2a}{w},$$

and the minimum is the smaller of these two. Thus, to minimize the time, she should run around the lake if $w < v/\pi$, and row straight across if $w > v/\pi$.

Exercises

Exercise 24.4. Find the local maxima and minima of $f(x) = -(x-1)^3(x+1)^2$.

Exercise 24.5. If we use x rather than y as the parameter in Example 24.1 we have to minimize the function

$$r^2 = f(x) = x^2 + (x^2 - b)^2$$

over the interval $-\infty < x < \infty$. Draw the graph of the function $f(x)$ for $b = 1/4$ and for $b = 1$.

Exercise 24.6. Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the value at the local maximum is less than the value at the local minimum.

Exercise 24.7. A train is moving along the curve $y = x^2 + 2$. A girl is at the point $(3, 2)$ At what point will the train be at when the girl and the train are closest? Hint: You will have to solve a cubic equation, but the numbers have been chosen so there is an obvious root.

Exercise 24.8. Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.

Exercise 24.9. A 12×12 piece of sheet metal is to be cut along the solid lines to form a closed rectangular box. (The bends are indicated by the dashed lines.) Find the dimensions of the box of largest volume which can be constructed in this manner. (See Thomas page 278.)

Exercise 24.10. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

Exercise 24.11. Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

Exercise 24.12. Find the largest possible volume of a right circular cylinder inscribed in a hemisphere of radius r.

Exercise 24.13. Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

Exercise 24.14. Given $\triangle ABC$ and let AH be an altitude to side BC. If AB and CH both have length 1, find the length x of BH which will maximize the area of $\triangle ABC$.

Exercise 24.15. The following questions all refer to the curve $x^2 - xy + y^2 = 9$.

1. Find a formula for $\dfrac{dy}{dx}$ (in terms of x and y).

2. Find all points (x, y) on the curve where $x = 0$ and find $\dfrac{dy}{dx}$ at each of them.

3. Find all points (x, y) on the curve $x^2 - xy + y^2 = 9$ where $x = \pm 3$ and find $\dfrac{dy}{dx}$ at each of them.

4. Find all points on the curve $x^2 - xy + y^2 = 9$ where the tangent is horizontal; then find all the points where it is vertical.

5. Find all points of the curve that are closest to and farthest from the origin. Hint: The square of the distance to the origin is $r^2 = x^2 + y^2$ and $r^2 = xy + 9$ on the curve. (Finding where $dr/dx = 0$ requires solving two quadratic equations in two unknowns, but the algebra is not difficult if you don't make a mistake.)

6. Sketch the curve. Hint: The curve is an ellipse centered at the origin.

25 Exponential Growth

See Thomas pages 502-510.

§25.1. Suppose that a quantity N grows exponentially in time. This means that the value of N at time t is given by a formula of form

$$N = N_0 a^t$$

where a and N_0 are constants. Notice that $N = N_0$ when $t = 0$ so that N_0 is the **initial value** of N. The derivative is

$$\frac{dN}{dt} = N_0 a^t \ln a = kN, \qquad k = \ln a.$$

Consider a tiny time interval from t to $t+\Delta t$. The change in N over this interval is

$$\Delta N = N(t + \Delta t) - N(t).$$

The average growth rate on this interval is $\Delta N/\Delta t$ and the instantaneous growth rate at time t is the derivative dN/dt. The average percentage change in this interval is the change ΔN divided by the amount N present.[13] The average percentage rate in this interval is the percentage change divided by the change in time Δt. The instantaneous percentage rate is the limit as $\Delta t \to 0$. Thus

[13] "Per" means divide and "cent" means 100. Thus % means 1/100. For example, 0.04=4%.

When a quantity N grows exponentially, its percentage growth rate

$$k = \frac{dN/dt}{N}$$

is a positive constant, i.e. the growth rate is proportional to the amount present:

$$\frac{dN}{dt} = kN$$

The constant of proportionality is $k = \ln a$.

(When $a < 1$ the value of N decreases as t increases and we say that N decays exponentially.)

§25.2. The **doubling time** of a quantity $N = N_0 e^{kt}$ which is increasing exponentialy is the time t such that $N = 2N_0$. Since

$$2N_0 = N_0 e^{kt} \implies 2 = e^{kt} \implies \ln 2 = kt$$

the doubling time is $t = (\ln 2)/k$. Similarly, the **half life** of a quantity $N = N_0 e^{kt}$ which is decreasing exponentialy is the time t such that $N = N_0/2$, i.e. $t = -(\ln 2)/k$.

§25.3. Many phenomena are governed by exponential growth laws, To name a few:

1. Money invested in a bank account grows exponentially at 6% per year (or whatever the interest rate is).
2. The population of the world grows exponentially at 1.5% per year. (This will not continue forever.)
3. The amount of radioactivity in a radioactive material decays exponentially (at a rate that depends on the material). For example carbon-14 decays at 0.012% per year and polonium-210 decays at 0.495% per day.
4. Some chemical reactions occur at a rate proportional to the amount of the chemical present.

§25.4. Here is how a few different banks compound interest.

(i) The Alaska Bank pays interest as follows. Every year it looks at the balance in an account and adds 6%. Thus an initial deposit of B_0 dollars grows to

$$B = B_0(1.06)^t$$

dollars after t years (assuming no other deposits are made during this period.)

(ii) The Montana Bank pays interest as follows. Every month it looks at the balance in an account and adds $6/12\% = 0.5\%$. Thus an initial deposit of B_0 dollars grows to

$$B = B_0(1.005)^{12t}$$

dollars after $12t$ months $= t$ years (assuming no other deposits are made during this period.)

(ii) The Delaware Bank pays interest as follows. Every day it looks at the balance in an account and adds $6/365\%$. Thus an initial deposit of B_0 dollars grows to

$$B = B_0\left(1 + \frac{0.06}{365}\right)^{365t}$$

dollars after $365t$ days $= t$ years (assuming no other deposits are made during this period.)

Theorem 25.5. $\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{mt} = e^{rt}$.

Proof. Using the law that $\ln(b^c) = c\ln b$ and changing the dummy variable shows that

$$\lim_{m\to\infty}\ln\left(1 + \frac{r}{m}\right)^{mt} = \lim_{m\to\infty} mt\ln\left(1 + \frac{r}{m}\right) = \lim_{h\to 0+}\frac{t}{h}\ln(1 + rh)$$

so it is enough to prove that

$$\lim_{h\to 0+}\frac{t}{h}\ln(1 + rh) = rt.$$

Now let $f(x) = \ln(1 + rx)$ and note that the last limit is the difference quotient $(f(h) - f(0))/h$. Thus the limit is $tf'(0)$ By the Chain Rule, $f'(x) = r/(1+rx)$ so $f'(0) = r$ so so the limit is rt. Now exponentiate and use the continuity of the exponential. $\square$

§25.6. The general formula for the balance B in a bank account after t years if the balance is initially B_0, the interest rate is r per year, and the interest is **compounded** m times per year is

$$B = B_0\left(1 + \frac{r}{m}\right)^{mt}.$$

(The formula is used even if t is not an integer.) When m becomes infinite we say that the interest is **compounded continuously** and the theorem says that

$$B = B_0e^{rt}.$$

In all cases the formula is of the form

$$B = B_0a^t$$

where $a = (1 + r/m)^m$ if the compounding period is $1/m$ years and $a = e^r$ if the compounding is continuous. The following table shows the values of $a^t \left(1 + \frac{r}{m}\right)^{mt}$ for $t = 2$, $r = 0.05$, $n = mt$, and various values of the number m of compounding periods per year.

m	$\left(1+\dfrac{r}{m}\right)^{mt}$
1	1.102500000000000
12	1.104941335558328
52	1.105117820169223
365	1.105163349128883
∞	1.105170918075648

Exercises

Exercise 25.7. Find k such that $dB/dt = kB$ if $B = B_0 \left(1 + \dfrac{r}{m}\right)^{mt}$ and if $B = B_0 e^{rt}$.

Exercise 25.8. Polonium-210 has a half life of 140 days. (a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days. (b) Find the mass after 100 days. (c) When will the mass be reduced to 10 mg? (d) Sketch the graph of the mass as a function of time.

Exercise 25.9. The number N of bacteria in laboratory container t hours after the start of an experiment is

$$N = 1000 \cdot 2^t$$

where t is the number of hours since the beginning of the experiment. (a) How many bacteria are present at time $t = 0$? (b) When will the number N be twice that initial amount? Four times? Eight times? (c) At what (instantaneous) rate is N increasing at time t? (d) At what (instantaneous) percentage rate is N increasing at time t?

Exercise 25.10. After 3 days a sample of radon-222 decayed to 58% of its original amount. (a) What is the half life of radon-222? (b) How long would it take the sample to decay to 10% of its original amount?

Exercise 25.11. Radiocarbon dating works on the principle that ^{14}C decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ^{14}C as does plant material on earth today. Estimate the age of the parchment.

Exercise 25.12. The population of the country of Slobia grows exponentially. (a) If its population in the year 1990 was 1,990,000 and its population in the year 2000 was 2,000,000 what will be its population in the year 2010? (b) How long will it take the population to double?

Exercise 25.13. (Archer Daniels Midland) According to a recent TV commercial from the Archer Daniels Midland Corporation, "in fifty years the world will have to set ten billion places at the table." Another ADM commercial says "when this baby is old enough to vote, the world will have a billion more mouths to feed." What is the present population of the world and how fast is it increasing?[14]

26 Indeterminate Forms (l'Hôpital's Rule)

See Thomas pages 292-299.

§26.1. Often when we try to evaluate a derivative by just plugging in we get nonsense like

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

these are called **indeterminate forms**. A special technique called *l'Hôpital's Rule* can be used to evaluate the limit in this case. We will de-emphasize this technique because it encourages blind calculation and leads students to forget what a limit is. However, we will touch on it briefly. Here is the simplest case.

l'Hôpital's Rule. *Suppose that the functions $f(x)$ and $g(x)$ are differentiable and that $f(a) = g(a) = 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

We won't give a careful proof, but roughly speaking, the reason why the rule is true is that

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \approx \frac{f'(a)}{g'(a)}$$

when $x \approx a$.

§26.2. l'Hôpital's Rule also works for the indeterminate form ∞/∞: If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$$

[14]This problem cannot be solved algebraically. You can however find two equations relating the present world population and the rate of increase, eliminate the latter to get a single equation in the former, and draw a graph with a computer to estimate the solution.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right exists. Both forms $0/0$ and ∞/∞ also work when $a = \pm\infty$. Moreover, l'Hôpital's Rule holds even when $f(x)$ and $g(x)$ are not defined at $x = a$: it is enough to assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

§ 26.3. Warning. l'Hôpital's Rule doesn't work unless plugging in gives an indeterminate form. For example, with $f(x) = x^3$, $g(x) = x^2$, and $a = 1$ we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 1}\frac{x^3}{x^2} = 1, \qquad \text{but} \qquad \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to 1}\frac{3x^2}{2x} = \frac{3}{2}.$$

Example 26.4. Exponentials defeat powers. Setting $x = \infty$ in x/e^x gives the indeterminate form ∞/∞ so by l'Hôpital's Rule

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0.$$

Applying this argument twice gives

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$$

Applying this argument n times gives

$$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty}\frac{n!}{e^x} = 0.$$

Example 26.5. Powers defeat logarithms. Setting $x = \infty$ in $(\ln x)/x$ gives the indeterminate form ∞/∞ so by l'Hôpital's Rule

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Example 26.6. Other indeterminate forms can be treated by performing some algebra first. For the indeterminate form $0\cdot\infty$ try dividing by the second factor to get the indeterminate form ∞/∞. For example,

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-1/x^2} = -\lim_{x\to 0+}\frac{x^2}{x} = 0.$$

Example 26.7. For the indeterminate forms 0^0 or 1^∞ try taking the logarithm to get $0\cdot\infty$ as in Example 26.6. Then exponentiate the answer to solve the original problem. For example,

$$\ln\lim_{x\to 0+} x^x = \lim_{x\to 0+}\ln(x^x) = \lim_{x\to 0+} x\ln x = 0$$

so

$$\lim_{x\to 0+} x^x = \exp\left(\ln\lim_{x\to 0+} x^x\right) = \exp(0) = 1.$$

Exercises

Exercise 26.8. Illustrate the indeterminate form ∞/∞ by finding four large numbers A, B, C, D, all larger than one million, with $A/B = 0.37$ and $C/D = 458$.

Exercise 26.9. (i) Evaluate A/B when $A = 0.01$ and $B = 0.00001$. (ii) Evaluate A/B when $A = 0.00001$ and $B = 0.01$. (iii) Which indeterminate form does this illustrate?

Exercise 26.10. (i) Evaluate A^B when $A = 1.01$ and $B = 1000$. (ii) Evaluate A^B when $A = 1.00001$ and $B = 1000$. (iii) Which indeterminate form does this illustrate? (This problem requires a calculator.)

Exercise 26.11. (i) Evaluate A^B when $A = 10^{1000}$ and $B = 0.001$. (ii) Evaluate A^B when $A = 10^{1000}$ and $B = 0.01$. (iii) Which indeterminate form does this illustrate? (This problem does not require a calculator.)

Exercise 26.12. Evaluate each of the following.

(i) $\lim_{x\to\infty} x\,3^x$ (ii) $\lim_{x\to-\infty} x\,3^x$ (iii) $\lim_{x\to 0} x\,3^x$ (iv) $\lim_{x\to\infty} \frac{3^x}{x}$

(v) $\lim_{x\to-\infty} \frac{3^x}{x}$ (vi) $\lim_{x\to\infty} \frac{x}{3^x}$ (vii) $\lim_{x\to-\infty} \frac{x}{3^x}$ (viii) $\lim_{x\to\infty} \frac{\ln x}{x}$

27 Antiderivatives

See Thomas pages 307-318.

Definition 27.1. An **antiderivative** of a function f is a function F such that $F' = f$. Note that

- If F is an antidericative of f so is $F + c$ for any constant c. (Proof: the derivative of a constant is zero.)
- Any two antiderivatives of f differ by a constant. (Proof: If the derivative of a function is zero then that function is constant. See Theorem 22.6)

Remark 27.2. Antidifferentiation suffices to solve differential equation where only the derivative of the unknown function appears (e.g. constant gravity), but not other differential equations where both the unknown function and its derivative appear. Later you will learn that antiderivatives can be used to compute areas.

Example 27.3. If the position of a particle at time t is denoted by y then the quantities

$$v = \frac{dy}{dt}, \qquad a = \frac{d^2y}{dt^2},$$

are called respectively the **velocity** and **acceleration** of the particle. According to Newton, if a ball is thrown into the air its height y at time t satisfies

$$\frac{d^2y}{dt^2} = -g$$

where $g = 32\text{ft/sec}^2$. Hence there is a constant v_0 (the initial velocity) such that

$$\frac{dy}{dt} = -gt + v_0, \qquad v_0 = \left.\frac{dy}{dt}\right|_{t=0}$$

and there is a constant y_0 (the initial height) such that

$$y = -\frac{gt^2}{2} + v_0 t + y_0, \qquad y_0 = y\big|_{t=0}.$$

Exercises

Exercise 27.4. Find the most general antiderivative of each of the following functions. Check your answer by differentiating.

(a) $f(x) = x^3 + 3x^2 + 7 + x^{-2}$ (b) $f(t) = \sin(t)$

(c) $f(x) = \dfrac{x^4 + 3x^3 + 7x + x^{-1}}{x}$ (d) $f(\theta) = \sec^2(\theta)$

(e) $f(x) = e^x$ (f) $f(x) = x + 1/x$.

Exercise 27.5. Find $F(x)$ if $F'(x) = x^3 + 3x^2 + 7 + x^{-2}$ and $F(1) = 12$.

Exercise 27.6. Find $F(t)$ if $F'(t) = \sin(t)$ and $F(0) = 7$.

Exercise 27.7. Find $F(\theta)$ for $-\pi/2 < \theta < \pi/2$ if $F'(\theta) = \sec^2(\theta)$ and $F(0) = 7$.

Exercise 27.8. A ball is thrown upward with an initial speed of 48 ft/sec from a roof which is 432 feet above the ground. Find its height t seconds later. When does it reach its maximum height. What is its maximum height? When does it hit the ground? How fast is it going when it hits the ground?

