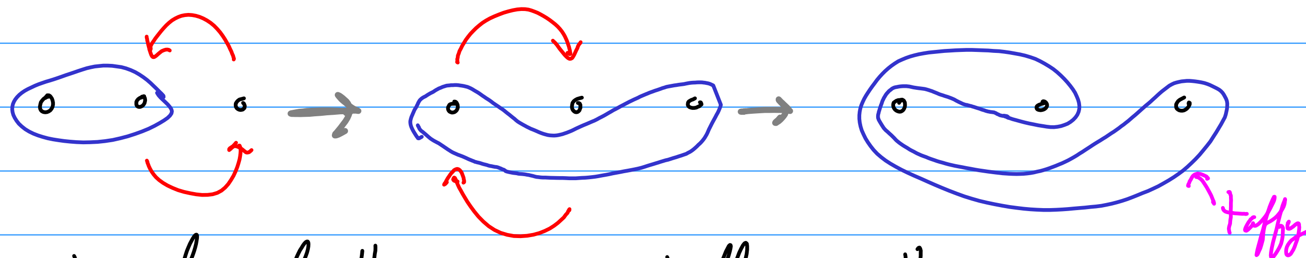
IMA Tutorial: Transport & Mixing

2010/04/01

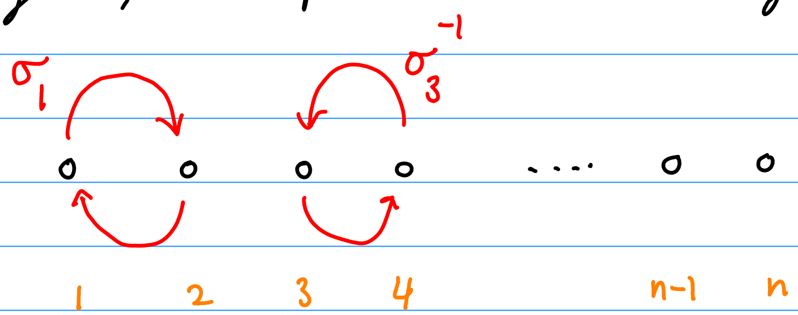
Lecture 6: Topological Mixing

Stirring by moving rods [movie] $\left\{\begin{array}{l}\text{fluids (viscous)} \\ \text{elastic bodies (bread, taffy)}\end{array}\right.$



Repeat: line length grows exponentially in this case.

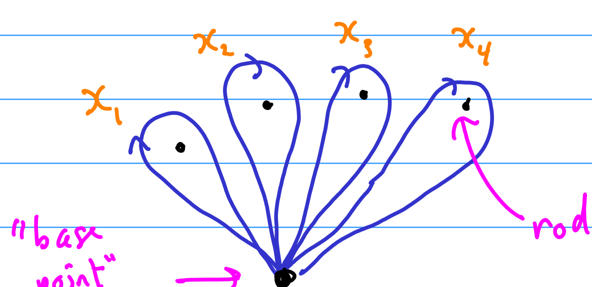
In general, can represent rod motions using generators of braid group:



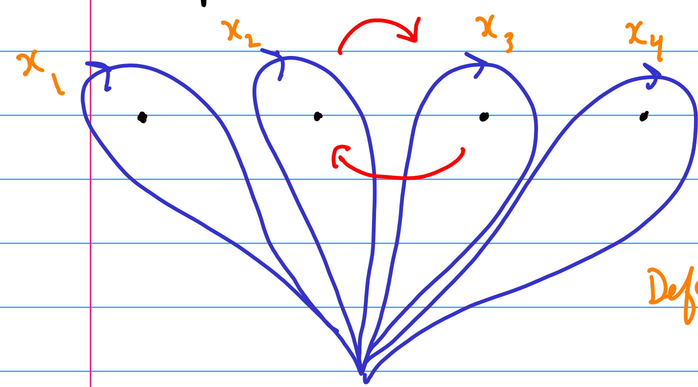
$n-1$ generators, $\{\sigma_1, \dots, \sigma_{n-1}\}$ + inverses.

