IMA Tutorial: Transport & Mixing

2010/01/28

Lecture 1: Stirring & Mixing

Stirring: mechanical action (cause)
Mixing: homogenization of a scalar (effect)

$\theta(\underline{x},t)$ = concentration, $\underline{u}(\underline{x},t)$ given

Advection Diffusion eq. (AD)

$$\frac{\partial\theta}{\partial t} + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta, \qquad \nabla\cdot\underline{u} = 0 \quad \text{in } \Omega$$

Boundary conditions: $\left.\begin{array}{l}\hat{n}\cdot\nabla\theta = 0\\ \hat{n}\cdot\underline{u} = 0\end{array}\right\}$ on boundary $\partial\Omega$

Let $\langle\cdot\rangle = \int_\Omega \cdot \, dV$

Multiply AD by $m\,\theta^{m-1}$, integrate:

$$\langle m\theta^{m-1}\partial_t\theta\rangle = \partial_t\langle\theta^m\rangle$$

$$\langle m\theta^{m-1}\underline{u}\cdot\nabla\theta\rangle = \langle\underline{u}\cdot\nabla\theta^m\rangle = \langle\nabla\cdot(\underline{u}\theta^m)\rangle \quad (\nabla\cdot\underline{u}=0)$$

$$= \int_{\partial\Omega}\theta^m\,\underbrace{\underline{u}\cdot\hat{n}}_{0!}\,dS = 0$$

$$\langle m\theta^{m-1}\kappa\nabla^2\theta\rangle = \kappa m\langle\nabla\cdot(\theta^{m-1}\nabla\theta) - \nabla\theta^{m-1}\cdot\nabla\theta\rangle$$

$$= \kappa m\int_{\partial\Omega}\theta^{m-1}\,\underbrace{\nabla\theta\cdot\hat{n}}_{0!}\,dS - \kappa\, m(m-1)\langle\theta^{m-2}|\nabla\theta|^2\rangle$$

$$\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle$$

$m=0$ is trivial

$m=1$: $\partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

$m=2$: $\partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle\theta^2\rangle$ non-increasing!

Let variance $Var = C_2 = \langle \theta^2 \rangle - \langle \theta \rangle^2$ (↑ constant: $\langle\theta\rangle$)

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

Scenario:

- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = $ constant. (↑ in some sense)

Hence the system is "driven" towards a homogeneous state where

$$\theta(\underline{x}, t) = \langle \theta \rangle = \text{constant}. \quad (C_2 = 0,\ \langle\theta^2\rangle = \langle\theta\rangle^2)$$

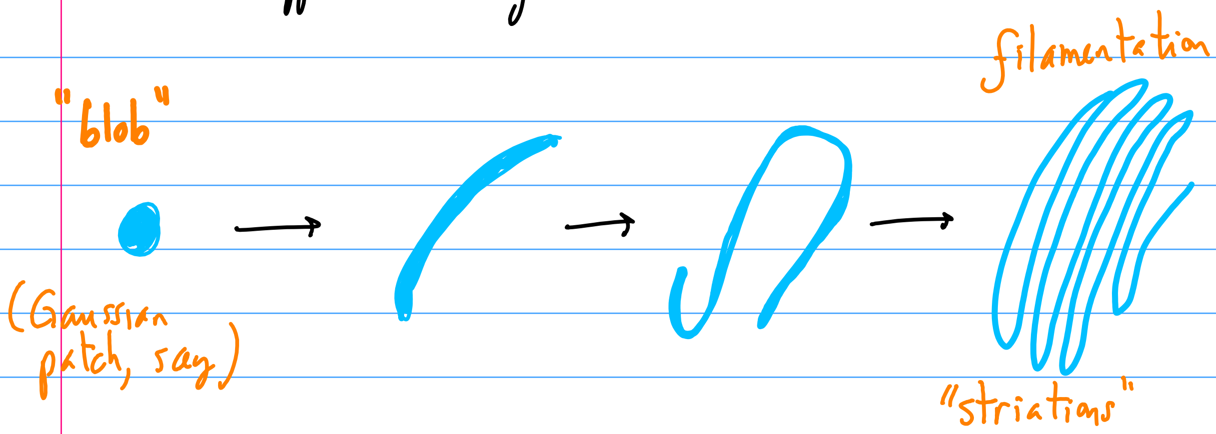
Assume $\langle\theta\rangle = 0$ WLOG

No fluctuations from the mean! When C_2 is small "enough", we say the system is mixed.