Exercise 27.9. One second after the ball in the previous problem is thrown another ball is thrown upward with an initial speed of 24 ft/sec. Are the balls ever at the same height? (Before they hit the ground of course.)

Exercise 27.10. A graph $y = f(x)$ passes through the point $(1, 6)$ and its slope at the point $(x, f(x))$ is $2x + 1$. What is $f(2)$?

Exercise 27.11. Find a function $f(x)$ such that $f'(x) = x^3$ and the line $x+y = 0$ is tangent to the graph of f.

Exercise 27.12. A car traveling 50 miles per hour when the brakes are applied producing a a constant deceleration of 40 ft/sec^2. What is the distance covered before the car comes to a stop. Hint: One mile= 5280 feet.

Exercise 27.13. A stone is dropped from a roof and hits the ground with a velocity of 120 feet per second. What is the height of the roof?

28 Additional Problems

Exercise 28.1. For the function $y = xe^x$ make a table showing all horizontal asymptotes, all vertical asymptotes, all local extrema, all points of inflection, intervals on which the function is increasing, intervals on which the function is decreasing, intervals on which the function is concave up, and intervals on which the function is concave down. Draw a graph of the function and on the is graph draw the tangent line at each point of inflection.

Chapter V

Integration

29 The Definite Integral

See Thomas pages 325-355.

The definite integral $\int_a^b f(x)\,dx$ of a nonnegative function $f(x)$ is the area of the region $a \le x \le b$ and $0 \le y \le f(x)$. The precise definition involves approximating this region by skinny rectangles. The sum of the areas of these rectangles is called a Riemman[15] sum and the definite integral is the limit of the Riemann sums as the rectangles become skinnier and skinnier. To say this precisely we first need some terminology.

§29.1. Sigma Notation. The notation $\sum_{j=m}^n a_j$ is short for the sum of the numbers $a_m, a_{m+1}, \ldots, a_n$, i.e.

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n.$$

For example, $\sum_{j=2}^5 j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54$. In the notation $\sum_{j=m}^n a_j$ the integers m and n are called the **limits of summation**, the expression a_j is called the jth **summand**, and the variable j is called the **index of summation**. The index of summation is a dummy variable (see §7.7), i.e.

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i$$

but the limits of summation are free variables (see §7.8), i.e.

$$\sum_{j=m}^{n} a_j \neq \sum_{j=p}^{q} a_j \qquad \text{(usually).}$$

Here are some obvious laws.

(Constants Law) $$\sum_{j=m}^{n} 1 = n - m + 1.$$

(Linearity Law I) $$\sum_{j=m}^{n} (a_j \pm b_j) = \left(\sum_{j=m}^{n} a_j\right) \pm \left(\sum_{j=m}^{n} b_j\right).$$

(Linearity Law II) $$\sum_{j=m}^{n} k a_j = k \sum_{j=m}^{n} a_j.$$

[15] pronounced "Reemann"

(Additivity Law) $\sum_{j=m}^{n} a_j = \left(\sum_{j=m}^{p} a_j\right) + \left(\sum_{j=p+1}^{n} a_j\right) \qquad$ for $m \leq p \leq n$.

(Order Law) $\sum_{j=m}^{n} a_j \leq \sum_{j=m}^{n} b_j \qquad$ if $a_j \leq b_j$ for $j = m, m+1, \ldots, n$.

§29.2. Recall that $[a, b]$ denotes the **closed interval** $a \leq x \leq b$ with endpoints a and b. A **partition** of the interval $[a, b]$ is a finite sequence

$$P = (x_0, x_1, \ldots, x_n)$$

such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The partition divides the interval $[a, b]$ into subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n].$$

The length of the kth interval is

$$\Delta x_j = x_j - x_{j-1}.$$

The **mesh** $\|P\|$ of the partition P is the length of the largest subiniterval:

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$

When $\|P\| \approx 0$ all of the subintervals are small and there are a large number of them (i.e. $n \approx \infty$). A **Riemann partition** of the interval $[a, b]$ is a partition P equipped with additional numbers

$$C = (c_1, c_2, \ldots, c_n)$$

such that c_j lies in the jth interval, i.e.

$$x_{j-1} \leq c_j \leq x_j$$

for $j - 1, 2, \ldots, n$.

Definition 29.3. Let $f(x)$ be a function which is continuous on the interval $[a, b]$ and (P, C) be a Riemann partition of $[a, b]$ as in 29.2. The number

$$S(f, P, C) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$$

is called the **Riemann sum** of f for (P, C). In the notation of §29.2 this can be written

$$S(f, P, C) = \sum_{j=1}^{n} f(c_j)\Delta x_j.$$

Definition 29.4. The **definite integral** of $f(x)$ from $x = a$ to $x = b$ is the limit

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(f, P, C)$$

of the Riemann sums as the mesh $\|P\|$ of the partition goes to 0. In other words

$$\sum_{j=1}^{n} f(c_j)\Delta x_j \approx \int_a^b f(x)\,dx$$

when $\|P\| \approx 0$. The function f is called **integrable** on the interval $[a, b]$ iff this limit exists.

§29.5. In the notation $\int_a^b f(x)\,dx$ the numbers a and b are called the **limits of integration**, the function f is called the **integrand**, and variable x is called the **variable of integration**. The variable of integration is a dummy variable:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

but the limits of summation are free variables:

$$\int_a^b f(x)\,dx \neq \int_p^q f(x)\,dx \qquad \text{(usually)}.$$

§29.6. Integration Laws. Here are some important laws which which integrable functions f and g satisfy. They are all proved the same way, namely by showing that an analogous law holds for Riemann sums and then passing to the limit.

Constants Law. *The integral of a constant function is*

$$\int_a^b k\,dx = k(b-a).$$

This is because for any partition as in 29.2 we have

$$\sum_{j=0}^{n} k\Delta x_j = k\big((x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})\big) = c(b-a).$$

(This is a “collapsing sum”: the x_j in each summand cancels the x_j in the next one. The two terms which don't cancel are $x_0 = a$ from the first summand $x_n = b$ from the last.)

Linearity Law. *The definite integral is linear in the integrand. This means that*

$$\int_a^b \Big(f(x) + g(x) \Big) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx,$$

and, for any constant k,

$$\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx.$$

This is because for any Riemann partition as in 29.2 we have

$$\sum_{j=1}^{n} \Big(f(c_j) + g(c_j) \Big) \Delta x_j = \left(\sum_{j=1}^{n} f(c_j) \Delta x_j \right) + \left(\sum_{j=1}^{n} g(c_j) \Delta x_j \right)$$

and

$$\sum_{j=1}^{n} k f(c_j) \Delta x_j = k \sum_{j=1}^{n} f(c_j) \Delta x_j.$$

Additivity Law. *If $a < c < b$, then*

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

This is because any Riemann partition of $[a, b]$ as in 29.2 with $x_m = c$ we have

$$\sum_{j=1}^{n} f(c_j) \Delta x_j = \left(\sum_{j=1}^{m} f(c_j) \Delta x_j \right) + \left(\sum_{j=m+1}^{n} f(c_j) \Delta x_j \right)$$

Remark 29.7. It is convenient to define

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$$

Then the Additivity Law holds for all a, b, c not just $a < c < b$.

Order Law. *The definite integral preserves order, i.e. if $f(x) \le g(x)$ for all x, then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

This is because any Riemann partition as in 29.2 we have $f(c_j) \le g(c_j)$ so

$$\sum_{j=1}^n f(c_j)\Delta x_j \le \sum_{j=1}^n g(c_j)\Delta x_j.$$

Key Estimate.

$$\left(\min_{a\le x\le b} f(x)\right)(b-a) \le \int_a^b f(x)\,dx \le \left(\max_{a\le x\le b} f(x)\right)(b-a)$$

Since $\min_{a\le x\le b} f(x) \le f(x) \le \max_{a\le x\le b} f(x)$ the Key Estimate follows immediately the Constants Law and the Order Law.

Theorem 29.8. *A continuous function is integrable.*

§29.9. This theorem is normally proved in more advanced courses like Math 521, but we can indicate the idea. Suppose that f is continuous on $[a,b]$. A partition P determines two Riemann sums as follows. On each interval $x_{j-1} \le x \le x_j$ the function f assumes its maximum at some point $\overline{c}_j$ and its minimum at some other point $\underline{c}_j$. (Usually $\overline{c}_j$ will be one of the endpoints and $\underline{c}_j$ will be the other.) The Riemann sum

$$U(f,P) = \sum_{j=1}^n f(\overline{c}_j)\Delta x_j$$

is called the **upper sum** and the Riemann sum

$$L(f,P) = \sum_{j=1}^n f(\underline{c}_j)\Delta x_j$$

is called the **lower sum**. (See Thomas page 345; there are pictures of upper sums and lower sums in Exercise 77-80 in Thomas pages 354-5.) By the choice of $\overline{c}_j$ and $\underline{c}_j$ we have $f(\underline{c}_j) \le f(x) \le f(\overline{c}_j)$ for $x_{j-1} \le x \le x_j$. In particular,

$$f(\underline{c}_j) \le f(c_j) \le f(\overline{c}_j)$$

for any Riemann partition (P, C) and hence $L(f, P) \le S(f, P, C) \le U(f, P)$, i.e.

$$\sum_{j=1}^{n} f(\underline{c}_j)\Delta x_j \le \sum_{j=1}^{n} f(c_j)\Delta x_j \le \sum_{j=1}^{n} f(\overline{c}_j)\Delta x_j.$$

The proof of Theorem 29.8 rests on the following fact:

Theorem 29.10. *The definite integral of a continuous function is the unique number which lies between every lower sum and every upper sum, i.e.*

$$\sum_{j=1}^{n} f(\underline{c}_j)\Delta x_j \le \int_a^b f(x)\,dx \le \sum_{j=1}^{n} f(\overline{c}_j)\Delta x_j$$

This is true because when the mesh is small the lower sum and the upper sum are approximately the same. See the pictures in Thomas pages 354-355.

Example 29.11. Figure 3 shows an upper sum and a lower sum approximating the integral $\int_1^3 x^{-1}\,dx$. The partition P is

$$a = x_0 = 1 < x_1 = 1.5 < x_2 = 2 < x_3 = 2.5 < x_4 = 3 = b$$

so $n = 4$ and all the intervals $[x_{j-1}, x_j]$ all have the same length:

$$\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = 0.5.$$

Since the function $f(x) = x^{-1}$ is decreasing, its minimum on any interval is assumed at the right endpoint and its maximum is assumed at the left endpoint, i.e. $\underline{c}_j = x_j$ and $\overline{c}_j = x_{j-1}$. Hence the lower sum is

$$L = \sum_{j=1}^{4} \frac{\Delta x}{\underline{c}_j} = \frac{0.5}{1.5} + \frac{0.5}{2.0} + \frac{0.5}{2.5} + \frac{0.5}{3.0} = 0.95$$

and the upper sum is

$$U = \sum_{j=1}^{4} \frac{\Delta x}{\overline{c}_j} = \frac{0.5}{1.0} + \frac{0.5}{1.5} + \frac{0.5}{2.0} + \frac{0.5}{2.5} = 1.2833\ldots$$

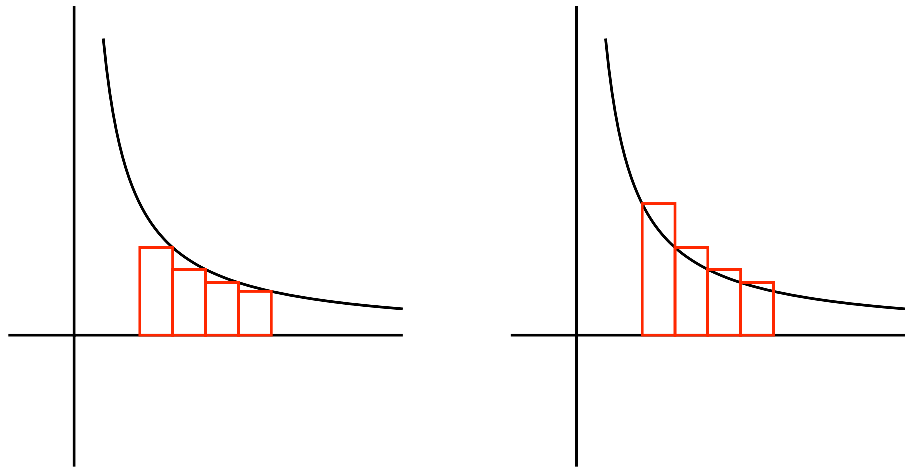


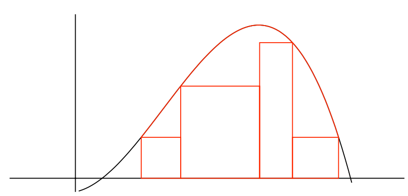
Figure 3: A lower sum and an upper sum

so Theorem 29.10 tells us that

$$0.95 \leq \int_1^3 \frac{dx}{x} \leq 1.2833\ldots.$$

The exact value of the integral is $\ln 3$ which, according to my calculator, is $1.098612289\ldots$.

Example 29.12. The picture at the right shows shows a lower Riemann sum for a function which has an interior maximum. In the first two intervals of the partition the minimum occurs at the left endpoint and in the other two the minimum occurs at the right endpoint.



Exercises

Exercise 29.13. Use Riemann sums with four intervals of length one to find positive numbers L and U with

$$3 < L \leq \int_1^5 \left(3 + \frac{1}{x}\right)\,dx \leq U.$$

Exercise 29.14. Let P denote the partition

$$P = (x_0, x_1, x_2, x_3, x_4) = (0, \pi/6, \pi/4, \pi/3, \pi/2)$$

of the interval $[0, \frac{\pi}{2}]$. Find the lower sum L and the upper sum U for the function $f(x) = \sin(x)$ on this interval and show (using a calculator) that $L < 1 < U$.

Exercise 29.15. Let $f(x) = 3x + 4$, $a = 2$, $b = 5$. The region $a \le x \le b$, $0 \le y \le f(x)$ is a trapezoid.

1. Draw the trapezoid and find the area $\int_a^b f(x)\,dx$ of this trapezoid by by elementary geometry.

2. Evaluate the approximation $\sum_{j=1}^{6} f(c_j)\Delta x_j$ and the mesh $\|P\|$ when

$$\begin{array}{lllll} x_0 = 2, & c_1 = 2.2, & x_1 = 2.5, & c_2 = 2.8, & x_2 = 3.0 \\ & c_3 = 3.2, & x_3 = 3.5, & c_4 = 3.9, & x_4 = 4.1 \\ & c_5 = 4.2, & x_5 = 4.4, & c_6 = 4.8, & x_6 = 5.0. \end{array}$$

3. Draw a graph illustrating the trapezoid and the six rectangles $x_{j-1} \le x \le x_j$, $0 \le y \le f(c_j)$ whose areas $f(c_j)\,\Delta x_j$ sum to the approximation just computed.

4. Evaluate the approximation $\sum_{j=1}^{8} f(c_j)\Delta x_j$ when

$$x_j = a + \frac{k}{8}(b - a), \qquad c_j = \frac{x_{j-1} + x_j}{2}.$$

   If you like, you may use the formula

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Exercise 29.16. Let $f(x) = x^2$, $a = 1$, $b = 3$, and

$$c_j = x_j = a + \frac{k}{8}(b - a)$$

for $k = 0, 1, \ldots, 8$.

1. Draw a graph showing the graph $y = f(x)$ for $a \le x \le b$ and also showing the the eight rectangles $x_{j-1} \le x \le x_j$, $0 \le y \le f(c_j)$.

2. Calculate the approximation

$$\sum_{j=1}^{8} f(c_j)\Delta x_j.$$

   If you like, you may use the formula

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. What is $1^2 + 2^2 + 3^2 + \cdots + 20^2$?

Exercise 29.17. Here is a question from an old 221 exam. *Find a number smaller than* $\int_1^3 e^{-x^2}\,dx$. *The answer* 10^{-100} *is correct if you can prove it.*

Exercise 29.18. Redraw the graph in Example 29.12 showing the upper sum corresponding to the partition rather than the lower sum.

