IMA Tutorial: Transport & Mixing

2010/03/11

Lecture 5: Stochastic Models (part 2)

More refs:
Falkovich et al. 2001
Zeldovich et al. 1984

(moment of inertia tensor)

$$\dot{M} = M\cdot A + A^T\cdot M + 2\kappa I, \qquad M = RDR^T$$

R orthogonal, D diagonal

Eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d$.

$$\boxed{\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}}$$

$\tilde{A}_{ii} = R^T A R$, evolves independent of ρ_i for $\rho_1 \gg \rho_2 \gg \cdots \gg \rho_d$.

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\left[1 + 2\kappa e^{-2\rho_{0i}}\int_0^t \exp\left(-2\mathcal{A}_i(t')\right)dt'\right]$$

where

$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')\,dt'$$

(see Falkovich et al. for description as SDE.)

For $\underline{\kappa = 0}$, we argued that if $\tilde{A}_{ii}$ is a random var., then ρ_i are distributed according to large deviation form (for large t).

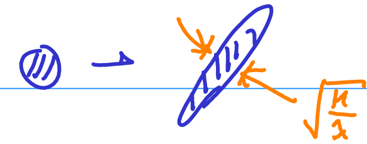
$$P(\rho_1,\rho_2,t) \sim \exp\left(-t\,S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right)\underbrace{\theta(\rho_1)}_{\substack{\text{ordering}\\ \rho_1 \geq \rho_2}}\underbrace{\delta(\rho_1+\rho_2)}_{\text{incompressibility}}$$

in 2D ($d=2$). (return 3D later)

$\theta(x) =$ step function

$$\lambda_1 = \lim_{t\to\infty} \rho_1/t = \text{Lyapunov exp.} > 0 \quad \text{(for chaotic flows)}$$

What happens with diffusion? Recall "filament":



The contracting direction "stabilizes" near the Batchelor width $\sqrt{\frac{\kappa}{\lambda_1}}$.

or "freezes"

Shraiman & Siggia 1994

Chertkov et al. 1997

Balkovsky & Fouxon 1999

$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{P_{stab}(\rho_2)}_{\text{stationary distribution.}}$$

If we assume, say, an initial Gaussian "patch" of passive scalar, then the concentration at a point scales as

$$\theta(\underline{x}, t) \sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2}$$

indep. of $\underline{x}$

$$= \exp\left(-\sum_i \rho_i\right)$$

Expected value:

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha \sum \rho_i} \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) P_{stab}(\rho_2)\, d\rho_1\, d\rho_2$$

Non-exponential function of t (neglect)

$$\sim \int e^{-\alpha \rho_1} \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) d\rho_1$$

← Do the ρ_2 integral

Use $h_i = \rho_i / t$ as variable:

expected value, not integral

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha h_1 t}\, e^{-t\, S(h_1 - \lambda_1)}\, dh_1$$

$h_1 \to h$

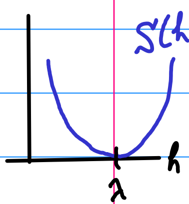
$\lambda_1 \to \lambda$

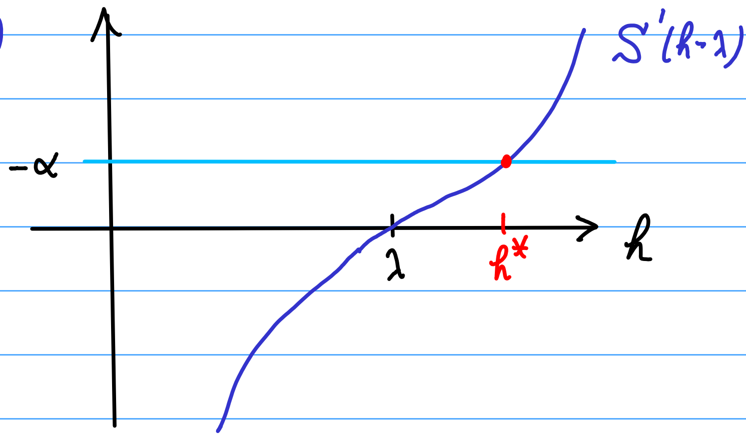
$$\boxed{\langle \theta^\alpha \rangle(t) \sim \int e^{-t(\alpha h + S(h - \lambda))}\, dh}$$

Let $H(h) = \alpha h + S(h-\lambda)$.

