IMA Tutorial: Transport & Mixing

2010/01/28

Lecture 1: Stirring & Mixing

Stirring: mechanical action (cause)
Mixing: homogenization of a scalar (effect)

$\theta(\underline{x},t)$ = concentration, $\underline{u}(\underline{x},t)$ given

Advection Diffusion eq. (AD)

$$\frac{\partial\theta}{\partial t} + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta, \qquad \nabla\cdot\underline{u} = 0 \quad \text{in } \Omega$$

Boundary conditions: $\left.\begin{array}{l}\hat{n}\cdot\nabla\theta = 0\\ \hat{n}\cdot\underline{u} = 0\end{array}\right\}$ on boundary $\partial\Omega$

Let $\langle\cdot\rangle = \int_\Omega \cdot \, dV$

Multiply AD by $m\,\theta^{m-1}$, integrate:

$$\langle m\theta^{m-1}\partial_t\theta\rangle = \partial_t\langle\theta^m\rangle$$

$$\langle m\theta^{m-1}\underline{u}\cdot\nabla\theta\rangle = \langle\underline{u}\cdot\nabla\theta^m\rangle \overset{\nabla\cdot\underline{u}=0}{=} \langle\nabla\cdot(\underline{u}\theta^m)\rangle$$

$$= \int_{\partial\Omega}\theta^m\,\underbrace{\underline{u}\cdot\hat{n}}_{0!}\,dS = 0$$

$$\langle m\theta^{m-1}\kappa\nabla^2\theta\rangle = \kappa m\langle\nabla\cdot(\theta^{m-1}\nabla\theta) - \nabla\theta^{m-1}\cdot\nabla\theta\rangle$$

$$= \kappa m\int_{\partial\Omega}\theta^{m-1}\,\underbrace{\nabla\theta\cdot\hat{n}}_{0!}\,dS - \kappa\,m(m-1)\langle\theta^{m-2}|\nabla\theta|^2\rangle$$

$$\boxed{\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle}$$

$m=0$ is trivial

$m=1$: $\partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

$m=2$: $\partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle \theta^2 \rangle$ non-increasing!

Let variance $Var = C_2 = \langle \theta^2 \rangle - \langle \theta \rangle^2$

constant

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

Scenario:

- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = $ constant.

in some sense

Hence the system is "driven" towards a homogeneous state where

$$\theta(\underline{x}, t) = \langle \theta \rangle = \text{constant}. \quad (C_2 = 0,\ \langle \theta^2 \rangle = \langle \theta \rangle^2)$$

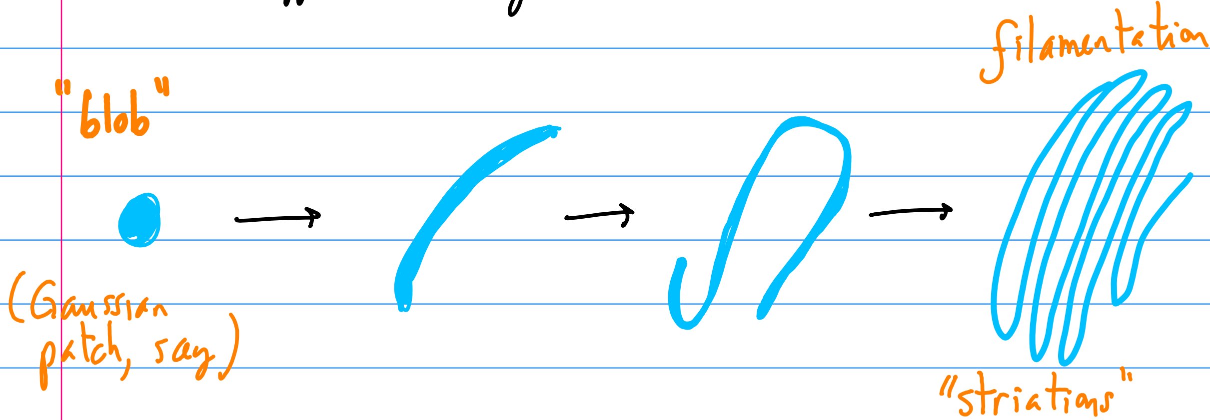
Assume $\langle \theta \rangle = 0$ WLOG

No fluctuations from the mean! When C_2 is small "enough", we say the system is mixed.