Exercise 29.19. The continuous function f takes the values

$$f(1) = 3, \quad f(1.6) = 6.7, \quad f(2.8) = 11.2, \quad f(3.3) = 9.9, \quad f(4) = 3,$$

and is increasing for $1 \le x \le 2.8$ and decreasing for $2.8 \le x \le 4$. Find the lower Riemann sum L for the partition

$$x_0 = 1 < x_1 = 1.6 < x_2 = 2.8 < x_3 = 3.3 < x_4 = 4$$

of the interval $[1, 4]$. Sketch a possible graph and also draw the area represented by the Riemann sum. (On an exam you would be instructed to leave the addition and multiplication undone so as to make your work easier to grade.)

Exercise 29.20. Repeat the previous exercise with the upper Riemann sum rather than the lower.

30 The Fundamental Theorem of Calculus

See Thomas pages 355-368.

Theorem 30.1 (The Fundamental Theorem). *Suppose that the function* $f(x)$ *is continuous on the closed interval* $[a, b]$. *Define a function* $I(x)$ *on* $[a, b]$ *by*

$$I(x) = \int_a^x f(t)\,dt.$$

Let F *be* any *antiderivative of* f *on* $a \le x \le b$, *i.e.*

$$F'(x) = f(x)$$

Then

(*I*) $$I'(x) = f(x);$$

and

(*II*) $$\int_a^b f(t)\,dt = F(b) - F(a).$$

Proof: For part (I) we compute the difference quotient:

$$\frac{I(x+h)-I(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

By the Key Estimate in §29.6 the right hand side satisfies

$$\min_{x\le t\le x+h} f(t) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \max_{x\le t\le x+h} f(t)$$

if $h > 0$. The inequality also holds for $h < 0$ if we replace $x \le t \le x+h$ by $x+h \le t \le x$. Because the function f is continuous, the max and the min are both close to $f(x)$ when h is small. Hence

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx f(x)$$

for $h \approx 0$. This proves (I).

Part (II) is an easy consequence of part (I) as follows. If $I'(x) = F'(x)$ then $F(x) = I(x) + C$ for some constant C so

$$F(b) - F(a) = I(b) - I(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt.$$

§30.2. We often write

$$\int f(x)\,dx = F(x) + C$$

to mean that $F(x)$ is an antiderivative of $f(x)$, i.e.

$$\int f(x)\,dx = F(x) + C \iff F'(x) = f(x).$$

For example,

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

The function $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$. One should think of it as a convenient notation for antiderivatives. The "$+C$" indicates that it should be viewed as a set of functions: one for each choice of the constant C. When evaluating the definite integral the choice of the constant C doesn't matter since it cancels:

$$\int_a^b f(x)\,dx = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$

§30.3. Another handy notation is $F(x)\Big|_a^b$ defined by

$$F(x)\Big|_a^b = F(b) - F(a).$$

With this notation the Fundamental Theorem takes the form

$$\int_a^b F'(x)\,dx = F(x)\Big|_a^b.$$

For example,

$$\int_a^b x^3\,dx = \frac{x^4}{4}\Big|_a^b = \frac{b^4}{4} - \frac{a^4}{4}.$$

Exercises

Exercise 30.4. Evaluate the following:

(i) $\int_1^2 x^{-2}\,dx$ (ii) $\int_0^1 (1-2x-3x^2)\,dx$

(iii) $\int 2^x\,dx$ (iv) $\int_0^4 \sqrt{x}\,dx$

(v) $\int_0^1 x^{3/7}\,dx$ (vi) $\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right)\,dt$

(xii) $\int_1^2 \frac{x^2+1}{\sqrt{x}}\,dx$ (viii) $\int_1^4 (\sqrt{t} - 2/\sqrt{t})\,dt$

(ix) $\int_0^2 (x^3-1)^2\,dx$ (x) $\int_0^1 u(\sqrt{u} + \sqrt[3]{u})\,du$

(xi) $\int_1^{-1} (x-1)(3x+2)\,dx$ (xii) $\int_1^2 (x+1/x)^2\,dx$

Exercise 30.5. (1) Water flows into a container at a rate of three gallons per minute for two minutes, five gallons per minute for seven minutes and eleven gallons per minute for two minutes. How much water is in the container? (2) Water flows into a container at a rate of t^2 gallons per minute for $2 \leq 3 \leq 5$. How much water is in the container?

Exercise 30.6. Evaluate $\int x^3 + 3x^2 + 7 + x^{-2}\,dx$.

Exercise 30.7. True or false? $\int_{-1}^1 \frac{3}{t^4}\,dt = \frac{-1}{t^3}\Big|_{-1}^1 = -1 + 1 = 0.$

Exercise 30.8. Evaluate $\frac{d}{dx}\int_0^x e^{-t^2}\,dt$. Hint: Do not try to evaluate the integral.

Exercise 30.9. Evaluate

(i) $\int_3^3 (x^4 + x^2 + 1)\,dx$ (ii) $\frac{d}{dx}\int_3^x (u^4 + u^2 + 1)\,du$

(iii) $\int_3^3 \sqrt{x^5+2}\,dx$ (iv) $\frac{d}{dx}\int_3^x \sqrt{u^5+2}\,du$

(v) $\displaystyle \frac{d}{dx}\int_3^{\sin x}\sqrt{u^5+2}\,du$ (vi) $\displaystyle \frac{d}{dx}\int_3^{\sqrt{x^5+2}}\sin u\,du$

(vii) $\displaystyle \frac{d}{dx}\int_x^{3}\sqrt{u^5+2}\,du$ (viii) $\displaystyle \frac{d}{dx}\int_{\sin x}^{3}\sqrt{u^5+2}\,du$

Exercise 30.10. Find $\displaystyle\int_0^5 f(x)\,dx$ where $f(x)$ is defined by

$$f(x) = \begin{cases} x+2 & \text{for } x < 1 \\ 3^x & \text{for } x \geq 1 \end{cases}$$

Hint: Additivity Law.

31 Averages

See Thomas page 351.

Definition 31.1. The **average value** of the function f on the interval $[a, b]$ is

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

§31.2. The average velocity is the average of the velocity. Remember the trip to Milwaukee? (See §6.3.) The position (=reading on the mile post) at time t was $s = f(t)$. The distance

$$\Delta s = f(t+\Delta t) - f(t)$$

travelled over the time interval from t to $t + \Delta t$ is called the **displacement**. The ratio

$$v_{\text{av}} = \frac{\Delta s}{\Delta t}$$

is called the **average velocity** over this time interval, while the derivative

$$v = \frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = f'(t)$$

is called the **instantaneous velocity** at time t. Now by the Fundamental Theorem

$$\Delta s = f(t+\Delta t) - f(t) = \int_t^{t+\Delta t} f'(\tau)\,d\tau.$$

But $v = f'(\tau)$ is the instantaneous velocity at time τ so the average velocity (from §6.3) is the average of the velocity function (which we just learned), i.e.

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{1}{\Delta t}\int_t^{t+\Delta t} f'(\tau)\,d\tau = \frac{1}{\Delta t}\int_t^{t+\Delta t} v(\tau)\,d\tau.$$

Exercises

Exercise 31.3. Find the average value of $f(x) = x^p$ over the interval $[0, 1]$ for $p = 1/2, 1, 2$.

Exercise 31.4. Repeat the previous exercise for the interval $[0, \frac{1}{5}]$ and the interval $[0, 5]$.

Exercise 31.5. Prove that if the function $f(x)$ is continuous on the interval $[a, b]$ the there is a point c in the interval $[a, b]$ such that $f(c) =$ the average value of $f(x)$ on the interval, i.e.

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This is called the **Mean Value Theorem for Integrals**. Hint: Apply the Mean Value Theorem 22.4 to $F(x) = \int_a^x f(t)\,dt$.

32 Change of Variables

See Thomas pages 307-317.

Theorem 32.1 (Change of Variables Formula). *Suppose that the function $u = g(x)$ is continuously differentiable on the closed interval $[a, b]$ and that the function $f(x)$ is continuous of the range of g. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \qquad (*)$$

§32.2. The Change of Variables Formula is an easy consequence of the Fundamental Theorem. If $F(u)$ is an antiderivative of $f(u)$, then the right hand side of $(*)$ is

$$\int_{g(a)}^{g(b)} f(u)\,du = F(g(b)) - F(g(a)).$$

By the Chain Rule

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x),$$

i.e. $(F \circ g)(x)$ is an antiderivative of the integrand in $(*)$. Hence by the Fundamental Theorem again

$$\int_a^b f(g(x))g'(x)\,dx = (F \circ g)(x)(b) - (F \circ g)(a) = F(g(b)) - F(g(a)).$$

Remark 32.3. The Change of Variables Formula is the "Chain Rule backwards".

§32.4. Here is another proof of the Change of Variables Formula which is more like the proof of the analogous formula for multiple integrals which we will study in Math 234. For this proof we must assume that g is increasing. Choose a partition $P = (x_0, x_1, \ldots, x_n)$ of the interval $[x_0, x_n] = [a, b]$. By the Mean Value Theorem 22.4 each interval $[x_{k-1}, x_k]$ in the partition contains a point c_k such that

$$g(x_k) - g(x_{k-1}) = g'(c_k)(x_k - x_{k-1}). \qquad (i)$$

The partition P together with the points c_k determine a Riemann partition of the interval $[a, b]$ so by the definition of the indefinite integral we have

$$\int_a^b f(g(x))g'(x)\,dx \approx \sum_{k=1}^{n} f(g(c_k))g'(c_k)(x_k - x_{k-1}) \qquad (ii)$$

when the mesh $\|P\|$ of the partition is small. Now $Q = \big(g(x_0), g(x_1), \ldots, g(x_n)\big)$ is a partition of the interval $[g(x_0), g(x_n)] = [g(a), g(b)]$ and $g(c_k)$ lies in the interval $[g(x_{k-1}), g(x_k)]$ so

$$\int_{g(a)}^{g(b)} f(u)\,du \approx \sum_{k=1}^{n} f(g(c_k))(g(x_k) - g(x_{k-1})) \qquad (iii)$$

By (i) the right hand sides of (ii) and (ii) are equal, and the approximate equalities are arbitrarily accurate so the left hand sides of (ii) and (ii) are also equal.

§ 32.5. The differential notation from §11.4 is very handy when computing with the Change of Variables Formula. I like to write the formula with the dummy variable appearing in the superscript and subscript of the integral sign to remind myself to change the limits of integration. For example, let's evaluate $\int_{x=0}^{x=2} \sqrt{1+x^2}\, 2x\,dx$. We use the change of variables $u = 1 + x^2$. Then

$$du = 2x\,dx, \qquad x = 0 \implies u = 1, \qquad x = 2 \implies u = 5$$

so

$$\int_{x=0}^{x=2} \sqrt{1+x^2}\, 2x\,dx = \int_{u=1}^{u=5} \sqrt{u}\,du = \frac{2u^{3/2}}{3}\bigg|_{u=1}^{u=5} = \frac{2(5^{3/2})}{3} - \frac{2(1^{3/2})}{3}.$$

Example 32.6. When you use the Change of Variables Formula to compute an indefinite integral you must substitute back so that the answer is expressed in the same variables as the original problem. For example, using the substitution $u = 3x$, $du = 3\,dx$ we get

$$\int \cos(3x)\,dx = \int \cos u\,\frac{du}{3} = \frac{\sin u}{3} + C = \frac{\sin 3x}{3} + C.$$

§32.7. Let s be the position of a particle at time t as in §6.3 and §31.2. and suppose $s = s_0$ at some time $t = t_0$ and $s = s_1$ at some time $t = t_1$. Let $v = ds/dt$ be the velocity at time t. Then by the Change of Variables formula

$$\int_{t_0}^{t_1} v\,dt = \int_{t_0}^{t_1} \frac{ds}{dt}\,dt = \int_{s_0}^{s_1} ds = s_1 - s_0.$$

This says that the total displacement $s_1 - s_0$ is the sum of all the infinitesimal displacements ds. On the other hand the total distance travelled by the particle on this time interval is

$$\int_{t_0}^{t_1} |v|\,dt = \int_{t_0}^{t_1} \left|\frac{ds}{dt}\right|\,dt = \int_{s_0}^{s_1} |ds|.$$

This will be different from the absolute value $|s_1 - s_0|$ of the total displacement $s_1 - s_0$ if the particle changes direction (i.e. if v changes sign) on the time interval.

Exercises

Exercise 32.8. Evaluate the indefinite integral by making the indicated substitution.

(i) $\displaystyle\int x(x^2-1)^{99}\,dx, \quad u = x^2 - 1.$

(ii) $\displaystyle\int \frac{x^2}{\sqrt{5+x^3}}\,dx, \quad u = 5 + x^3.$

(iii) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx, \quad u = \ln x.$

(iv) $\displaystyle\int \frac{x\,dx}{1+x^2}, \quad u = x^2.$

(v) $\displaystyle\int x\sqrt{x-1}\,dx, \quad u = x - 1.$

(vi) $\displaystyle\int e^x(1+e^x)^5\,dx, \quad u = 1 + e^x.$

Exercise 32.9. Evaluate $\displaystyle\int_1^2 (x^5+1)^2 5x^4\,dx$ in two ways, first by expanding and evaluating the integral directly and then via the change of variables $u = x^5 + 1$.

Exercise 32.10. Evaluate the definite integral.

(i) $\displaystyle\int_0^1 x(x^2-1)^{99}\,dx.$

(ii) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{5+x^3}}\,dx.$

(iii) $\displaystyle\int_1^3 \frac{(\ln x)^2}{x}\,dx.$

(iv) $\displaystyle\int_2^3 \frac{x\,dx}{1+x^2}.$

(v) $\displaystyle\int_3^5 x\sqrt{x-1}\,dx.$

(vi) $\displaystyle\int_0^1 e^x(1+e^x)^5\,dx.$

(vii) $\displaystyle\int_0^2 \frac{1-x}{1+x}\,dx.$

(viii) $\displaystyle\int_0^2 \frac{1-x}{1+x^2}\,dx.$

Exercise 32.11. A particle moves along a straight line. Its velocity at time t is $v = t^2 - t - 6$. What is the distance between its position at time $t = 0$ and its position at time $t = 4$? How far does it travel between time $t = 0$ and time $t = 4$? Hint: $v = ds/dt$. The second question is different from the first since the particle turns around at time $t = 3$.

Chapter VI

Applications of Definite Integrals

The basic principle in this chapter is that the

whole is equal to the sum of its parts.

Every application rests on a formula of the form

$$Q = \int dQ.$$

Riemann sums play a key role in the reasoning. In each application we reason with Riemann sums to achieve an approximate equality

$$Q \approx \sum \Delta Q$$

and then pass to the limit to find Q exactly.

33 Plane Area

See Thomas pages 376-386.

To find the area of a plane figure by calculus we use the formula

$$A = \int dA.$$

Example 33.1. We find the area of the area bounded by the two curves $y = x^3$ and $x = y^2$. The curves intersect at $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$. The region between the two curves is defined by the inequalities

$$0 \leq x \leq 1, \qquad x^3 \leq y \leq \sqrt{x}.$$

Break the area into strips parallel to the y-axis of width dx. The strip corresponding to a given value of x has area

$$dA = (\sqrt{x} - x^3)\, dx$$

and the total area is

$$A = \int dA = \int_0^1 (\sqrt{x} - x^3)\, dx = \left(\frac{2x^{3/2}}{3} - \frac{x^4}{4} \right)\Bigg|_0^1 = \frac{5}{12}.$$

Remark 33.2. The reasoning in the previous example really involves Riemann sums. The area was approximated by the Riemann sum

$$\sum_{j=1}^{n} \Delta A_j = \sum_{j=1}^{n} (\sqrt{c_j} - c_j^3)\Delta x_j$$

where $0 = x_0 \leq c_1 \leq x_1 \leq c_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq c_n \leq x_n = 1$. and $\Delta x_j = x_j - x_{j-1}$. The limit of the Riemann sums for $\max_j \Delta_j \approx 0$ is the the desired area.

Example 33.3. We evaluate the area bounded by the curve $y = x^2$ and the line $y = 2x$ in two ways.