For large time, the integral is dominated by saddle point h^*:

$$H'(h^*) = 0 = \alpha + S'(h^* - \lambda)$$





Because of convexity of S, h^* is unique.

We then have $H(h) = H(h^*) + \frac{1}{2} H''(h^*)(h-h^*)^2 + \ldots$

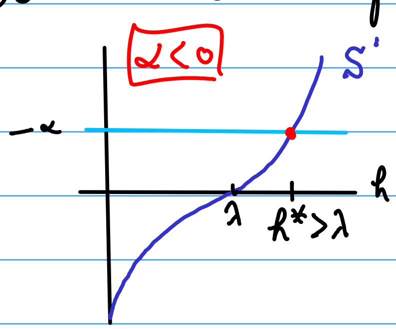
which we use to evaluate the integral. Find:

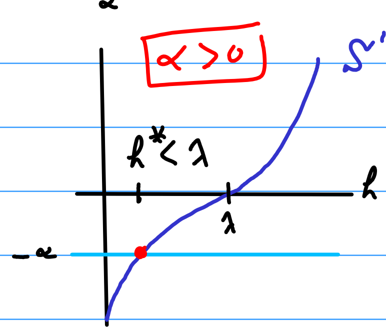
$$\langle \theta^\alpha \rangle (t) \sim e^{-\gamma_\alpha t}, \text{ where } \gamma_\alpha = H(h^*)$$

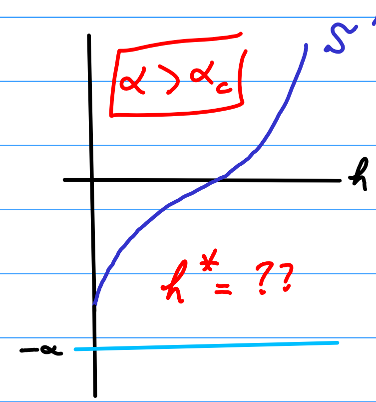
Note that we do not have $\langle \theta^\alpha \rangle \sim e^{-\alpha \gamma t}$, which would be the case if θ decayed the same pointwise everywhere.

$$\text{Kurtosis} \sim \frac{\langle \theta^\alpha \rangle}{\langle \theta \rangle^\alpha} \sim e^{-\gamma_\alpha t}$$

So how do we expect γ_α to behave?







We have $\gamma_0 = 0$, since $S'(h-\lambda) = 0$ at $h=\lambda$, and $S(0)=0$.

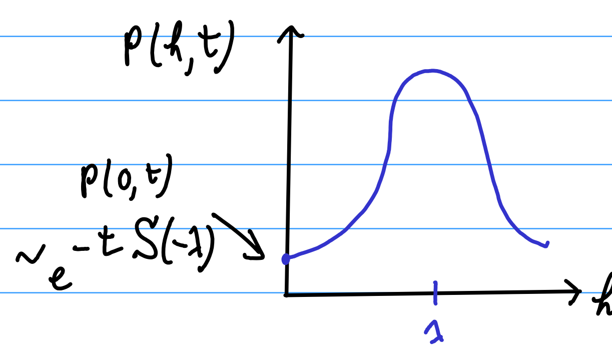
↳ $\langle\theta^0\rangle = 0$ ok!

Hence, γ_α changes sign at $\alpha = 0$.

What happens for $\alpha > \alpha_c$? No saddle point, since would require $h^* < 0$ (not allowed). Hence, take $h^* = 0$ (slowest decay)

The moments $\langle\theta^\alpha\rangle$ decay slower than expected, all the more so for larger α: INTERMITTENCY

Why the leveling-off? For large α, $\langle\theta^\alpha\rangle$ is dominated by realizations with large θ, that is, having experienced little stretching. For $\alpha > \alpha_c$, these are all that matter, so γ_α is the rate of decay of realizations with no stretching,



All this was for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.