Big Q: Where is $\underline{u}(\underline{x}, t)$!? (stirring)

It doesn't appear in the variance equation!

But of course the variance equation is not closed: it depends on $\nabla\theta$.
What happens when you stir?



This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle \theta^2 \rangle = -2\kappa \underbrace{\langle |\nabla\theta|^2 \rangle}_{\text{this becomes larger as we stir}}$$

By how much are gradients increased? After all, if $|\nabla\theta|$ becomes too large, then $\langle\theta^2\rangle \to 0$, so there are no gradients anymore!

Answer: for "good" stirring, the system is driven to a state where

$$\kappa \langle |\nabla\theta|^2 \rangle \longrightarrow \boxed{\text{independent of } \kappa}$$

Hence, $\boxed{\nabla\theta \sim \kappa^{-1/2}}$

This is the chaotic/turbulent mixing scenario:

$\dfrac{\partial\langle\theta^2\rangle}{\partial t}$ becomes independent of κ after a "short" transient

(How short? Typically $\sim \log \kappa$)

This is the Platonic ideal of mixing

Furthermore, the smallest scales visible in the concentration field $\theta(\underline{x}, t)$ have size $\sim \sqrt{\kappa}$. (missing a dimensional factor → see later)

Note that $\partial_t \langle \theta^2 \rangle$ independent of κ is crucial: in most applications, κ is tiny!

Heat: $\kappa = 2.2160 \times 10^{-5}\ m^2/s$ at 300K

10 m room: diffusion time $\sim \dfrac{L^2}{\kappa} = \dfrac{(10m)^2}{(2\times10^{-5} m^2/s)} \sim 4.5 \times 10^6$ sec

~ 1300 hours

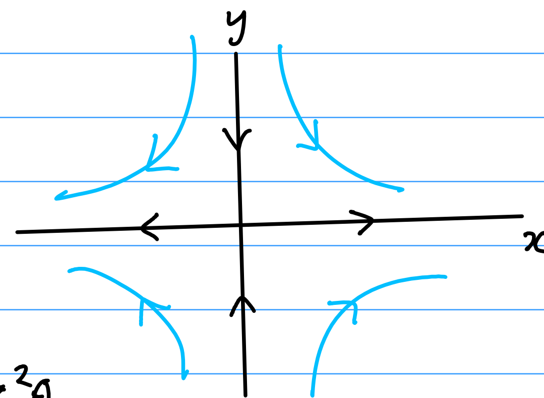
$\boxed{\sim 53 \text{ days !}}$

So we better stir!
Even thermal convection is often enough.

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t \theta + \lambda x \partial_x \theta - \lambda y \partial_y \theta = \kappa \nabla^2 \theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

$$-\lambda f - \lambda y f' = \kappa f''$$

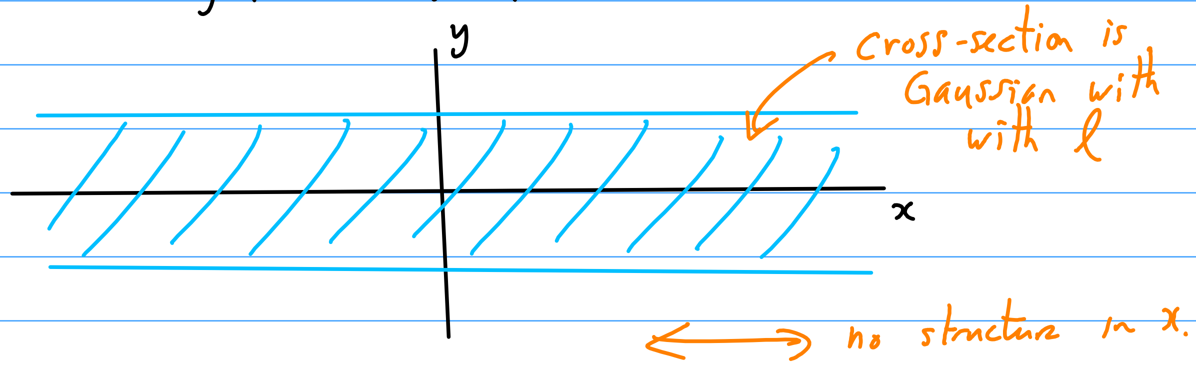
Boundary condition:

$f \to 0$ as $y \to \pm\infty$.