(i) Using vertical strips: $dA = (2x - x^2)\, dx$, $0 \leq x \leq 2$, so

$$A = \int dA = \int_{x=0}^{x=2} (2x - x^2)\, dx = \left(x^2 - \frac{x^3}{3} \right)\Bigg|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

(ii) Using horizontal strips: $dA = (\sqrt{y} - y/2)\, dy$, $0 \leq y \leq 4$,

$$A = \int dA = \int_{y=0}^{y=4} \left(\sqrt{y} - \frac{y}{2} \right) dy = \left(\frac{2y^{3/2}}{3} - \frac{y^2}{4} \right)\Bigg|_0^4 = \frac{16}{3} = \frac{16}{4} = \frac{4}{3}$$

Remark 33.4. The curve $y = x^2$ is concave up. The Secant Concavity Theorem 22.10 tells us that the line $y = 2x$ is above the curve $y = x^2$ for $0 \leq x \leq 2$.

Example 33.5. Exercise 33.6 asks us to find the area bounded by the curve $y = x(4 - x)$ and the line $y = x$ in two ways: using vertical strips and using horizontal strips. Here is how to get started on the latter method. The curve and line intersect in the two points $(x, y) = (0, 0)$ and $(x, y) = (3, 3)$. The curve $y = x(4 - x)$ can also be written as $4 - y = (x - 2)^2$. We have two kinds of horizontal strips:

$$\begin{aligned} dA &= (y - 2 + \sqrt{4 - y})\, dy \qquad && \text{for } 0 \leq y \leq 3 \\ dA &= 2\sqrt{4 - y}\, dy && \text{for } 3 \leq y \leq 4. \end{aligned}$$

By the Additivity Rule $\int_a^b = \int_a^c + \int_c^b$ we have

$$A = \int dA = \int_0^3 \big(y - 2 + \sqrt{4 - y}\big)\, dy + \int_3^4 2\sqrt{4 - y}\, dy.$$

Exercises

Exercise 33.6. Find the area bounded by the curve $y = x(4 - x)$ and the line $y = x$ in two ways, first by using horizontal strips and the Additivity Rule (see Example 33.5) and then using vertical strips.

Exercise 33.7. Find the area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$. (Thomas Example 4 Page 380.)

Exercise 33.8. Find the area of the region in the first quadrant bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$. (Thomas Example 5 Page 381.)

Exercise 33.9. Find the area in the previous exercise by integrating with respect to y. (Thomas Example 6 Page 382.)

Exercise 33.10. Find the area of the region in the first quadrant above the curve $y = x^2$ and below the curve $y = x^3$

Exercise 33.11. Find the area of the region in the first quadrant above the curve $x = y^2$ and below the curve $x = y^3$.

Exercise 33.12. Find a so that the curves $y = x^2$ and $y = a \cos x$ intersect at the points $(x, y) = (\pm\frac{\pi}{4}, \frac{\pi^2}{16})$. Then find the area between these curves.

34 Volumes

See Thomas pages 396-415.

To find the volume V of a body by calculus we use the formula

$$V = \int dV.$$

§34.1. Suppose the body results by revolving the region

$$0 \le y \le f(x), \qquad a \le x \le b$$

about the x-axis. Then the line segment from $(x, 0)$ to $(x, f(x))$ is the radius of a disk whose area is $A(x) = \pi f(x)^2$ so the volume is

$$V = \int dV = \int A(x)\,dx = \int_a^b \pi f(x)^2\,dx.$$

The term $dV = \pi f(x)^2\,dx$ represents the volume of an infinitely thin circular disk of width dx and radius $f(x)$.

Example 34.2. If we revolve the half disk

$$0 \le y \le \sqrt{a^2 - x^2}, \qquad -a \le x \le a$$

about the x-axis we get a sphere of radius a. Its volume is

$$V = \int_{-a}^{a} \pi(\sqrt{a^2 - x^2})^2\,dx = \pi\left(a^2 x - \frac{x^3}{3}\right)\Bigg|_{-a}^{a} = \pi\left(2a^3 - \frac{2a^3}{3}\right) = \frac{4\pi a^3}{3}.$$

Example 34.3. Consider the triangle bounded by the x-axis, the y-axis, and the line $ay + hx = ah$, Revolve this triangle about the y-axis. (Note: y-axis *not* x-axis.) The result is a right circular cone whose apex is at the point $(0, h)$ and whose base is a disk of radius a. The disk perpendicular to the y-axis centered at the point $(0, y)$ has radius $x = a(h-y)/h$ and hence area $A(y) = \pi a^2(h-y)^2/h^2$. Hence the volume is

$$V = \int_0^h \frac{\pi a^2(h-y)^2}{h^2}\,dy = \frac{\pi a^2}{h^2}\left(-\frac{(h-y)^3}{3}\right)\Bigg|_0^h = \frac{\pi a^2 h}{3}.$$

Thus the volume of a right circular cone is one third the area of its base time its altitude.

§ 34.4. The Slice Principle. Suppose that we want to find the volume of some solid body $\mathcal{B}$. We choose a line L and for each point x on L we find find the area $A(x)$ of the intersection of $\mathcal{B}$ with the plane through x perpendicular to L. Then

$$V = \int dV \qquad \text{where} \qquad dV = A(x)\,dx.$$

The disk method for the volume of a surface of revolution is an instance of the Slice Principle with the line L being the axis of rotation and $A(x) = \pi f(x)^2$.

Example 34.5. Here is another instance of the Slice Principle. A plane passes through a diameter of the base of a right circular cylinder of radius a and makes an angle of β with the base. It cuts a wedge out of the cylinder: we will find the volume of this wedge. Represent the base of the cylinder as the disk $x^2 + y^2 \le a^2$ and let the diameter through which the plane passes be the x-axis. A plane perpendicular to this diameter and passing through the point $(x, 0)$ cuts the wedge in a right triangle of base $b = \sqrt{a^2 - x^2}$ and altitude $h = \sqrt{a^2 - x^2}\tan\beta$ so the area of the triangle is $A(x)\frac{1}{2}bh = \frac{1}{2}(a^2 - x^2)\tan\beta$. The volume of the wedge is

$$V = \int dV = \int_{-a}^{a} A(x)\,dx = \frac{\tan\beta}{2}\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{2a^3\tan\beta}{3}$$

§34.6. Cylinders. The body formed by passing parallel lines through a plane figure is called an (infinite) **cylinder**. The parallel lines are called the **generators** of the cylinder. We do not assume that the generators are perpendicular to the plane of the original figure. A second plane parallel to the first and at a distance h from it cuts off a cylinder of **altitude** h. By the Slice Principle the volume of this cylinder is $F = Ah$ where A is the area of the base. This is because every plane parallel to the original plane cuts cylinder in a figure of the same area A.

Remark 34.7. The diameter of a quarter dollar coin is about one inch so its area is about $(\pi/4)\text{in}^2$. A stack of 80 quarters forms a cylinder about five inches tall. This right circular cylinder has volume $20\pi\text{in}^3$. If we place a ruler along the side of the stack and then tilt the ruler (keeping the quarters touching it) we get a cylinder which is not a right cylinder. Its altitude is still five inches and its volume is still $20\pi\text{in}^3$, the total volume of the 80 quarters. In fact, even if the edges of the coins do not align along a straight line the volume of the quarters is unchanged. (This illustrates the Slice Principle.)

§34.8. Cones. Consider a plane figure of area A and a point P not on the plane. The body formed by drawing line segments from the points of the figure to the point P is called a **cone**. The point P is called the **apex** of the cone and the distance from P to the plane is called the **altitude** of the cone. Let L be the line though P perpendicular to the plane and x be the distance from P to a variable point X on L. By similarity the plane through X and perpendicular L cuts the cone in a figure of area

$$A(x) = \frac{x^2}{h^2}A$$

so the volume of the cone is

$$V = \int dV = A\int_0^h \frac{x^2}{h^2}\,dx = \frac{Ah}{3}.$$

For example, the volume of a pyramid of altitude h whose base is a square of side a is $V = a^2h/3$. It does not matter if the apex of the pyramid is directly over the center of the square. See Exercises 34.21-34.23.

§34.9. Shells. Consider a body formed by rotating around the y-axis a figure lying in the right half plane. Suppose that a line parallel to the y-axis and at a distance x from it cuts the figure in a segment of length $f(x)$. When this line segment is rotated around the y-axis, it sweeps out a right circular cylinder of radius x and height $f(x)$. The area of this cylinder is $A(x) = 2\pi x f(x)$. A second line segment at a distance of dx from the first sweeps out a slightly larger cylinder of about the same area and the cylindrical **shell** between these two cylinders has volume

$$dV = 2\pi x f(x)\,dx.$$

Example 34.10. We recalculate the volume of the cone from Example 34.3 using shells. This cone was constructed by rotating the triangle bounded by the x-axis, the y-axis, and the line $y = h(a-x)/a$ about the y-axis. The volume of the shell with coordinate x and width dx is

$$dV = 2\pi x \cdot \frac{h(a-x)}{a}\,dx$$

so the volume of the cone is

$$V = \int_0^a 2\pi x\frac{h(a-x)}{a}\,dx = \frac{\pi h}{a}\left(ax^2 - \frac{2x^3}{3}\right)\Bigg|_0^a = \frac{\pi a^2 h}{3}$$

in agreement with the answer we found in Example 34.3.

Example 34.11. We calculate the volume of a sphere of radius a using shells. The sphere is obtained by rotating the half disk

$$0 \le y \le \sqrt{a^2 - x^2}, \qquad -a \le x \le a$$

about the x-axis. A line parallel to the x-axis at height y intersects the half disk in a segment of length $2\sqrt{a^2 - y^2}$ and sweeps out a cylinder of area $2\pi y \cdot 2\sqrt{a^2 - y^2}$. Hence the volume is

$$V = \int_0^a 4\pi y\sqrt{a^2 - y^2}\, dy = -2\pi \int_{a^2}^0 \sqrt{u}\, du = -2\pi \left. \frac{2u^{3/2}}{3} \right|_{a^2}^0 = \frac{4\pi a^3}{3}$$

To evaluate the integral we used the substitution $u = a^2 - y^2$, so $du = -2y\, dy$, and $x = a \implies u = 0$ and $x = 0 \implies u = a^2$.

Exercises

Exercise 34.12. Find the volume generated of a pyramid that has an altitude of h and a base that is a square of side a. Hint: This is an example of a cone as defined in §34.8. A plane parallel to the base and at a distance x from the apex cuts the pyramid in a square. Determine the side length of the square using similar triangles. (See Thomas Example 1 Page 398.)

Exercise 34.13. Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis. (See Thomas Example 7 Page 401.)

Exercise 34.14. Find the volume of the solid generated by revolving the region bounded by $y = 1 - \sqrt{x}$, the x-axis, and the line $x = 4$ about the x-axis.

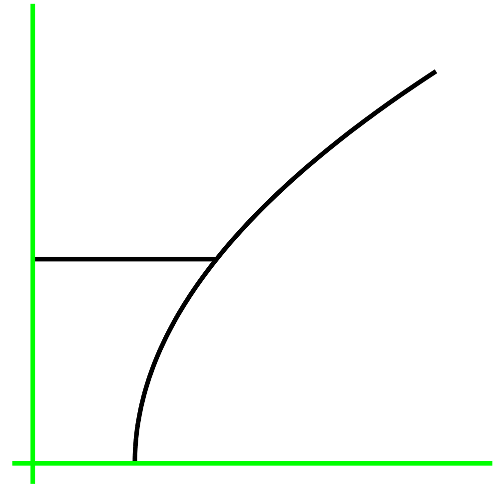
Exercise 34.15. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$. Hint: The radius of the circle swept out by by the point $(x, \sqrt{x})$ is $\sqrt{x} - 1$. (See Thomas Example 6 Page 401.)

Exercise 34.16. Find the volume of the solid generated by revolving the region between the parabola $x = 2 - y^2$ and the y-axis about the x-axis.

Exercise 34.17. Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$. Hint: The radius of the circle swept out by by the point $(y^2 + 1, y)$ is $2 - y^2$. (See Thomas Example 8 Page 402.)

Exercise 34.18. Find the volume of the solid generated by revolving the solid between the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant about the y-axis. Hint: A horizontal slice intersects the solid in a "washer", i.e. the area between two concentric circles. (See Thomas Example 10 Page 404.)

Exercise 34.19. A vase is constructed by rotating the curve $y = \sqrt{x-1}/100$ for $0 \leq y \leq 6$ around the y axis. It is filled with water to a height $y = h$ where $h < 6$. (a) Find the volume of the water in terms of h. (b) If the vase is filling with water at the rate of 2 cubic units per second, how fast is the height of the water increasing when this height is 5 units?



Exercise 34.20. A triangle is formed by drawing lines from the two endpoints of a line segment of length b to a vertex V which is at a height h above the line of the line segment. Its area is then $A = \int_{y=0}^{h} dA$ where dA is the area of the strip cut out by two parallel lines separated by a distance of dz and at a height of z above the line containing the line segment. Find a formula for dA in terms of b, z, and dz and evaluate the definite integral.

Exercise 34.21. A pyramid is formed by drawing lines from the four vertices of a rectangle of area A to a apex P which is at a height h above the plane of the rectangle. (The apex need not be above the center of the rectangle.) Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the rectangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

Exercise 34.22. A tetrahedron is formed by drawing lines from the three vertices of a triangle of area A to a apex P which is at a height h above the plane of the triangle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the triangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

Exercise 34.23. A skew cone is formed by drawing lines from the perimeter of a circle of area A to an apex P which is at a height h above the plane of the circle. (The apex need not be above the center of the circle.) Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the circle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

35 Arc Length

See Thomas pages 416-423.

To find the arc length s of a curve by calculus we use the formula

$$s = \int ds.$$

§35.1. Consider a parametric curve Γ defined by the equations

$$x = f(t), \qquad y = g(t), \qquad a \le t \le b,$$

where the functions f and g are continuously differentiable. The endpoints of the curve are the points

$$A = (f(a), g(a)), \qquad B = (f(b), g(b)).$$

A partition $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$ determines a polygonal arc L with vertices

$$P_j = (x_j, y_j) = (f(t_j), g(t_j)).$$

The polygonal arc L has the same endpoints $A = P_0$ and $B = P_n$ as the original curve. Bt the Pythogorean Theorem the length L of the polygonal arc is

$$L = \sum_{j=1}^{n} \Delta s_j$$

where

$$\Delta s_j = \sqrt{(\Delta x_j)^2 + (\Delta y_j)^2}, \qquad \Delta x_j = x_j - x_{j-1}, \qquad \Delta y_j = y_j - y_{j-1}. \quad (\dagger)$$

Note that

$$\Delta s_j = \frac{\Delta s_j}{\Delta t_j} \cdot \Delta t_j = \sqrt{\left(\frac{\Delta x_j}{\Delta t_j}\right)^2 + \left(\frac{\Delta y_j}{\Delta t_j}\right)^2} \cdot \Delta t_j$$

and by the definition of the derivative

$$\sqrt{\left(\frac{\Delta x_j}{\Delta t_j}\right)^2 + \left(\frac{\Delta y_j}{\Delta t_j}\right)^2} \approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

when $\Delta t_j \approx 0$ and the right hand side is evaluated at any value of t in the interval $[t_{j-1}, t_j]$. The integral

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

is defined to be the **arclength** of the original parameterized curve

Theorem 35.2. **(i)** *The length L of the polygonal arc is approximately equal to the arclength when mesh is small:*

$$\max_j \Delta t_j \approx 0 \implies L \approx s.$$

(ii) *The value of the arclength s is independent of the parametrization of the curve.*

Part (i) requires proof because the formula for L is not obviously a Riemann sum. The proof is not difficult, but we leave it for a more advanced course. Part (ii) is an easy consequence of the Change of Variables Theorem 32.1 as follows. Any other parameterization can be obtained from the original parameterizarion via a substitution $t = T(\tau)$ where T is an increasing function whose domain is an interval $[\alpha, \beta]$ and whose range is the domain $[a, b]$ of the original parameterization. Thus

$$dt = T'(\tau), \qquad d\tau = T'(\tau) = \frac{dt}{d\tau}\, d\tau$$

and

$$\tau = \alpha \implies t = a, \qquad \tau = \beta \implies t = b$$

so by the Chain Rule and the Change of Variables Theorem

$$\begin{aligned}
\int_\alpha^\beta \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\, d\tau &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\cdot\frac{dt}{d\tau}\right)^2 + \left(\frac{dy}{dt}\cdot\frac{dt}{d\tau}\right)^2}\, d\tau \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot\frac{dt}{d\tau}\, d\tau \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt
\end{aligned}$$

as required.

