Lecture 2: Linear flows

(AD) $\quad \partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta.$

For this lecture, think of θ as a "patch"

Last time we examined $\underline{u} = (\lambda x, -\lambda y)$. Let's try something more general:

$$\underline{u} = \underline{U} + \underline{x}\cdot A, \qquad \nabla\cdot\underline{u} = \operatorname{trace} A = 0.$$

const.

Let $\langle f\rangle = \int_\Omega f\, dV \qquad (\Omega = \mathbb{R}^2 \text{ or } \mathbb{R}^3)$

Solve (AD) using moments: $\boxed{c_i = \dfrac{\langle x_i\theta\rangle}{\langle\theta\rangle}}$ $\qquad (\partial_t\langle\theta\rangle = 0)$

(AD) → $\partial_t\langle x_i\theta\rangle + \langle x_i\nabla\cdot((\underline{U}+\underline{x}\cdot A)\theta)\rangle = \kappa\langle x_i\nabla^2\theta\rangle$

$\partial_t\langle x_i\theta\rangle - \langle (U_j + x_\ell A_{\ell j})\theta\cdot\underbrace{\partial_j x_i}_{\delta_{ji}}\rangle = \kappa\ (0)$

$\langle\theta\rangle\partial_t c_i - U_i\langle\theta\rangle - A_{\ell i}\langle\theta\rangle c_\ell = 0$

$$\boxed{\partial_t \underline{c} = \underline{U} + \underline{c}\cdot A}$$

Motion of center of mass

Next moments: $\boxed{m_{ij} = \dfrac{\langle x_i x_j\theta\rangle}{\langle\theta\rangle} - c_i c_j}$

Again, multiply (AD) by $x_i x_j$ and $\langle\cdot\rangle$.

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = \langle x_i x_j \partial_k((U_k + x_l A_{lk})\theta)\rangle$$

$$= -\langle (U_k + x_l A_{lk})(\delta_{ik} x_j + x_i \delta_{jk})\theta\rangle$$

$$= -U_i c_j \langle\theta\rangle - U_j c_i \langle\theta\rangle - A_{li}\underbrace{\langle x_l x_j \theta\rangle}_{\langle\theta\rangle(m_{lj}+c_l c_j)} - A_{lj}\underbrace{\langle x_l x_i \theta\rangle}_{\langle\theta\rangle(m_{li}+c_l c_i)}$$

$$\partial_t(c_i c_j) = c_i \partial_t c_j + c_j \partial_t c_i$$
$$= c_i(U_j + A_{lj} c_l) + c_j(U_i + A_{li} c_l)$$

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = -\left(\partial_t(c_i c_j) + A_{li} m_{lj} + A_{lj} m_{li}\right)\langle\theta\rangle$$

That's the hard part! Next:

$$\langle x_i x_j \nabla^2\theta\rangle = \langle \theta\, \nabla^2(x_i x_j)\rangle = 2\langle\theta\rangle\delta_{ij}$$

So finally:

$$\partial_t m_{ij} = A_{li} m_{lj} + A_{lj} m_{il} + 2\kappa\,\delta_{ij}$$

Let $(M)_{ij} = m_{ij}$ (symmetric matrix)

$$\boxed{\partial_t M = M\cdot A + A^T\cdot M + 2\kappa I}$$

Moment of inertia equation. "spread" of patch

Time to solve these equations!

$$\underline{c}(t) = \underline{c}(0)\cdot e^{At} + \underline{U}\cdot\int_0^t e^{A(t-\tau)}\,d\tau$$

$$M(t) = e^{A^T t} \cdot M(0) \cdot e^{At} + 2\kappa \int_0^t \underbrace{e^{A^T(t-\tau)} \cdot e^{A(t-\tau)}} d\tau$$

Can't write as $e^{(A+A^T)(t-\tau)}$

unless $[A, A^T] = 0$ Normal matrix

Shear flow: $A = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$, $A^2 = 0$

$$e^{At} = I + At + 0$$

$$\int_0^t e^{A^T(t-\tau)} \cdot e^{A(t-\tau)} d\tau = \int_0^t \left(I + A^T(t-\tau)\right) \cdot \left(I + A(t-\tau)\right) d\tau$$

$$= \int_0^t \left(I - (A+A^T)(\tau - t) + A^T \cdot A \, (\tau - t)^2 \right) d\tau$$

$$= tI + (A+A^T) \left.\frac{(\tau-t)^2}{2}\right|_0^t + \frac{1}{3} A^T \cdot A \left.(\tau - t)^3\right|_0^t$$

$$= tI + \frac{1}{2}(A+A^T)t^2 + \frac{1}{3} A^T \cdot A \, t^3$$

$$A^T A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A + A^T = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