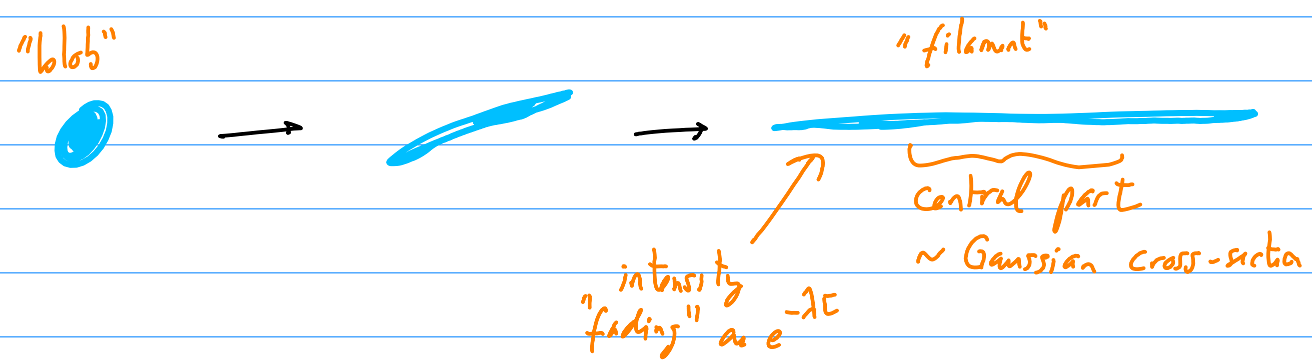
Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \frac{\kappa}{\lambda}$

Hence, $$\boxed{\theta(\underline{x}, t) \sim e^{-\lambda t} e^{-y^2/2\ell^2}}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

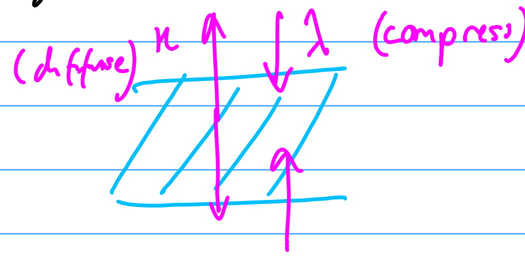
$$\boxed{\ell = \sqrt{\frac{\kappa}{\lambda}}}$$

Batchelor length

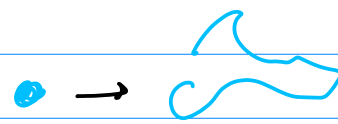
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

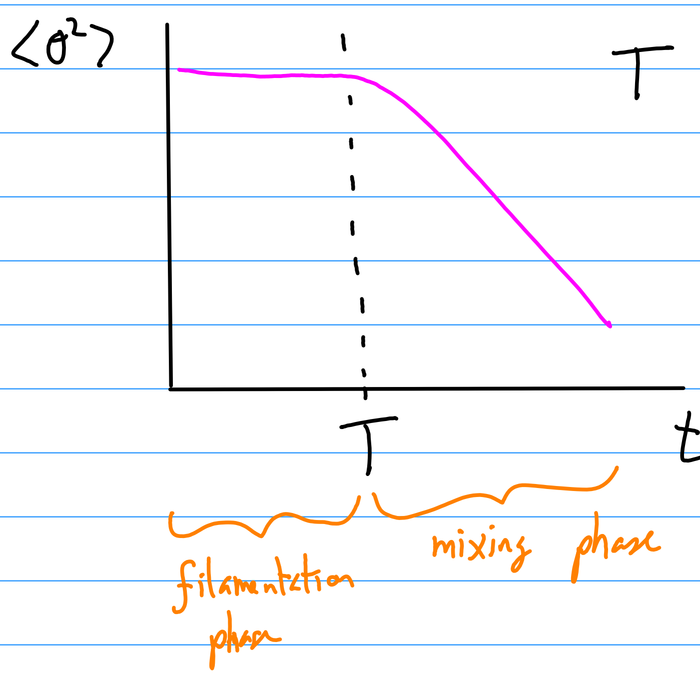
In practical applications, λ is often taken to be the local rate of strain.

ℓ is set by a balance between compression and diffusion



Summary: how mixing proceeds

- A blob is stirred 
- For a while, $\langle \theta^2 \rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order ℓ, diffusion takes over
- After that, $\langle \theta^2 \rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$$T \sim \lambda^{-1} \log \kappa$$