Remark 35.3. If we imagine the parameterization of the curve as describing the position of a moving point at time t, then

$$s = s(t) = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

represents the distance travelled by the particle during the time interval $[a, t]$. By the Fundamental Theorem the derivative of s is

$$s'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This is often written

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

in analogy with the notation (†) used above in the formula for the length of the polygonal arc.

Remark 35.4. In the special case where the curve is the graph $y = f(x)$ of a function we may take the parameterization

$$x = t, \qquad y = f(t)$$

and the formula for the arclength reduces to

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example 35.5. For the curve $y = (4\sqrt{2}/3)x^{3/2} - 1$, $0 \le x \le 1$, we have $dy/dx = 2\sqrt{2}x^{1/2}$ so

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 8x}\, dx$$

and

$$s = \int ds = \int_0^1 \sqrt{1+8x}\, dx = \frac{2}{3} \cdot \frac{1}{8}(1+8x)^{3/2}\Big|_0^1 = \frac{13}{6}$$

Example 35.6. We find the arclength of the arc of the circle of radius a subtended by the lines $y = (\tan\beta)x$ and $y = (\tan\alpha)x$ where $0 < \beta < \alpha < \pi/2$. This curve is given partametерically by the equations

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad \beta \le \theta \le \alpha.$$

We compute

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = a\cos\theta, \qquad ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = a\, d\theta$$

so the arclength is $s = \int_\beta^\alpha d\theta = a(\alpha - \beta)$. Using the formula

$$y = \sqrt{a^2 - x^2}, \qquad a\cos\alpha \le x \le a\cos\beta$$

we get

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{a}{\sqrt{a^2 - x^2}}$$

and hence

$$s = \int_{a\cos\alpha}^{a\cos\beta} \frac{a\, dx}{\sqrt{a^2 - x^2}} = -a\cos^{-1}\left(\frac{x}{a}\right)\Big|_{a\cos\alpha}^{a\cos\beta} = -a\alpha + a\beta$$

as before.

Exercises

Exercise 35.7. Find the length of each of the following curves. Warning: The functions in these problems have been carefully chosen so that the resulting integral can be done. The slightest error in algebra may lead to an integral which cannot be done.

(i) $x = 1 - t$, $y = 32 + 3t$, $-2/3 \leq t \leq 1$.

(ii) $x = t^3$, $y = 3t^2/2$, $0 \leq t \leq \sqrt{3}$.

(iii) $x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$, $0 \leq t \leq \pi/2$.

(iv) $y = x^{3/2}$, $0 \leq x \leq 4$.

(v) $y = (x^{3/2}/3) - x^{1/2}$, $1 \leq x \leq 3$.

(vi) $x = (y^3/6) + 1/(2y)$, $2 \leq y \leq 3$.

(vii) $y = \int_0^x \sqrt{u^2 + 2u}\, du$, $1 \leq x \leq 4$.

Exercise 35.8. Find a definite integral whose value is the length of each of the following curves. Do *not* evaluate the integral. *Do* specify the limits of integration.

(i) $x = a\cos\theta$, $y = b\sin\theta$, $0 \leq \theta \leq 2\pi$.

(ii) $x = t^2$, $y = t^3$, $0 \leq t \leq 3$.

(iii) $y = x^4$, $2 \leq x \leq 3$.

(iv) $y = \int_0^x \sqrt{u^2 + 1u}\, du$, $1 \leq x \leq 4$.

36 Surface Area

See Thomas pages 436-446. This topic was not covered in Fall 2006.

To find the area S of a surface by calculus we use the formula

$$S = \int dS.$$

§36.1. The region cut out of a right circular cone by two planes parallel to the base is called a **frustrum**. Let P_1 and P_2 be the points in which a generator of the cone intersects the two planes: the frustrum is swept out by rotating the line segment P_1P_2 about the axis of the cone. Let P be the midpoint of P_1P_2 and r_1, r_2, and $r = (r_1 + r_2)/2$, be the radii of the circles swept out by P_1, P_2, and P respectively. If we cut the cone and lay it flat we see that the area A of the frustrum is the difference of the areas of two sectors with a common central angle γ and radii r_1 and r_2, i.e.

$$A = \left(\frac{\gamma r_2^2}{2} - \frac{\gamma r_1^2}{2}\right) = \gamma\left(\frac{r_1 + r_2}{2}\right) \cdot (r_2 - r_1) = \gamma r \cdot (r_2 - r_1).$$

The second factor $(r_2 - r_1)$on the right is length of the line segment P_1P_2 and the first factor γr is the length of the circular are swept out by the midpoint P. But this circular arc has the same length as the circle swept out by the midpoint P in the original cone, i.e. $\gamma r = 2\pi x$ where x is the radius of this circle, i.e. the distance from P to the axis of the cone. Hence

$$A = 2\pi x\Delta, \qquad \Delta = r_2 - r_1. \tag{‡}$$

§36.2. Now we compute the surface area swept out when a curve Γ in the right half plane is rotated around the y-axis. As in §35.1 we consider a polygonal arc L with vertices $P_0, P_1, \ldots, P_n$ on Γ and having the same endpoints $A = P_0$ and $B = P_n$ as Γ. By Equation (‡) the polygonal arc L sweeps out a surface (a union of frustrums) with area

$$S(L) = \sum_{j=1}^{n} x_j \Delta s_j$$

where x_j is the x-coordinate of the midpoint of the line segment $P_{j-1}P_j$ and Δs_j is the length of this segment. If we choose a parameterization of the original curve Γ the sum $S(L)$ becomes a Riemann sum approximating the area $S(\Gamma)$ swept out by Γ. Hence

$$S(\Gamma) = \int dS, \qquad dS = 2\pi x\, ds$$

where ds is the arclength element of the original curve Γ.

Example 36.3. The surface area of of the sphere of radius a swept out when the circular arc $x = \sqrt{a^2 - y^2}$ of radius a is rotated about the y-axis is

$$S = \int_{-a}^{a} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

The quantity under the radical sign is

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{\sqrt{a^2 - y^2}}\right)^2 = \frac{a^2}{a^2 - y^2}$$

so the integral evaluates to

$$S = \int_{-a}^{a} \frac{2\pi a x}{\sqrt{a^2 - y^2}}\, dy = \int_{-a}^{a} 2\pi a\, dy = 4\pi a^2$$

where we used $x = \sqrt{a^2 - y^2}$ in the penultimate step.

Remark 36.4. We can also find the area of a sphere of radius a using the standard parameterization

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

For this parameterization $ds = a\, d\theta$ so

$$S = \int 2\pi x\, ds = \int_{-\pi/2}^{\pi/2} (2\pi a\cos\theta) a\, d\theta = 2\pi a^2 \sin\theta \Big|_{-\pi/2}^{\pi/2} = 4\pi a^2.$$

Exercises

Exercise 36.5. Find the area of the surface generated by rotating each of the following curves about the indicated axis. Warning: The functions in these problems have been carefully chosen so that the resulting integral can be done. The slightest error in algebra will lead to an integral which cannot be done.

(i) $y = x^3/9$, $0 \le x \le 2$, x-axis.

(ii) $y = \sqrt{x+1}$, $1 \le x \le 5$, x-axis.

(iii) $x = (1/3)y^{3/2} - y^{1/2}$, $1 \le y \le 3$, y-axis.

(iv) $x^{2/3} + y^{2/3} = 1$, $-1 \le x \le 1$, $y \ge 0$, x-axis.

(v) $y = \int_0^x \sqrt{u^2 - 1}\, du$, $1 \le x \le 2$, y-axis.

Exercise 36.6. Find a definite integral whose value is the area of the surface generated by rotating each of the following curves about the indicated axis. Do *not* evaluate the integral. *Do* specify the limits of integration.

(i) $y = x^3/9$, $0 \le x \le 2$, y-axis.

(ii) $y = \sqrt{x^2+1}$, $1 \le x \le 5$, x-axis.

(iii) $x = y^{3/2} - y^{1/2}$, $1 \le y \le 3$, y-axis.

(iv) $x^{1/3} + y^{1/3} = 1$, $-1 \le x \le 1$, $y \ge 0$, x-axis.

(v) $y = \int_0^x \sqrt{u^3 - 1}\, du$, $1 \le x \le 2$, y-axis.

Exercise 36.7. Find the surface area generated by revolving the circular arc $y = \sqrt{1 - x^2}$, $a \le x \le b$ about the x-axis. (The result depends only on $b - a$, i.e. all slices of the same width cut from a spherical loaf of bread have the same amount of crust.)

37 Center of Mass

See Thomas pages 424-435.

§37.1. Every physical body has a **center of mass**. Every geometrical figure has a **centroid**. The centroid of a figure is (by definition) the center of mass of the body which results on giving the figure a uniform mass density. If a body B with center of mass $\bar{P}$ is suspended from a point Q on its boundary, it comes to rest with the line $Q\bar{P}$ vertical. If the body is a bar (lies in a line) or a plate (lies in a plane) and is supported from any point other than the center of mass, it will not balance.

§37.2. Consider the case where the body consists of n collinear mass points $P_1, P_2, \ldots, P_n$ with masses $m_1, m_2, \ldots, m_n$. (Imagine that the straight line is a teeter-totter and the mass points are children.) Let x be a coordinate on the line, i.e. $|x|$ is the distance from a point O (the origin) on the line with $x > 0$ for points to the right of O and $x < 0$ for points to the left. If the coordinate of the mass point P_j is x_j, then the coordinate $\bar{x}$ of the center of mass $\bar{P}$ is the weighted average

$$\bar{x} = \frac{1}{M}\sum_j m_j x_j, \qquad M = \sum_j m_j. \tag{$*$}$$

The denominator M in the weighted average is the total mass and each mass point receives a weight m_j/M which represents the fraction of the total mass of the mass point P_j. (These weights sum to one.)

§37.3. Here is an argument which explains why the bar doesn't balance if it is supported at some point other than the center of mass. Imagine that the bar is supported at the origin O and let the bar make an angle of θ with the horizontal axis. Then the potential energy of the jth mass point is its mass m_j times its height $x_j \sin\theta$. (When $0 \le \theta \le \pi/2$ the points to the right of the origin have positive height and those to the left have negative height. Remember that we have defined x_j to be the distance from P_j to O along the bar, not the horizontal coordinate of P_j.) Hence the total potential energy is

$$U = \left(\sum_j m_j x_j\right)\sin\theta.$$

Physicists tell us that at equilibrium the potential energy is a local minimum which implies that

$$0 = \frac{dU}{d\theta} = \left(\sum_j m_j x_j\right)\cos\theta$$

which means that either $\cos\theta = 0$ (the bar is vertical) or else that $\sum_j m_j x_j = 0$ (the center of mass is at the origin).

§37.4. Decomposition Principle. In many situations a set of mass points behaves as if it is a single mass point of mass M concentrated at the center of mass of the set. For example, imagine that the mass points are distributed among k sets $B_1, B_2, \ldots, B_k$. Let the total mass of the ith set be M_i and the center of mass of the points in B_i have coordinate $\tilde{x}_i$. Then $M = \sum_i M_i$ is the total mass and the center of mass $\bar{P}$ is

$$\bar{x} = \frac{1}{M}\sum_i M_i\tilde{x}_i \qquad M = \sum_i M_i.$$

For example imagine two boys with masses m_1, m_2 at $x_1, x_2 > 0$ on one side of the teeter-totter and two girls with masses m_3, m_4 at $x_3, x_4 < 0$ on the other side. The boys have total mass $M_1 = m_1 + m_2$ and center of mass $\tilde{x}_1 = (m_1x_1 + m_2x_2)/M_1$ and the girls have total mass $M_2 = m_3 + m_4$ and center of mass $\tilde{x}_2 = (m_3x_3 + m_4x_4)/M_2$. If a man of weight M_1 sits at $\tilde{x}_1$ and a woman of weight M_2 sits at $\tilde{x}_2$ the center of mass is at

$$\bar{x} = \frac{M_1\tilde{x}_1 + M_2\tilde{x}_2}{M_1 + M_2} = \frac{(m_1x_2 + m_2x_2) + (m_3x_3 + m_4x_4)}{(m_1 + m_2) + (m_3 + m_4)}$$

which is the same as the center of mass of the boys and girls. I like to remember the Decomposition Principle with the following dumb slogan:

The center of mass of the centers of mass is the center of mass.

§37.5. Next we treat a solid bar. We assume that the bar has a mass density μ which means that the amount dm mass in an infinitely small interval of width dx about the point with coordinate x is

$$dm = \mu(x)\,dx.$$

(The mass density at a point of the bar might depend on position of the point if (say) the bar would made of an alloy of two metals with more of one metal at one end than at the other.) Suppose that the coordinate of a point on the bar lies between the two values a and b. In analogy with Equation $(*)$ above we have

The total mass of the bar with coordinate x in $[a, b]$ and mass density $\mu(x)$ is

$$M = \int dm = \int_a^b \mu(x)\,dx$$

and the coordinate of the center of mass $\bar{P}$ is

$$\bar{x} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_a^b x\mu(x)\,dx.$$

Example 37.6. Suppose that the bar has a uniform mass density, i.e. the amount of mass in a segment of the bar is proportional to the width of the segment. Then the mass density μ is constant and the amount of mass in a tiny segment of width dx is $dm = \mu\, dx$. The center of mass is

$$\bar{x} = \frac{\displaystyle\int x\, dm}{\displaystyle\int dm} = \frac{\displaystyle\int_a^b x\mu\, dx}{\displaystyle\int_a^b \mu\, dx} = \frac{\mu\, \dfrac{x^2}{2}\Big|_a^b}{\mu x\Big|_a^b} = \frac{\mu(b^2 - a^2)}{2\mu(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}.$$

Thus the center of mass is at the midpoint of the bar. Since the mass distribution is uniform, the center of mass and centroid coincide.

§37.7. Similar considerations hold for systems which are not confined to one dimension. To specify a point in space it takes three coordinates (x, y, z) . For a system of mass points $P_j = (x_j, y_j, z_j)$ of mass m_j connected by weightless rigid rods the center of mass is the point $\bar{P} = (\bar{x}, \bar{y}, \bar{z})$ whose coordinates are

$$\bar{x} = \frac{1}{M}\sum_j m_j x_j, \qquad \bar{y} = \frac{1}{M}\sum_j m_j y_j, \qquad \bar{z} = \frac{1}{M}\sum_j m_j z_j,$$

where $M = \sum_j m_j$ is the total mass as before. If the system of mass points lies in the (x, y)-plane, the z-coordinates of all the points (and hence of the center of mass) are zero. For a body with a mass distribution μ the center of mass has coordinates

$$\bar{x} = \frac{1}{M}\int x\, dm, \qquad \bar{y} = \frac{1}{M}\int y\, dm, \qquad \bar{z} = \frac{1}{M}\int z\, dm,$$

where $M = \int dm$ is the total mass. For a wire the infinitesimal mass dm in an infinitesimal piece of arc of length ds at the point P is given by

$$dm = \mu\, ds,$$

where $\mu = \mu(P)$ is the mass density at the point P. For a plate the infinitesimal mass dm in an infinitesimal piece of the plate of area dA at the point P is given by

$$dm = \mu\, dA,$$

where $\mu = \mu(P)$ is the mass density at the point P. For a body the infinitesimal mass dm in an infinitesimal piece of the body of volume dV at the point P is given by

$$dm = \mu\, dV,$$

where $\mu = \mu(P)$ is the mass density at the point P. Appropriate units for the density μ are grams per centimeter for a wire, grams per square centimeter for a plate, and grams per cubic centimeter for a body. Using Calculus 221 we can find the center of mass of a wire and, with the aid of the Decomposition Principle (see below), some plates and bodies. Multiple integrals (taught in Calculus 234) are required for the general case.