Assume an initial circular blob of size ρ: $M(0) \sim \rho^2 I$

$$e^{A^T t} \cdot M(0) \cdot e^{At} = \rho^2 \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha t & 1 \end{pmatrix} = \rho^2 \begin{pmatrix} 1 + \alpha^2 t^2 & \alpha t \\ \alpha t & 1 \end{pmatrix}$$

The different components of M have different asymptotic growth rates:

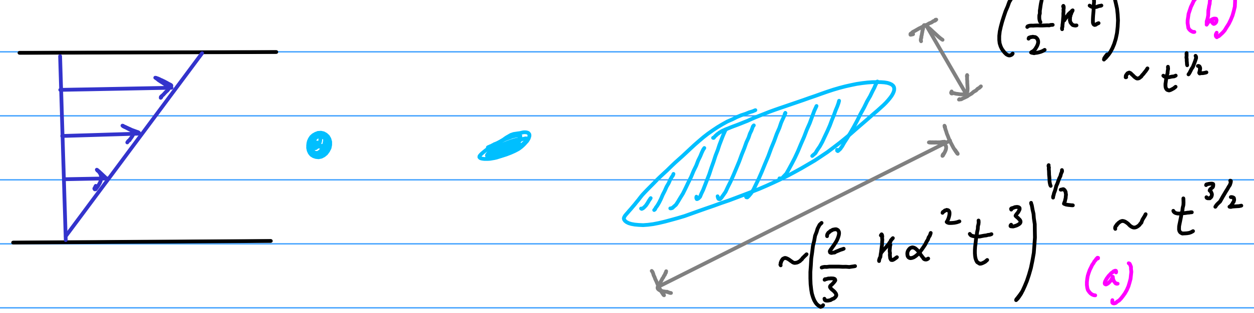
$$M_{11} = \rho^2 (1 + \alpha^2 t^2) + 2\kappa t + \underline{\frac{2}{3} \kappa \alpha^2 t^3}$$

$$M_{22} = \rho^2 + \underline{2\kappa t}$$

$$M_{12} = \rho^2 \alpha t + \underline{\kappa \alpha t^2}$$

So for large time, $M \sim \begin{pmatrix} \frac{2}{3}\kappa\alpha^2 t^3 & \kappa\alpha t^2 \\ \kappa\alpha t^2 & 2\kappa t \end{pmatrix}$ $\quad \det M \sim \frac{1}{3}\kappa^2\alpha^2 t^4$

$\underline{x}\cdot A = (x \;\; y)\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} = (\alpha y, 0)$



Sort of like a filament, except keeps "fattening".

Can use this to predict decay rate: $\text{area}^2 \sim \det M = \frac{1}{3}\kappa^2\alpha^2 t^4$

Concentration at a point $\sim \frac{\langle\theta\rangle}{\text{area}} \sim \boxed{\frac{3\rho^2\theta_0}{\alpha\kappa t^2}}$

faster!

Compare to purely diffusive case: $M = (\rho^2 + 2\kappa t) I$

concentration $\sim \frac{\rho^2\theta_0}{2\kappa t}$

This speedup is known as Taylor-Aris dispersion or shear dispersion

General 2x2 matrix: $\operatorname{tr} A = 0, \quad \det A = -\lambda^2$

Need to compute e^{At}.

Trick: $A^2 - (\operatorname{tr} A) A + (\det A) I = 0 \Rightarrow A^2 = \lambda^2 I$

Cayley-Hamilton theorem

λ can be imaginary

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I \sum_{n \text{ even}} \frac{(\lambda t)^n}{n!} + \sum_{n \text{ odd}} \frac{A \lambda^{n-1} t^n}{n!}$$

$$\boxed{e^{At} = \cosh(\lambda t)\, I + A \lambda^{-1} \sinh(\lambda t)}$$

$$A = \begin{pmatrix} \lambda & 0 \\ \alpha & -\lambda \end{pmatrix} : \quad \boxed{e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ \frac{\alpha}{\lambda} \sinh(\lambda t) & e^{-\lambda t} \end{pmatrix}}$$