IMA Tutorial: Transport & Mixing

2010/03/03

Lecture 4: Stochastic Models (part 1)

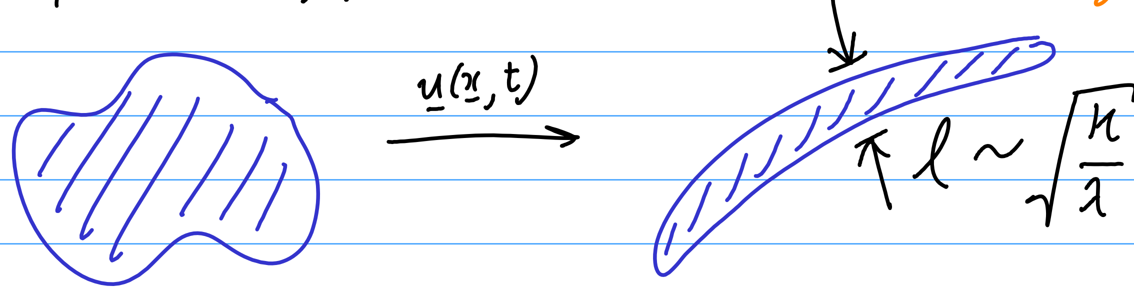
Antonsen et al. '96
Balkovsky & Fouxon '99

(AD) $\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa \nabla^2 \theta.$

Ω = domain = $\mathbb{R}^d$

Linear flow: $\underline{u}(\underline{x},t) = \underline{x}\cdot A(t)$

Batchelor regime



$\langle f \rangle = \int_\Omega f \, dV$, $\qquad \langle \underline{x}\theta\rangle = 0$ (center of mass)

(moment of inertia tensor)

$(M_{ij}) = m_{ij} = \dfrac{\langle x_i x_j \theta\rangle}{\langle\theta\rangle}$, $\qquad \dot{M} = M\cdot A + A^T\cdot M + 2\kappa I$

Let $M = RDR^T$, $\quad R$ orthogonal, D diagonal

$$\dot{M} = \dot{R}DR^T + RD\dot{R}^T + R\dot{D}R^T = RDR^TA + A^TRDR^T + 2\kappa I$$

$$R^T\dot{R}D + D\dot{R}^TR + \dot{D} = \underbrace{DR^TAR}_{\tilde{A}} + \underbrace{R^TA^TRD}_{\tilde{A}^T} + 2\kappa I$$

Now: $\dfrac{d}{dt}(R^TR) = \dot{R}^TR + R^T\dot{R} = \dfrac{d}{dt}(I) = 0$, so $(R^T\dot{R})^T = \dot{R}^TR = -R^T\dot{R}$

$\Rightarrow R^T\dot{R}$ is antisymmetric

$[R^T\dot{R}D]_{ii} = (R^T\dot{R})_{ik}D_{ki} = (R^T\dot{R})_{ii}D_{ii} = 0$

(no sum)

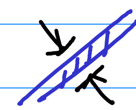
$$\dot D_{ii} = D_{il}\tilde A_{li} + \tilde A_{li} D_{li} + 2\kappa$$

$$\dot D_{ii} = 2\tilde A_{ii} D_{ii} + 2\kappa$$

Write $D_{ii} = e^{2\rho_i}$, with $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_d$.

$\dot D_{ii} = 2e^{2\rho_i}\dot\rho_i$, $\quad \boxed{\dot\rho_i = \tilde A_{ii} + \kappa e^{-2\rho_i}}$

Great equation: • $\tilde A = R^T A R$ → rotated velocity gradient matrix.

• $e^{-2\rho_i}$ → negligible unless $\rho_i < 0$ 

compression

Moral: the directions of contraction or compression play an important role.

Now we need an equation for R: off-diagonal terms.

$$[R^T\dot R D]_{ij} = (R^T\dot R)_{il} D_{lj} = (R^T\dot R)_{ij} D_{jj}, \quad i \neq j$$

(no sum over i,j)

$$[D\dot R^T R]_{ij} = D_{il}(\dot R^T R)_{lj} = D_{ii}(\dot R^T R)_{ij} = -(R^T\dot R)_{ij} D_{ii}$$

$$(D_{jj} - D_{ii})(R^T\dot R)_{ij} = D_{ii}\tilde A_{ij} + \tilde A_{ji} D_{jj}$$

$$(R^T\dot R)_{ij} = \Omega_{ij} \iff \boxed{\dot R = R\Omega}$$

$$\boxed{\Omega_{ij} = \frac{e^{2\rho_i}\tilde A_{ij} + e^{2\rho_j}\tilde A_{ji}}{e^{2\rho_j} - e^{2\rho_i}}}$$

Not completely obvious what this means...