Example 37.8. (The Centroid of a Semicircle.) We find the center of mass of the semicircle

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad 0 \le \theta \le \pi$$

with a uniform mass distribution μ. The element of arclength is

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = a\, d\theta.$$

The amount of mass in it is $dm = \mu\, ds = \mu a\, d\theta$. The coordinates of the center of mass are

$$\bar{x} = \frac{\displaystyle\int x\,\mu\, ds}{\displaystyle\int \mu\, ds} = \frac{\displaystyle\int_0^\pi \cos\theta\,\mu\, d\theta}{\displaystyle\int_0^\pi \mu\, ds} = \frac{\mu\sin\theta\Big|_0^\pi}{\mu\pi} = 0$$

and

$$\bar{y} = \frac{\displaystyle\int y\,\mu\, ds}{\displaystyle\int \mu\, ds} = \frac{\displaystyle\int_0^\pi \sin\theta\,\mu\, d\theta}{\displaystyle\int_0^\pi \mu\, ds} = \frac{-\mu\cos\theta\Big|_0^\pi}{\mu\pi} = \frac{2}{\pi}.$$

Note that the constant μ appears in the numerator and denominator and thus cancels. Thus when we want to find the the center of mass of a uniform mass distribution (i.e. constant mass density), we may as well assume that $\mu = 1$.

§ 37.9. Integral Decomposition Principle. The Decomposition Principle holds quite generally, whenever a body is decomposed into subsets in almost any way imaginable. Suppose that a plate in the (x, y)-plane is decomposed into infinitely many infinitely thin strips parameterized by a variable u. The mass of the strip is dm (it may depend on the parameter u) and its center of mass is $(\tilde{x}, \tilde{y})$ which also depends on u. An integral version of the Decomposition Principle applies and the center of mass of the plate has coordinates

$$\bar{x} = \frac{1}{M}\int \tilde{x}\, dm, \qquad \bar{y} = \frac{1}{M}\int \tilde{y}\, dm, \qquad M = \int dm.$$

A similar principle holds for a three dimensional solid body.

Example 37.10. (The Centroid of a Half Disk.) We find the center of mass of the half disk

$$0 \le y \le \sqrt{a^2 - x^2}, \qquad -a \le x \le a$$

of radius a with a uniform mass distribution μ. The mass density is constant so, by the same reasoning as in Example 37.8 , we might as well take $\mu = 1$. Then the total mass is the area of the half disk, i.e.

$$M = \int dA = \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \pi a^2/2$$

(half the area of the disk of radius a) and the center of mass of an infinitely thin strip with end points $(x, 0)$ and $(x, \sqrt{a^x - x^2})$ is its midpoint $(\tilde{x}, \tilde{y})$:

$$\tilde{x} = x, \qquad \tilde{y} = \frac{\sqrt{a^x - x^2}}{2}.$$

By the Integral Decomposition Principle 37.9, the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\displaystyle\int \tilde{x}\, dm}{\displaystyle\int dm} = \frac{\displaystyle\int_{-a}^{a} x\Big(\sqrt{a^2 - x^2}\Big)\, dx}{\pi a^2/2} = -\frac{1}{\pi a^2}\int_0^0 u\, du = 0$$

(where we we made the substitution $u = a^2 - x^2$, $du = -2x\, dx$) and

$$\begin{aligned}
\bar{y} &= \frac{\displaystyle\int \tilde{y}\, dm}{\displaystyle\int dm} = \frac{\displaystyle\int_{-a}^{a}\left(\frac{\sqrt{a^2 - x^2}}{2}\right)\cdot\Big(\sqrt{a^2 - x^2}\Big)\, dx}{\pi a^2/2} \\
&= \frac{1}{\pi a^2}\int_{-a}^{a}(a^2 - x^2)\, dx = \frac{1}{\pi a^2}\left(a^2 x - \frac{x^3}{3}\right)\Bigg|_{-a}^{a} = \frac{4a}{3\pi}.
\end{aligned}$$

Note that the center of mass is on the y-axis as could be predicted by symmetry.

§ 37.11. Here is a more general formulation of the argument used in Example 37.13. Consider a thin plate with constant mass density bounded by the curves $y = f(x)$ and $y = g(x)$ and between the lines $x = a$, and $x = b$. Assume $g(x) < f(x)$ for $a < x < b$. Then the center of mass of an infinitely thin strip with end points $(x, g(x))$ and $(x, f(x))$ is its midpoint $(\tilde{x}, \tilde{y})$:

$$\tilde{x} = x, \qquad \tilde{y} = \frac{f(x) + g(x)}{2}.$$

The mass density is constant and thus cancels in the expression for the center of mass so we take $\mu = 1$. Hence the height of the strip is $f(x) - g(x)$ so, if the strip has with dx, its mass (area) is

$$dm = \Big(f(x) - g(x)\Big)\, dx.$$

By the Integral Decomposition Principle 37.9, the center of mass $\bar{P} = (\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\displaystyle\int \tilde{x}\, dm}{\displaystyle\int dm} = \frac{\displaystyle\int_a^b x\Big(f(x) - g(x)\Big)\, dx}{\displaystyle\int_a^b \Big(f(x) - g(x)\Big)\, dx}$$

and

$$\bar{y} = \frac{\displaystyle\int \tilde{y}\,dm}{\displaystyle\int dm} = \frac{\displaystyle\int_a^b \frac{f(x)+g(x)}{2}\cdot\Big(f(x)-g(x)\Big)\,dx}{\displaystyle\int_a^b \Big(f(x)-g(x)\Big)\,dx}.$$

Theorem 37.12. *The medians*[16] *of a triangle are concurrent: they intersect in the centroid.*

Proof: First we give a physics argument and then we give a calculus proof. Decompose the triangle into infinitely many infinitely thin rectangular strips parallel to one of the sides. Each strip has its center of mass at the midpoint. By similar triangles each of these midpoints lies on the median. The mass of the strip is proportional to its length and we can find the coordinates of the center of mass using the Integral Decomposition Principle. This is equivalent to finding the center of mass of a variable mass distribution on the median so the centroid lies on the median. (See Exercise 37.21.) Since the argument works for each of the medians the centroid lies on all three medians.

Example 37.13. (The Centroid of a Triangle.) We prove Theorem 37.12 using calculus. Let the vertices be (a, h), (b, h), and $(0, 0)$ with $a < b$ and $h > 0$. The sides of the triangle lie on the three lines $x = ay/h$, $x = by/h$, and $y = h$. We view the horizontal line $y = h$ as the base of the triangle so h is the altitude and the area is $h(b-a)/2$. Since the mass density μ is constant so we might as well take take $\mu = 1$. The base of the triangle has length $b - a$ so, by similar triangles, the length of the horizontal line segment at height y is $(b - a)y)/h$ so the the mass (i.e. the area) of the thin horizontal rectangular strip of width dy and height y is

$$dm = \frac{(b-a)y}{h}\,dy$$

The total mass is

$$M = \int dm = \frac{(b-a)y}{h}\,dy = -\frac{(b-a)y^2}{2h}\bigg|_0^h = \frac{(b-a)h}{2}.$$

(No surprise: the base of the triangle is $b-a$ and the height is h.) The center of mass of the horizontal strip at height y is the midpoint $(\tilde{x}, \tilde{y})$ of the horizontal line segment so

$$\tilde{x} = \frac{(a+b)y}{2h}, \qquad \tilde{y} = y.$$

By the Integral Decomposition Principle 37.9,

$$\bar{y} = \frac{1}{M}\int \tilde{y}\,dm = \frac{2}{(b-a)h}\int_0^h y\frac{(b-a)y}{h}\,dy = \frac{2}{h^2}\int_0^h y^2\,dy = \frac{2h}{3}$$

[16] A **median** of a triangle is a line joining a vertex to the midpoint of the opposite side.

and

$$\bar{x} = \frac{1}{M}\int \tilde{x}\,dm = \frac{2}{(b-a)h}\int_0^h \frac{(a+b)y}{2h}\cdot\frac{(b-a)y}{h}\,dy = \frac{a+b}{3}.$$

The horizontal line through the centroid is $y = 2h/3$ and it intersects the sides $x = ay/h$ and $x = by/h$ in the points $(2a/3, 2h/3)$ and $(2b/3, 2h/3)$. The midpoint of the line segment joining these two points is the centroid.

Exercises

Exercise 37.14. Two rods of lengths a and b and the same constant mass density are welded together to form a right angle. Assume that the endpoints of the first rod are $O = (0,0)$ and $A = (a,0)$ and the endpoints of the second rod are O and $B = (0,b)$. Find the center of mass $\bar{P} = (\bar{x}, \bar{y})$.

Exercise 37.15. Find the center of mass of a bar located on the x-axis with end points $x = 0$ and $x = 2$ and mass density $\mu(x) = x^3$.

Exercise 37.16. Find the center of mass of a thin plate with constant mass density bounded by the parabola $y = x^2$ and the line $y = 3$.

Exercise 37.17. Find the center of mass of a thin plate (a half disk) with constant mass density mass bounded by the x-axis and the circle $x^2 + y^2 = a^2$. Is the answer the same as for the semicircle of Example 37.8?

Exercise 37.18. Find the center of mass of a thin plate (a quarter disk) with constant mass density mass bounded by the x-axis, the y-axis, and the circle $x^2 + y^2 = a^2$.

Exercise 37.19. Find the center of mass of a thin plate with constant mass density mass in the first quadrant bounded by the lines $y = 3$, $x = 3$, and the circle $x^2 + y^2 = 9$. (You can avoid some calculation by using geometry and the previous problem to find the area.)

Exercise 37.20. Find the center of mass of a thin plate with constant mass density bounded by the curves $y = x + 1$, $y = x^2$ between the lines $x = 0$, and $x = 1$.

Exercise 37.21. Show that if a system of mass points in space all lie on the same line, then the center of mass also lies on that line. (Hint: Choose coordinates so that the line is the x-axis. Then the y_j and and z_j all vanish: you must show that $\bar{y}$ and $\bar{z}$ also vanish.)

Exercise 37.22. Find the center of mass of a system of three mass points of the same mass at the points $A = (a,h)$, $B = (b,h)$, $C = (0,0)$. Is the center of mass of the three mass points the same as the centroid of the triangle ABC?

Exercise 37.23. Find the center of mass of a system of three bars with the same uniform mass density connecting the vertices $A = (a,h)$, $B = (b,h)$, $C = (0,0)$. Hint: By the Decomposition Principle this is the same as the center of mass of a system of three possibly unequal mass points located at the midpoints of the sides. Is the center of mass of the three bars the same as the centroid of the triangle ABC?

Chapter VII
Loose Ends

This chapter contains material which we probably won't have enough time to cover in the course.

38 The Natural Log Again*

See Thomas pages 476-485.

In this section we prove Theorem 16.1 which defined the exponential function $\exp_a(x) = a^x$ and also prove that this function is differentiable.

§38.1. It is easy to find the antiderivative of a power, at least when the exponent is not -1, namely

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

To handle the exceptional case we make a

Definition 38.2. The **natural logarithm** function is defined by

$$\ln x = \int_1^x \frac{dt}{t}.$$

Its domain is the set of all positive numbers x.

Theorem 38.3. *There is a number e such that*

$$\ln x = \log_e x$$

for all positive x.

Proof:

§38.4. Now we can prove the Power Law

$$\frac{d}{dx}x^p = px^{p-1}$$

for any exponent p not just rational numbers.

39 Taylor Approximation*

This is a warmup for infinite series which we study in Math 222. See Thomas pages 807-810.

§39.1. When f is a function and $k \geq 0$ is an integer the notation $f^{(k)}$ denotes kth derivative of f. Thus

$$f^{(0)}(x) = f(x), \qquad f^{(1)}(x) = f'(x), \qquad f^{(2)}(x) = f''(x),$$

and so on. Given a number a in the domain of f and an integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \tag{\#}$$

is called the **degree** n **Taylor polynomial of** f **centered at** a. The Taylor polynomial $P_n(x)$ is the unique polynomial of degree n which has the same derivatives as f at a up to order n:

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

§39.2. The letter $\sum$ is the Greek S (for *sum*) and is pronounced *sigma* so the notation used in (#) is called **sigma notation**. It is a handy notation but if you don't like it you can indicate the summation with dots:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Hence the first few Taylor polynomials are

$$P_0(x) = f(a),$$

$$P_1(x) = f(a) + f'(a)(x-a),$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2},$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}.$$

§39.3. The Taylor polynomial $P_n(x)$ for $f(x)$ centered at a is the polynomial of degree n which best approximates $f(x)$ for x near a. To make this precise let

$$R_n(x) = f(x) - P_n(x)$$

denote the n**th Taylor Error of** f **at** a. When $R_n(x)$ is small $P_n(x)$ is a good approximation for $f(x)$. How small is small? The answer is given by

Theorem 39.4 (Taylor's Formula). *Suppose that f is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous. Let a be a point in the domain of f. Then*

$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0.$$

§ 39.5. Theorem 39.4follows from Theorem 22 on page 812 of Thomas; the proof is on pages 818-819. The Theorem tells us that not only is the error $R_n(x)$ small when x is close to a, it is so small that it is still small after being divied by the small number $(x-a)^n$.

In order to use Taylor's formula approximate a function f we pick a point a where the value of f and of its derivatives is known exactly. Then the Taylor polynomial $P_n(x)$ can be evaluated exactly for any x. We then need to "estimate the error" $R_n(x) = f(x) - P_n(x)$, i.e. to find an inequality

$$|R_n(x)| \le M|x-a|^{n+1}$$

which tells us how small the error $R_n(x)$ is, i.e. how close $P_n(x)$ is to $f(x)$. Theorem 23 on page 813 of Thomas tells us how to find M. We'll study this on Math 222.

Theorem 39.6 (Extended Mean Value Theorem). *Let f, R, and a be as in Theorem 39.4. Then for each b there is a number c_{n+1} between a and b such that*

$$f(b) - P_n(b) = \frac{f^{(n+1)}(c_{n+1})(b-a)^{n+1}}{(n+1)!}.$$

§39.7. Note that the formula for the error $f(b) - P_n(b)$ is the same as the next term in the series (#) except that the $n+1$st derivative $f^{(n+1)}$ is evaluated at the unknown point c_{n+1} instead of a. The Extended Mean Value Theorem is proved in problem 74 on page 174 of the text. Equation (♡) is an immediate consequence.

Exercises

Exercise 39.8. Evaluate $\sum_{k=3}^{5} \frac{1}{k}$.

Exercise 39.9. Let $f(x) = \sqrt{x}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(4) = f^{(k)}(4)$ for $k = 0, 1, 2, 3$.

Exercise 39.10. Let $f(x) = x^{1/3}$. Find the polynomial $P(x)$ of degree two which best approximates $f(x)$ near $x = 8$.

Exercise 39.11. Let $f(x)$ and $P(x)$ be as in §39.10. Evaluate $P(10)$ and use the Extended Mean Value Theorem to prove that

$$|10^{1/3} - P(10)| \le \frac{80}{27 \cdot 256}.$$

Hint: The function $g(x) = x^{-8/3}$ is decreasing so $g(10) \le g(8)$.

Exercise 39.12. Find a polynomial $P(x)$ of degree three such that

$$\lim_{x\to 0} \frac{\sin(x) - P(x)}{x^3} = 0.$$

Use the Extended Mean Value Theorem to show that

$$|\sin(x) - P(x)| \leq \frac{|x|^4}{24}.$$

40 Newton's Method*

Chapter VIII

More Problems

For graphing problems you may be asked to determine (a) where $f(x)$ is defined, (b) where $f(x)$ is continuous, (c) where $f(x)$ is differentiable, (d) where $f(x)$ is increasing and where it is decreasing, (e) where $f(x)$ is concave up and where it is concave down, (f) what the critical points of $f(x)$ are, (g) where the points of inflection are, (h) what (if any) the horizontal asymptotes to $f(x)$ are, and (i) what (if any) the vertical asymptotes to $f(x)$ are. (A horizontal line $y = b$ is called a *horizontal asymptote* if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$. A vertical line $x = a$ is called a *vertical asymptote* if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.)