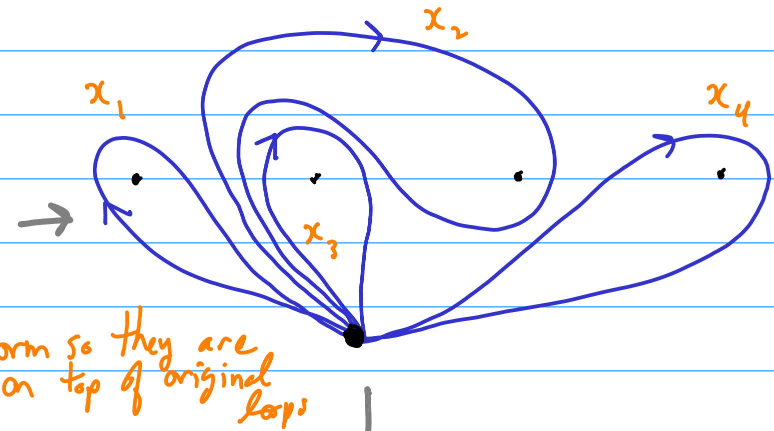
The previous example can be written $\sigma_2^{-1}\sigma_1$. (read left-to-right — conventions differ)

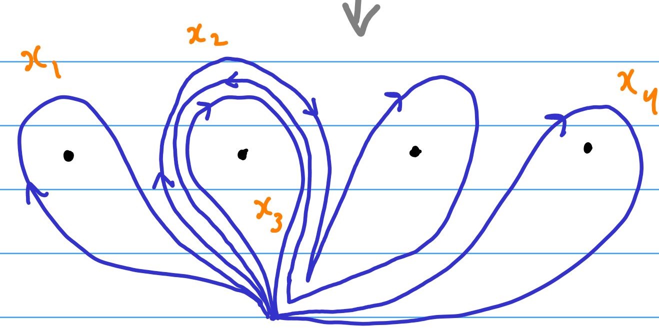
So at what rate will the taffy grow? One way is to look at action of braid group on loops, which are generators of the fundamental group, π_1.



Swap two rods:







Hence, σ_2 induces:

$$\left.\begin{aligned} x_1 &\mapsto x_1 \\ x_2 &\mapsto x_2 x_3 x_2^{-1} \\ x_3 &\mapsto x_2 \\ x_4 &\mapsto x_4 \end{aligned}\right\} \leftarrow \text{This is an automorphism of the free group } \pi_1$$

In general, σ_i induces

$$\boxed{\begin{aligned} x_i &\mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\mapsto x_i \\ x_j &\mapsto x_j, \quad j \neq i, i+1 \end{aligned}}$$

$\{x_1, \dots, x_n\}$ are the generators for the free group π_1(disc with n holes)

Alternate set of generators: $y_k = x_1 \cdots x_k$

For the above, easy to see $y_1 \mapsto y_1$

$$\begin{aligned} y_2 = x_1 x_2 &\mapsto x_1 (x_2 x_3 x_2^{-1}) \\ &= (x_1 x_2 x_3)(x_1 x_2)^{-1} x_1 \\ &= y_3 y_2^{-1} y_1 \end{aligned}$$

$$y_3 = x_1 x_2 x_3 \mapsto x_1 (x_2 x_3 \cancel{x_2^{-1}}) \cancel{x_2} = y_3$$

$$y_4 = x_1 x_2 x_3 x_4 \mapsto y_4$$

Hence, σ_i acts as $\boxed{\begin{array}{ll} y_i \mapsto y_{i+1} y_i^{-1} y_{i-1} & \\ y_j \mapsto y_j & , j \neq i \end{array}}$ slightly simpler

Now, the length of lines (similar to topological entropy) hooked on the rods will grow at the same rate as symbols:

Example: $\sigma_2^{-1} y_2 = y_1 y_2^{-1} y_3$,

$$\sigma_1 (\sigma_2^{-1} y_2) = \sigma_1(y_1)\,\sigma_1(y_2^{-1})\,\sigma_1(y_3)$$
$$= y_2 y_1^{-1} y_1^{-1} y_3$$

One symbol (y_2) went to 4 symbols ($y_2 y_1^{-1} y_1^{-1} y_3$) after $\sigma_2^{-1}\sigma_1$.