$(= 0$ for $i = j)$

Almost always true for long time, esp. in 2D, 3D with $\rho_1 + \rho_2 (+\rho_3) = 0$.
Usually a symmetry can break this, or fails locally.

Assume we have separation between the eigenvalues: $e^{2\rho_i} \gg e^{2\rho_j}, \quad i<j$

$$\Omega_{ij} \simeq \frac{e^{2\rho_i}\tilde{A}_{ij} + \cancel{e^{2\rho_j}\tilde{A}_{ji}}}{\cancel{e^{2\rho_j}} - e^{2\rho_i}} = -\tilde{A}_{ij}, \quad i<j$$

$$\boxed{\Omega_{ij} \simeq \begin{cases} -\tilde{A}_{ij}, & i<j \\ \tilde{A}_{ji}, & i>j \end{cases} \quad \text{(large } t\text{)}}$$

Independent of eigenvalues!

Can solve: $\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}$ since $\tilde{A}$ indep. of ρ_i

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\Big[1 + 2\underset{\uparrow \ \text{diffusion}}{\kappa} e^{-2\rho_{0i}} \int_0^t \exp\big(-2\mathcal{A}_i(t')\big)\,dt'\Big]$$

where
$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')\,dt'$$

When diffusion negligible: $\rho_i(t) = \rho_{i0} + \int_0^t \tilde{A}_{ii}(t')\,dt'$

In fact, solving the equations for ρ_i, R, $\kappa = 0$, is not a bad way of computing Lyapunov exponents:

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t}\rho_i(t)} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$$

Convergence: famous Oseledec Multiplicative ergodic theorem

(Some numerical issues regarding orthogonality of R.)

Now comes the stochastic part: could have formulated things in terms of an SDE. But we take a shortcut:

$$\rho_i(t) = \rho_{0_i} + \underbrace{\sum_t \tilde{A}_{ii}}_{\text{sum of uncorrelated random numbers (more later)}}$$

What is PDF of $\rho_i(t)$?

<u>Recall</u>: if x_i are i.i.d. and $X = \sum_{i=1}^{N} x_i$

$\overline{x_i} = \xi$

$\overline{x_i^2} - \overline{x_i}^2 = \sigma^2$

What is PDF of X? CENTRAL LIMIT THEOREM

$$P(X,N) \sim \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left(-\frac{(X - \underbrace{N\xi}_{\text{mean of } X})^2}{2N\sigma^2}\right)$$

Valid for: (i) $N \gg 1$; (ii) $X - N\xi < \sqrt{N}\sigma$

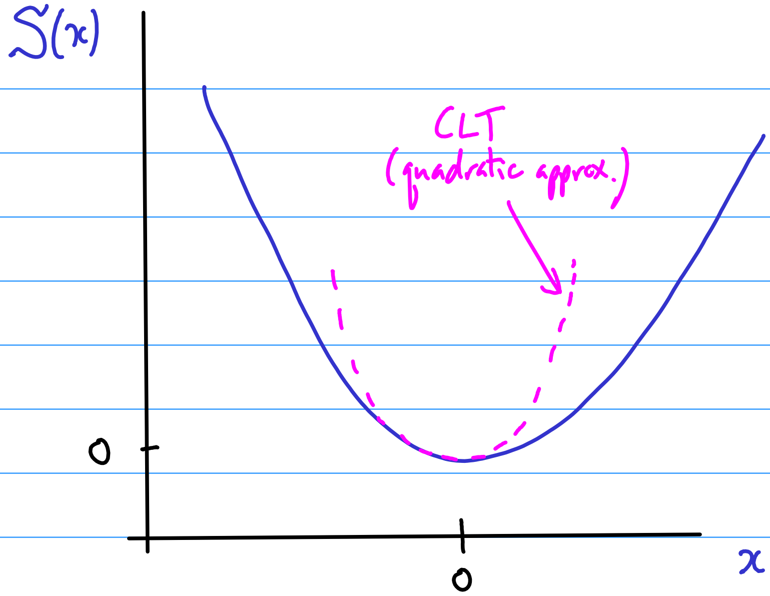
This second restriction is less commonly stated: it tells us that the CLT is not valid in the tails. The CLT tends to vastly underestimate the probability of rare events, or black swans as is trendy to call them these days. These tails matter for mixing.