For proofs the question will be carefully worded to indicate what you may assume in your proof. (See Problem 10 for example.) In this document you may use without proof any previously asserted fact. For example, you may use the fact that $\sin'(\theta) = \cos(\theta)$ to prove that $\cos'(\theta) = -\sin(\theta)$ since the former question precedes the latter below. (See Problems 12 and 13.) You may always use high school algebra (like $\cos(\theta) = \sin(\pi/2 - \theta)$) in your proofs.

1. State and prove the Sum Rule for derivatives. You may use (without proof) the Limit Laws.

2. State and prove the Product Rule for derivatives. You may use (without proof) the Limit Laws.

3. State and prove the Quotient Rule for derivatives. You may use (without proof) the Limit Laws.

4. State and prove the Chain Rule for derivatives. You may use (without proof) the Limit Laws. You may assume (as the proof in the Stewart text does) that the inner function has a nonzero derivative.

5. State the Sandwich[17] Theorem.

6. Prove that $\frac{dx^n}{dx} = nx^{n-1}$, for all positive integers n.

7. Prove that $\frac{dx^n}{dx} = nx^{n-1}$, for $n = 0$.

8. Prove that $\frac{dx^n}{dx} = nx^{n-1}$, for all negative integers n.

9. Prove that $\frac{de^x}{dx} = e^x$.

10. Prove that

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

[17] Also called the Squeeze Theorem

You may assume without proof the Sandwich Theorem, the Limit Laws, and that the sin and cos are continuous. Hint: See Problem 98.

11. Prove that
$$\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}=0.$$

12. Prove that $\dfrac{d\sin x}{dx}=\cos x$.

13. Prove that $\dfrac{d\cos x}{dx}=-\sin x$.

14. Prove that $\dfrac{d\tan x}{dx}=\sec^2 x$.

15. Prove that $\dfrac{d\cot x}{dx}=-\csc^2 x$.

16. Prove that $\dfrac{d\ln x}{dx}=\dfrac{1}{x}$.

17. Prove that $\dfrac{d\sin^{-1} x}{dx}=\dfrac{1}{\sqrt{1-x^2}}$.

18. Prove that $\dfrac{d\cos^{-1} x}{dx}=-\dfrac{1}{\sqrt{1-x^2}}$.

19. Prove that $\dfrac{d\tan^{-1} x}{dx}=\dfrac{1}{1+x^2}$.

20. True or false? A differentiable function must be continuous. If true, give a proof; if false, illustrate with an example.

21. True or false? A continuous function must be differentiable. If true, give a proof; if false, illustrate with an example.

22. Explain why $\lim\limits_{x\to 0} 1/x$ does not exist.

23. Explain why $\lim\limits_{\theta\to\pi/2}\tan\theta$ does not exist.

24. Explain why $\lim\limits_{\theta\to\pi/2}\sec\theta$ does not exist.

25. Explain why $\lim\limits_{\theta\to 0}\csc\theta$ does not exist.

26. Explain why $\lim\limits_{x\to 0}\sin(1/x)$ does not exist.

27. Explain why $\lim\limits_{\theta\to\infty}\cos\theta$ does not exist.

28. Let $\operatorname{sgn}(x)$ be the sign function. This function is given by
$$\operatorname{sgn}(x)=\begin{cases}1, & \text{if } x>0,\\ 0, & \text{if } x=0,\\ -1, & \text{if } x<0.\end{cases}$$

Explain why $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist.

29. Explain why $\lim_{y\to 0} 2^{1/y}$ does not exist.

30. Explain why $\lim_{x\to 1} 2^{1/(x-1)}$ does not exist.

31. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \sin 2x$.

32. Calculate $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{(x+h)-x}$ when $f(x) = \cos 2x$.

33. Calculate $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ when $f(x) = \sin(x^2)$.

34. Calculate $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ when $f(x) = \cos(x^2)$.

35. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \sqrt{\sin x}$.

36. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = x\sin x$.

37. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = e^{\sqrt{x}}$.

38. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = e^{\sin x}$.

39. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \ln(ax+b)$.

40. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = e^{\cos x}$.

41. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = x^x$.

42. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \dfrac{\sin x}{x}$.

43. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \sqrt{ax+b}$.

44. Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = (mx+c)^n$.

45. Use differentiation to estimate the number $\dfrac{127^{4/3}-125^{4/3}}{2}$ approximately without a calculator. Your answer should have the form p/q where p and q are integers. Hint: $5^3 = 125$.

46. What is the derivative of the area of a circle with respect to its radius?

47. What is the derivative of the volume of a sphere with respect to its radius?

48. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

49. Find the equations of the tangent and normal to the curve $y = x^3 - 2x + 7$ at the point $(1, 6)$.

50. Find the equation of the tangent line to the curve $3xy^2 - 2x^2y = 1$ at the point $(1, 1)$. Find d^2y/dx^2 at this point.

51. Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

52. Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$.

53. Find the equations of the tangent and normal to the curve $c^2(x^2+y^2) = x^2y^2$ at the point $(c/\cos\theta, c/\sin\theta)$.

54. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

55. Show that the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (p, q) is $\dfrac{xp}{a^2} - \dfrac{yq}{b^2} = 1$

56. Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

57. Find the linear and quadratic approximations to $f(x) = \dfrac{1}{\sqrt{4+x}}$ at $x = 0$.

58. Find the linear and quadratic approximations to $f(x) = \sqrt{1+x}$ at $x = 0$.

59. Find the linear and quadratic approximations to $f(x) = \dfrac{1}{(1+2x)^4}$ at $x = 0$.

60. Find the linear and quadratic approximations to $f(x) = (1+x)^3$ at $x = 0$.

61. Find the linear and quadratic approximations to $f(x) = \sec x$ at $x = 0$.

62. Find the linear and quadratic approximations to $f(x) = x\sin x$ at $x = 0$.

63. Find the linear and quadratic approximations to $f(x) = x^3$ at $x = 1$.

64. Find the linear and quadratic approximations to $f(x) = x^{1/3}$ at $x = -8$.

65. Find the linear and quadratic approximations to $f(\theta) = \sin\theta$ at $\theta = \pi/6$.

66. Find the linear and quadratic approximations to $f(x) = x^{-1}$ at $x = 4$.

67. Find the linear and quadratic approximations to $f(x) = x^3 - x$ at $x = 1$.

68. Find the linear and quadratic approximations to $f(x) = \sqrt{x}$ at $x = 4$.

69. Find the linear and quadratic approximations to $f(x) = \sqrt{x^2 + 9}$ at $x = -4$.

70. Use quadratic approximation to find the approximate value of $\sqrt{401}$ without a calculator. Hint: $\sqrt{400} = 20$.

71. Use quadratic approximation to find the approximate value of $(255)^{1/4}$ without a calculator. Hint: $256^{1/4} = 4$.

72. Use quadratic approximation to find the approximate value of $\dfrac{1}{(2.002)^2}$ without a calculator.

73. Approximate $(1.97)^6$ without a calculator. (Leave arithmetic undone.)

74. Let f be a function such that $f(1) = 2$ and whose derivative is known to be $f'(x) = \sqrt{x^3 + 1}$. Use a linear approximation to estimate the value of $f(1.1)$. Use a quadratic approximation to estimate the value of $f(1.1)$.

75. Find the second derivative of x^7 with respect to x.

76. Find the second derivative of $\ln x$ with respect to x.

77. Find the second derivative of 5^x with respect to x.

78. Find the second derivative of $\tan\theta$ with respect to θ.

79. Find the second derivative of x^2e^{3x} with respect to x.

80. Find the second derivative of $\sin 3x \cos 5x$ with respect to x.

81. Find the third derivative of u^4 with respect to u.

82. Find the third derivative of $\ln x$ with respect to x.

83. Find the second derivative of $\tan x$ with respect to x.

84. If $\theta = \sin^{-1} y$ show that $\dfrac{d^2\theta}{dy^2} = \dfrac{y}{(1-y^2)^{3/2}}$.

85. If $y = e^{-t}\cos t$ show that $\dfrac{d^2y}{dt^2} = 2e^{-t}\sin t$.

86. If $u = t + \cot t$ show that $\sin^2 t \cdot \dfrac{d^2u}{dt^2} - 2u + 2t = 0$.

87. If $y = e^{\tan x}$ show that $\cos^2 x \cdot \dfrac{d^2y}{dx^2} - (1 + \sin 2x)\dfrac{dy}{dx} = 0$.

88. State L'Hôpital's rule and give an example which illustrates how it is used.

89. Explain why L'Hôpital's rule works. Hint: Expand the numerator and the denominator in terms of Δx.

90. Give three examples to illustrate that a limit problem that looks like it is coming out to $0/0$ could be really getting closer and closer to almost anything and must be looked at a different way.

91. Give three examples to illustrate that a limit problem that looks like it is coming out to 1^∞ could be really getting closer and closer to almost anything and must be looked at a different way.

92. Give three examples to illustrate that a limit problem that looks like it is coming out to 0^0 could be really getting closer and closer to almost anything and must be looked at a different way.

93. Give three examples to illustrate that a limit problem that looks like it is coming out to $\infty - \infty$ could be really getting closer and closer to almost anything and must be looked at a different way.

94. Explain how limit problems that come out to ∞/∞ can always be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

95. Explain how limit problems that come out to $\infty - \infty$ can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

96. Explain how limit problems that come out to 0^0 can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

97. Explain how limit problems that come out to 1^∞ can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

98. Use calculus to show that the area A of a sector of a circle with central angle θ is $A = (\theta/2)R^2$ where R is the radius and θ is measured in radians. Hint: Divide the sector into n equal sectors of central angle $\Delta\theta = \theta/n$ and area ΔA. As in the proof (see Problem 10) that

$$\lim_{\Delta\theta \to 0} \frac{\sin(\Delta\theta)}{\Delta\theta} = 1,$$

the area ΔA lies between the areas of two right triangles whose areas can be expressed in terms of R and trig functions of $\Delta\theta$. Apply the Sandwich Theorem to $A = n\Delta A$ and use l'Hôpital's rule or Problem 10.

99. Use calculus to show that the area of a circle of radius R is πR^2. Hint: The area of a sector is a more general problem. (See problem 98.)

100. For which values of x is the function $f(x) = x^2 + 3x + 4$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

101. For which values of x is the function $f(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

102. For which values of x is the function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

103. Determine the value of k for which the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0, \end{cases}$$

is continuous at $x = 0$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

104. What does it mean for a function $f(x)$ to be continuous at $x = a$?

105. What does it mean for a function $f(x)$ to be differentiable at $x = a$?

106. What does $f'(a)$ indicate you about the graph of $y = f(x)$? Explain why this is true.

107. What does it mean for a function to be increasing? Explain how to use calculus to tell if a function is increasing. Explain why this works.

108. What does it mean for a function to be concave up? Explain how to use calculus to tell if a function is concave up. Explain why this works.

109. What is a horizontal asymptote of a function $f(x)$? Explain how to justify that a given line $y = b$ is a horizontal asymptote of $f(x)$.

110. What is a vertical asymptote of a function $f(x)$? Explain how to justify that a given line $x = a$ is an vertical asymptote of $f(x)$.

111. If $f(x) = |x|$, what is $f'(-2)$?

112. Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \leq 1, \\ bx + 2, & \text{if } x > 1, \end{cases}$$

is differentiable for all values of x.

113. Graph $f(x) = \begin{cases} 2 - x, & \text{if } x \geq 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$

114. Graph $f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0, \\ 2 - x, & \text{if } x < 0. \end{cases}$

115. Graph $f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ x^2 - 1, & \text{if } x \geq 1. \end{cases}$

116. Graph $f(x) = x + 1/x$.

117. Graph $f(x) = \dfrac{x^2 + 2x - 20}{x - 4}$ for $5 < x < 9$.

118. Graph $f(x) = \dfrac{1}{x^2 + 1}$.

119. Graph $f(x) = xe^x$.

120. State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

121. State the Mean Value Theorem and draw a picture which illustrates the statement of the theorem.

122. Explain why Rolle's theorem is a *special case* of the Mean Value Theorem.

123. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but that there is no number c in the interval $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

124. Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but that there is no number c in the interval $(0, 2)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

125. Show that the Mean Value Theorem is not applicable to the function $f(x) = |x|$ in the interval $[-1, 1]$.

126. Show that the Mean Value Theorem is not applicable to the function $f(x) = 1/x$ in the interval $[-1, 1]$.

127. Find a point on the curve $y = x^3$ where the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$.

128. Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

129. Find the local maxima and minima of $f(x) = (5x-1)^2 + 4$ without using derivatives.

130. Find the local maxima and minima of $f(x) = -(x-3)^2 + 9$ without using derivatives.

131. Find the local maxima and minima of $f(x) = -|x+4| + 6$ without using derivatives.

132. Find the local maxima and minima of $f(x) = \sin 2x + 5$ without using derivatives.

133. Find the local maxima and minima of $f(x) = |\sin 4x + 3|$ without using derivatives.

134. Find the local maxima and minima of $f(x) = x^4 - 62x^2 + 120x + 9$.

135. Find the local maxima and minima of $f(x) = (x-1)(x+2)^2$.

136. Find the local maxima and minima of $f(x) = -(x-1)^3(x+1)^2$.

137. Find the local maxima and minima of $f(x) = x/2 + 2/x$ for $x > 0$.

138. Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1, 3]$.

139. Find the local maxima and minima of $f(x) = \sin x + (1/2)\cos x$ in $0 \leq x \leq \pi/2$.

140. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

141. Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the value at the local maximum is less than the value at the local minimum.

142. Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

143. A train is moving along the curve $y = x^2 + 2$. A girl is at the point $(3, 2)$. At what point will the train be at when the girl and the train are closest? Hint: You will have to solve a cubic equation, but the numbers have been chosen so there is an obvious root.

144. Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.

145. Divide 15 into two parts such that the square of one times the cube of the other is maximum.

146. Suppose the sum of two numbers is fixed. Show that their product is maximum exactly when each one of them is half of the total sum.

147. Divide a into two parts such that the pth power of one times the qth power of the other is maximum.

148. Which number between 0 and 1 exceeds its pth power by the maximum amount?

149. Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

150. Show that the right circular cone with a given volume and minimum surface area has altitude equal to $\sqrt{2}$ times the radius of the base.

151. Show that the altitude of the right circular cone with maximum volume that can be inscribed in a sphere of radius R is $4R/3$.

152. Show that the height of a right circular cylinder with maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

153. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

154. An open box is to be made out of a given quantity of cardboard of area p^2. Find the maximum volume of the box if its base is square.

155. Find the dimensions of the maximum rectangular area that can be fenced with a fence 300 yards long.

156. Show that the triangle of the greatest area with given base and vertical angle is isosceles.

157. Show that a right triangle with a given perimeter has greatest area when it is isosceles.

158. What do distance, speed and acceleration have to do with calculus? Explain thoroughly.

159. A particle, starting from a fixed point P, moves in a straight line. Its position relative to P after t seconds is $s = 11 + 5t + t^3$ meters. Find the distance, velocity and acceleration of the particle after 4 seconds, and find the distance it travels during the 4th second.

160. The displacement of a particle at time t is given by $x = 2t^3 - 5t^2 + 4t + 3$. Find (i) the time when the acceleration is 8cm/s^2, and (ii) the velocity and displacement at that instant.

161. A particle moves along a straight line according to the law $s = t^3 - 6t^2 + 19t - 4$. Find (i) its displacement and acceleration when its velocity is 7m/s, and (ii) its displacement and velocity when its acceleration is 6m/s^2.

162. A particle moves along a straight line so that after t seconds its position relative to a fixed point P on the line is s meters, where $s = t^3 - 4t^2 + 3t$. Find (i) when the particle is at P, and (ii) its velocity and acceleration at these times t.

163. A particle moves along a straight line according to the law $s = at^2 - 2bt + c$, where a, b, c are constants. Prove that the acceleration of the particle is constant.

164. The displacement of a particle moving in a straight line is $x = 2t^3 - 9t^2 + 12t + 1$ meters at time t. Find (i) the velocity and acceleration at $t = 1$ second, (ii) the time when the particle stops momentarily, and (iii) the distance between two stops.