But we need the asymptotic growth (independent of choice of generators)
How?
"Abelianize" $\rightarrow$ treat like linear algebra.

$$y_i \mapsto y_{i+1} - y_i + y_{i-1}$$

$$\sigma_1 \rightarrow \tilde{K}_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \longrightarrow \tilde{K}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

From this we can get the growth in the # of symbols, but with lots of cancellations. In fact,

$$\sigma_2^{-1}\sigma_1 \rightarrow \tilde{K}_1 \tilde{K}_2^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ has eigenvalues on the unit circle $\rightarrow$ no growth!

But this was only a lower bound. For an upper bound, put absolute values everywhere!

$$\sigma_1 \to K_1 = \begin{pmatrix} +1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & +1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

But also $\sigma_1^{-1} \to K_1$, $\sigma_2^{-1} \to K_2$! (not a representation.)

$$\sigma_2^{-1}\sigma_1 \to K_1 K_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has eigenvalues $\phi_1^2, \phi_1^{-2}, 1$, where

$\phi_1 = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio

This is an upper bound. It happens to be sharp, which can easily be shown by other means (Burau representation: action on double cover)

OK, so what? Let's define an "efficiency": the topological entropy per generator (TEPG).

$$\text{TEPG} = \frac{\log(\text{growth induced by periodic motion of } n \text{ rods})}{\min \# \text{ of generators in sequence}}$$

Example above: $\text{TEPG}(\sigma_2^{-1}\sigma_1) = \dfrac{\log(\phi_1^2)}{2} = \log \phi_1$

Let's prove that this is optimal

Consider the set $\{K_i\}_{i=1,\dots,n}$ of the abelianized-absolute value action of σ_i.

Any sequence of σ's corresponds to a product of K's.

The growth of loops in π_1 is given by

$$\rho(M_1 \cdots M_k) \quad , \quad M_j \in \{K_i\}$$

spectral radius (largest eigenvalue in modulus)

If we normalize by how many generators, get $\rho^{1/k}(M_1 \cdots M_k)$.

Define: $$\rho_k(\{K_i\}) = \sup\left\{\rho(M_1 \cdots M_k) : M_j \in \{K_i\}\right\}$$

ρ_k gives the growth of the "best" product.

Define: $$\rho(\{K_i\}) = \limsup_{k\to\infty} \rho_k(\{K_i\})$$

JOINT SPECTRAL RADIUS

(Rota & Strang, 1962)

Hard to compute!

Easier: $$\hat{\rho}_k(\{K_i\}) = \sup\left\{\|M_1 \cdots M_k\|_1 : M_j \in \{K_i\}\right\}$$

1-norm of matrices

sup over |column-sums|

For any matrix,

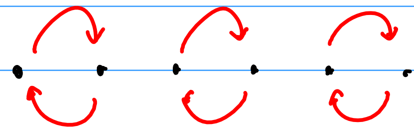
$M K_i$ changes the column sums by summing: $cs_{i\pm1} \mapsto cs_i + cs_{i\pm1}$

Hence, at best get a <u>Fibonacci sequence</u>: $\hat{\rho}_k(\{K_i\}) = F_{k+2}$.

Since $\lim_{k\to\infty} F_{k+2}^{1/k} = \phi_1$, we have an upper bound.

This upper bound can be realized.

There is a related problem where we count simultaneous motions as one:



1 operation

The entropy per operation in this case converges to the silver ratio!

$(1+\sqrt{2})$