More generally,

$$\boxed{P(X,N) \sim \exp\left(-N S\left(\frac{X - N\xi}{N}\right)\right)}$$

Large deviation form

$S(x)$ is a convex function with $S(0) = S'(0) = 0$.



$$S(x) = S(0) + S'(0)\,x + \frac{1}{2} S''(0)\,x^2 + \dots$$

$$S\left(\frac{X-N\xi}{N}\right) = \frac{1}{2} S''(0)\,\frac{(X-N\xi)^2}{N^2} + \dots$$

$$\exp\left(-N S\left(\frac{X}{N}-\xi\right)\right) \sim \exp\left(-S''(0)\,\frac{(X-N\xi)^2}{2N}\right)$$

Compare to CLT: $\boxed{S''(0) = \frac{1}{\sigma^2}}$

Can also express in terms of mean: $x = \frac{X}{N}$

$$P(x, N) \sim \exp\left(-N S(x-\xi)\right)$$

<u>Example:</u> Binomial distribution for x_i $(-1$ or 1, mean $0)$

$$p(x_i) = \frac{1}{2}\delta(x_i+1) + \frac{1}{2}\delta(x_i+1)$$

$$e^{-s(k)} = \int p(\xi)\, e^{-ik\xi}\, d\xi$$ characteristic function

$$= \frac{1}{2}\left(e^{ik} + e^{-ik}\right) = \cos k$$

For the mean $x = \frac{1}{N}\sum x_i$:

$$P(x,N) = \int p(x_1)\cdots p(x_N)\,\delta\!\left(\frac{x_1+\cdots+x_N}{N} - x\right) dx_1 \cdots dx_N$$

$$e^{-S(k)} = \int P(x,N)\, e^{-ikx}\, dx$$

$$= \int p(x_1)\cdots p(x_N)\, e^{-ik(x_1+\cdots+x_N)/N}\, dx_1 \cdots dx_N$$

$$= \prod_{i=1}^{N} \int p(x_i)\, e^{-ikx_i/N}\, dx_i = \left(\int p(\xi)\, e^{-ik\xi/N}\, d\xi\right)^N$$

$$= \left(e^{-s(k/N)}\right)^N = \cos^N(k/N)$$

Inverse Fourier

$$P(x,N) = \frac{1}{2\pi}\int e^{-S(k)}\, e^{ikx}\, dk = \frac{1}{2\pi}\int \cos^N\!\left(\frac{k}{N}\right) e^{ikx}\, dk$$

$$= \frac{N}{2\pi}\int \cos^N K\; e^{iNKx}\, dK \,, \quad K = k/N.$$

$$= \frac{N}{2\pi}\int e^{N(\log\cos K + iKx)}\, dK$$

For N large, look for saddle (stationary) point:

$$\frac{d}{dK}\underbrace{(\log\cos K + iKx)}_{H(K,x)} = -\tan K + ix = 0 \text{ when } K = K_{sp}.$$

$$\tan K_{sp} = -ix$$

$$H(k,x) = H(k_{sp},x) + \overset{0}{\widetilde{H'(k_{sp},x)}}(k-k_{sp}) + \frac{1}{2}H''(k_{sp},x)(k-k_{sp})^2 + \cdots$$

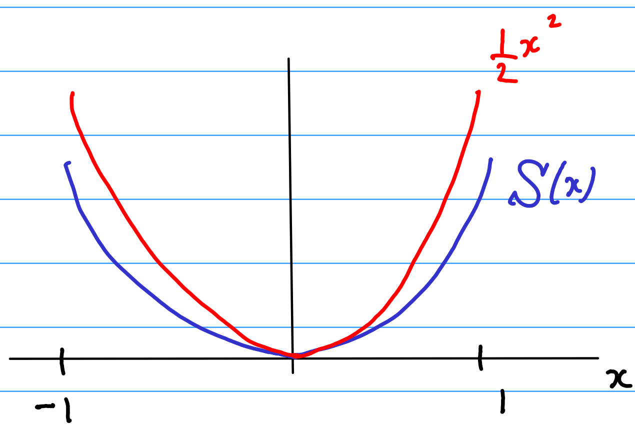
With this approximation the inverse transform is a Gaussian integral. Get finally (skip some steps... see Acosta lecture notes)

$$P(x,N) = \sqrt{\frac{N S''(0)}{2\pi}}\, e^{-N S(x)}, \quad \text{with}$$

$$\boxed{S(x) = -\frac{1}{2}(x+1)\log\left(\frac{1-x}{x+1}\right) + \log(1-x)} \qquad -1 \le x \le 1$$

Note $S(0) = 0$, $S'(x) = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right)$, so $S'(0) = 0$

$$S''(x) = \frac{1}{1-x^2}, \quad \text{so} \quad S''(0) = 1$$



$S(x)$ is called the
rate function
Cramer function
entropy function

Next lecture: what this has to do with mixing!