165. The distance s in meters travelled by a particle in t seconds is given by $s = ae^t + be^{-t}$. Show that the acceleration of the particle at time t is equal to the distance the particle travels in t seconds.

166. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$ radians?

167. A ladder 13 meters long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

168. A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet. (a) How fast is the distance from the television camera to the rocket changing at that moment? (b) How fast is the camera's angle of elevation changing at that same moment? (Assume that the telivision camera points toward the rocket.)

169. Explain why exponential functions arise in computing radioactive decay.

170. Explain why exponential functions are used as models for population growth.

171. Radiocarbon dating works on the principle that ^{14}C decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ^{14}C as does plant material on earth today. Estimate the age of the parchment.

172. After 3 days a sample of radon-222 decayed to 58% of its original amount. (a) What is the half life of radon-222? (b) How long would it take the sample to decay to 10% of its original amount?

173. Polonium-210 has a half life of 140 days. (a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days. (b) Find the mass after 100 days. (c) When will the mass be reduced to 10 mg? (d) Sketch the graph of the mass as a function of time.

174. If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0, find the number at time t.

175. If a radioactive substance disintegrates at a rate proportional to the amount present how much of the substance remains at time t if the initial amount is Q_0?

176. Current agricultural experts believe that the world's farms can feed about 10 billion people. The 1950 world population was 2.517 billion and the 1992 world population was 5.4 billion. When can we expect to run out of food?

177. The Archer Daniel Midlands company runs two ads on Sunday mornings. One says that "when this baby is old enough to vote, the world will have one billion new mouths to feed" and the other says "in thirty six years, the world will have to set eight billion places at the table." What does ADM think the population of the world is at present? How fast does ADM think the population is increasing? Use units of billions of people so you can write 8 instead of $8,000,000,000$. (Hint: $36 = 2 \times 18$.)

178. The population of California grows exponentially at an instantaneous rate of 2% per year. The population of California on January 1, 2000 was 20,000,000. (a) Write a formula for the population $N(t)$ of California t years after January 1, 2000. (b) Each Californian consumes pizzas at the rate of 70 pizzas per year. At what rate is California consuming pizzas t years after 1990? (c) How many pizzas were consumed in California from January 1, 2005 to January 1, 2009?

179. The population of the country of Slobia grows exponentially. (a) If its population in the year 1980 was 1,980,000 and its population in the year 1990 was 1,990,000, what is its population in the year 2000? (b) How long will it take the population to double? (Your answer may be expressed in terms of exponentials and natural logarithms.)

180. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

181. $\int (6x^5 - 2x^{-4} - 7x + 3/x - 5 + 4e^x + 7^x)\,dx$

182. $\int (x/a + a/x + x^a + a^x + ax)\,dx$

183. $\int \left(\sqrt{x} - \sqrt[3]{x^4} + \dfrac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right) dx$

184. $\int 2^x\,dx$

185. $\int_{-2}^{4} (3x-5)\,dx$

186. $\int_1^2 x^{-2}\,dx$

187. $\int_0^1 (1 - 2x - 3x^2)\,dx$

188. $\int_1^2 (5x^2 - 4x + 3)\,dx$

189. $\int_{-3}^{0} (5y^4 - 6y^2 + 14)\,dy$

190. $\int_0^1 (y^9 - 2y^5 + 3y)\,dy$

191. $\int_0^4 \sqrt{x}\,dx$

192. $\int_0^1 x^{3/7}\,dx$

193. $\int_1^3 \left(\dfrac{1}{t^2} - \dfrac{1}{t^4}\right) dt$

194. $\int_1^2 \dfrac{t^6 - t^2}{t^4}\,dt$

195. $\int_1^2 \dfrac{x^2+1}{\sqrt{x}}\,dx$

196. $\int_0^2 (x^3 - 1)^2\,dx$

197. $\int_0^1 u(\sqrt{u} + \sqrt[3]{u})\,du$

198. $\int_1^2 (x + 1/x)^2\,dx$

199. $\int_3^3 \sqrt{x^5 + 2}\,dx$

200. $\int_1^{-1} (x-1)(3x+2)\,dx$

201. $\int_1^4 (\sqrt{t} - 2/\sqrt{t})\,dt$

202. $\int_1^8 \left(\sqrt[3]{r} + \dfrac{1}{\sqrt[3]{r}}\right) dr$

203. $\int_{-1}^{0} (x+1)^3\,dx$

204. $\int_{-5}^{-2} \dfrac{x^4 - 1}{x^2 + 1}\,dx$

205. $\int_1^e \dfrac{x^2 + x + 1}{x}\,dx$

206. $\int_4^9 \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 dx$

207. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

208. $\int_1^8 \dfrac{x-1}{\sqrt[3]{x^2}}\,dx$

209. $\int_{\pi/4}^{\pi/3} \sin t\,dt$

210. $\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\,d\theta$

211. $\int_0^{\pi/2} (\cos\theta + \sin 2\theta)\,d\theta$

212. $\int_{2\pi/3}^{\pi} \sec x \tan x\,dx$

213. $\int_{\pi/3}^{\pi/2} \csc x \cot x\,dx$

214. $\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta$

215. $\int_{\pi/4}^{\pi/3} \sec^2\theta\,d\theta$

216. $\int_1^{\sqrt{3}} \dfrac{6}{1+x^2}\,dx$

217. $\int_0^{0.5} \dfrac{dx}{\sqrt{1-x^2}}$

218. $\displaystyle\int_4^8 (1/x)\,dx$

219. $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x\,dx$

220. $\displaystyle\int_8^9 2^t\,dt$

221. $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\,dx$

222. $\displaystyle\int_{-2}^{3} |x^2-1|\,dx$

223. $\displaystyle\int_{-1}^{2} |x-x^2|\,dx$

224. $\displaystyle\int_{-1}^{2} (x-2|x|)\,dx$

225. $\displaystyle\int_{0}^{2} (x^2-|x-1|)\,dx$

226. $\displaystyle\int_0^2 f(x)\,dx$ where $f(x)=\begin{cases} x^4, & \text{if } 0\le x<1, \\ x^5, & \text{if } 1\le x\le 2 \end{cases}.$

227. $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ where $f(x)=\begin{cases} x, & \text{if } -\pi\le x\le 0, \\ \sin x, & \text{if } 0<x\le \pi. \end{cases}$

228. True or false? $\displaystyle\int_{-1}^{1} \frac{3}{t^4}\,dt = \left.\frac{-1}{t^3}\right|_{-1}^{1} = -1+1=0.$

229. Explain what a Riemann sum is and write the definition of $\displaystyle\int_a^b f(x)dx$ as a limit of Riemann sums.

230. State the Fundamental Theorem of Calculus.

231. (1) Water flows into a container at a rate of three gallons per minute for two minutes, five gallons per minute for seven minutes and eleven gallons per minute for two minutes. How much water is in the container? (2) Water flows into a container at a rate of t^2 gallons per minute for $0\le t\le 5$. How much water is in the container?

232. Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

233. Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

234. Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this compute an area by summing up the areas of narrow rectangles and then show that applying the Fundamental Theorem of Calculus gives the same answer.

235. Sketch the graph of the curve $y=\sqrt{x+1}$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x=0$, $x=4$.

236. Make a sketch of the graph of the function $y=4-x^2$ and determine the area enclosed by the curve, the x-axis and the lines $x=0$, $x=2$.

237. Find the area under the curve $y=\sqrt{6x+4}$ and above the x-axis between $x=0$ and $x=2$. Draw a sketch of the curve.

238. Graph the curve $y = x^3$ and determine the area enclosed by the curve and the lines $y = 0$, $x = 2$ and $x = 4$.

239. Graph the function $f(x) = 9 - x^2$, $0 \le x \le 3$, and determine the area enclosed between the curve and the x-axis.

240. Graph the curve $y = 2\sqrt{1 - x^2}$, $x \in [0, 1]$, and find the area enclosed between the curve and the x-axis.

241. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ and between the lines $x = 0$ and $x = a$.

242. Graph the curve $y = 2\sqrt{9 - x^2}$ and determine the area enclosed between the curve and the x-axis.

243. Graph the area between the curve $y^2 = 4x$ and the line $x = 3$. Find the area of this region.

244. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

245. Find the area bounded by the curve $y = x(x - 3)(x - 5)$, the x-axis and the lines $x = 0$ and $x = 5$.

246. Find the area enclosed between the curve $y = \sin 2x$, $0 \le x \le \pi/4$ and the axes.

247. Find the area enclosed between the curve $y = \cos 2x$, $0 \le x \le \pi/4$ and the axes.

248. Find the area enclosed between the curve $y = 3\cos x$, $0 \le x \le \pi/2$ and the axes.

249. Find the area enclosed between the curve $y = \cos 3x$, $0 \le x \le \pi/6$ and the axes.

250. Find the area enclosed between the curve $y = \tan^2 x$, $0 \le x \le \pi/4$ and the axes.

251. Find the area enclosed between the curve $y = \csc^2 x$, $\pi/4 \le x \le \pi/2$ and the axes.

252. Find the area of the region bounded by $y = -1$, $y = 2$, $x = y^3$, and $x = 0$.

253. Find the area of the region bounded by the parabola $y = 4x^2$, $x \ge 0$, the y-axis, and the lines $y = 1$ and $y = 4$.

254. Find the area of the region bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

255. Graph $y^2 + 1 = x$, $x \le 2$ and find the area enclosed by the curve and the line $x = 2$.

256. Graph the curve $y = x/\pi + 2\sin^2 x$ and write a definite integral whose value is the area between the x-axis, the curve and the lines $x = 0$ and $x = \pi$. *Do not* evaluate the integral. *Do* specify the limits of integration.

257. Find the area bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$. Hint: Make a careful drawing to decide what area is intended.

258. Find the area bounded by the curve $y = \cos x$ and the x-axis between $x = 0$ and $x = 2\pi$.

259. Give an example which shows that $\int_a^b f(x)\,dx$ is not always the true area bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ even though $f(x)$ is continuous between a and b.

260. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

261. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

262. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

263. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

264. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

265. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

266. Use integration to find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

267. Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.

268. Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.

269. Find the area of the region between the parabolas $y^2 = x$ and $x^2 = 16y$.

270. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

271. Find a so that the curves $y = x^2$ and $y = a\cos x$ intersect at the points $(x, y) = (\frac{\pi}{4}, \frac{\pi^2}{16})$. Then find the area between these curves.

272. Write a definite integral whose value is the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: The part of a circle cut off by a line is a circular sector with a triangle removed.

273. Write a definite integral whose value is the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. *Do not* evaluate the integral. *Do* specify the limits of integration.

274. Write a definite integral whose value is the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$ *Do not* evaluate the integral. *Do* specify the limits of integration.

275. Write a definite integral whose value is the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: Divide the region into two parts.

276. Write a definite integral whose value is the area of the part of the first quadrant which is between the parabola $y^2 = x$ and the circle $x^2 + y^2 - 2x = 0$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: Draw a careful graph. Divide a semicircle in two.

277. Find the area bounded by the curves $y = x$ and $y = x^3$.

278. Graph $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/2$ and find the area enclosed by them and the x-axis.

279. Write a definite integral whose value is the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Evaluate this area. Hint: After a suitable change of variable, the definite integral becomes the definite integral whose value is the area of a circle.

280. Using integration find the area of the triangle with vertices $(-1, 1)$, $(0, 5)$ and $(3, 2)$.

281. Find the volume that results by rotating the triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$ around the x axis.

282. Find the volume that results by rotating the triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$ around the y axis.

283. Find the volume that results by rotating the triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$ around the line $x = -1$.

284. Find the volume that results by rotating the triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$ around the line $y = -1$.

285. Find the volume that results by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x-axis.

286. A triangle is formed by drawing lines from the two endpoints of a line segment of length b to a vertex V which is at a height h above the line of the line segment. Its area is then $A = \int_{y=0}^{h} dA$ where dA is the area of the strip cut out by two parallel lines separated by a distance of dz and at a height of z above the line containing the line segment. Find a formula for dA in terms of b, z, and dz and evaluate the definite integral.

287. A pyramid is formed by drawing lines from the four vertices of a rectangle of area A to an apex P which is at a height h above the plane of the rectangle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the rectangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

288. A tetrahedron is formed by drawing lines from the three vertices of a triangle of area A to an apex P which is at a height h above the plane of the triangle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the triangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

289. A cone is formed by drawing lines from the perimeter of a circle of area A to an apex V which is at a height h above the plane of the circle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the circle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

290. (a) A hemispherical bowl of radius a contains water to a depth h. Find the volume of the water in the bowl. (b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level rising when the water is 4 ft deep?

291. (Alternate wording for previous problem.) A hemispherical bowl is obtained by rotating the semicircle $x^2 + (y - a)^2 = a^2$, $y \leq a$ about the y-axis. It is filled with water to a depth of h, i.e. the water level is the line $y = h$. (a) Find the volume of the water in the bowl as a function of h. (b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level rising when the water is 4 ft deep? (Hint: Use the method of related rates and the Fundamental Theorem.)

292. A vase is constructed by rotating the curve $x^3 - y^3 = 1$ for $0 \leq y \leq 8$ around the y axis. It is filled with water to a height $y = h$ where $h < 8$. (a) Find the volume of the water in terms of h. (Express your answer as a definite integral. Do not try to evaluate the integral.) (b) If the vase is filling with water at the rate of 2 cubic units per second, how fast is the height of the water increasing when this height is 2 units?

Chapter IX

Notes for TA's

§40.1. My main objective is to get the students to use the notation correctly. When grading, point out incorrect syntax, both mathematical and grammatical.

§40.2. Read your email daily. After reading a message from me reply to me (OK suffices) so I know that you got it. I would be very happy if we all discussed what is going on in the course via email. Back in the stone age we used to have weekly meetings, but we can accomplish the same thing via email.

§40.3. Learning occurs only when students are active, not passive. For this reason I encourage students to interrupt the lecture with questions. I would be delighted if a TA would interrupt once in a while, even if that TA does not need to ask a question. It might encourage the students to speak up.

§40.4. The students should do the exercises from the notes as soon as I cover the corresponding material in lecture. If there are not enough problems, you can assign more (either from Chapter VIII or from the text) or you can ask me to assign more.

§40.5. Grade Exercises 1.4 and 1.5 at the end of Section 1 of these notes. The Math Department has an "Early Warning System" which aims to detect students who may get into trouble as early as possible so as to get them into the Tutorial Program. Weak algebra skills increase the likelihood that a student will have difficulty. Exercise 5.8 is also good preparation for word problems later in the course. Also use this exercise to look for at risk students.

§40.6. We don't allow calculators on exams. Encourage the students to leave arithmetic undone; it makes grading easier. Tell them that an answer like $x = 1 + 3$ is acceptable (if correct) but an answer like $x = 1 + 3 = 5$ will be penalized. On the other hand, sometimes a little algebraic simplification at the beginning of a problem makes the rest of the problem less error prone. For example, I would replace $3x + 1 + 5x$ by $8x + 1$ early in a calculation to avoid copying errors.

§40.7. I like to assign complicated computations as homework, especially when there is another easier way to do the problem which I also assign. However, I think the algebra we ask students to do on tests should not be so complicated.

§40.8. Inverse functions are one of the trickiest things in the course. I like to hammer on the fact that $y = f(x)$ and $x = f^{-1}(y)$ have the same graph, i.e.

$$y = f(x) \iff x = f^{-1}(y).$$

Rather than interchanging x and y to to draw the graph of the inverse function, I prefer to label the horizontal axis as y and the vertical axis as x. (The effect is still reflection in the diagonal.) Do Exercise 2.10 to reinforce this. I like to do the inverse trig functions right after the trig functions to accustom the students to inverse functions.

§40.9. The change of variables formula in §7.9 means that you don't need to do long division of polynomials when finding the limit of a rational function. Instead of dividing by $x - a$ you replace x by $a + h$ and then factor out h.

§40.10. There are several good reasons for the "Early Transcendentals" approach:

1. By introducing the most important functions early on we assure that the students get experience with them and are not short changed because we run out of time at the end of the semester.

2. We make the course more challenging in the first few weeks so that students are not lulled into overconfidence. (In the standard approach students usually find the first exam easy.)

3. We can do more interesting problems earlier.