Big Q: Where is $\underline{u}(\underline{x}, t)$!? (stirring)

It doesn't appear in the variance equation!

But of course the variance equation is not closed: it depends on $\nabla\theta$. What happens when you stir?



This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle \theta^2 \rangle = -2\kappa \underbrace{\langle |\nabla\theta|^2 \rangle}_{\text{this becomes larger as we stir}}$$

By how much are gradients increased? After all, if $|\nabla\theta|$ becomes too large, then $\langle\theta^2\rangle \to 0$, so there are no gradients anymore!

Answer: for "good" stirring, the system is driven to a state where

$$\kappa \langle |\nabla\theta|^2 \rangle \longrightarrow \boxed{\text{independent of } \kappa}$$

Hence, $\boxed{\nabla\theta \sim \kappa^{-1/2}}$

This is the chaotic/turbulent mixing scenario:

$\dfrac{\partial \langle\theta^2\rangle}{\partial t}$ becomes independent of κ after a "short" transient

(How short? Typically $\sim \log \kappa$)

This is the Platonic ideal of mixing

Furthermore, the smallest scales visible in the concentration field $\theta(\underline{x}, t)$ have size $\sim \sqrt{\kappa}$. (missing a dimensional factor → see later)

Note that $\partial_t \langle \theta^2 \rangle$ independent of κ is crucial: in most applications, κ is tiny!

Heat: $\kappa = 2.2160 \times 10^{-5}\ m^2/s$ at 300K

10 m room: diffusion time $\sim \frac{L^2}{\kappa} = \frac{(10m)^2}{(2\times10^{-5} m^2/s)} \sim 4.5 \times 10^6$ sec

~ 1300 hours

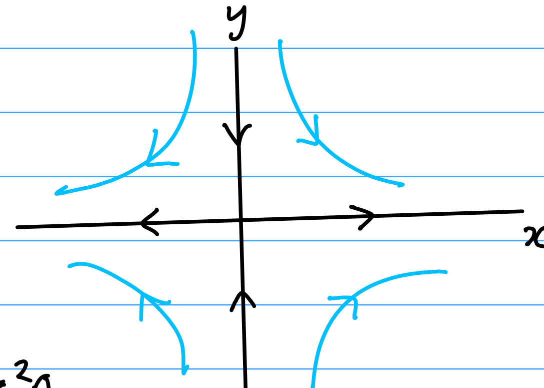
~ 53 days!

So we better stir! Even thermal convection is often enough.

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t \theta + \lambda x \partial_x \theta - \lambda y \partial_y \theta = \kappa \nabla^2 \theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

$$-\lambda f - \lambda y f' = \kappa f''$$

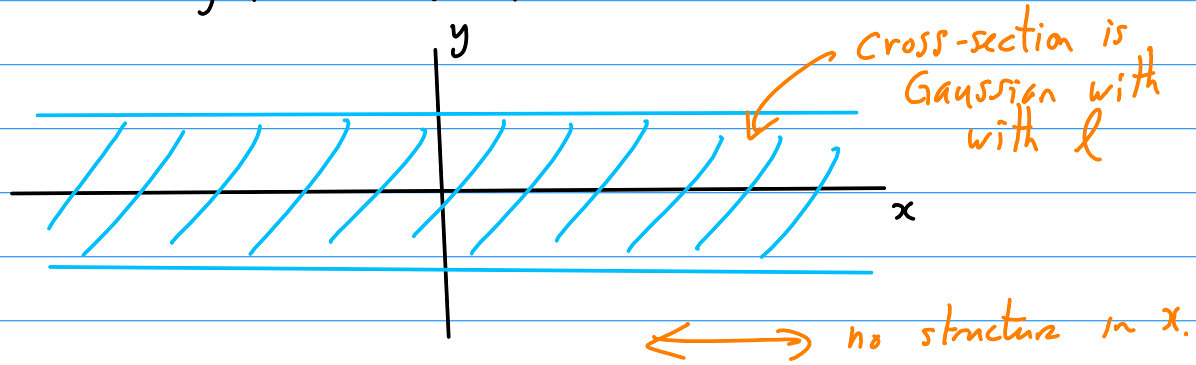
Boundary condition: $f \to 0$ as $y \to \pm\infty$.

Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \frac{\kappa}{\lambda}$

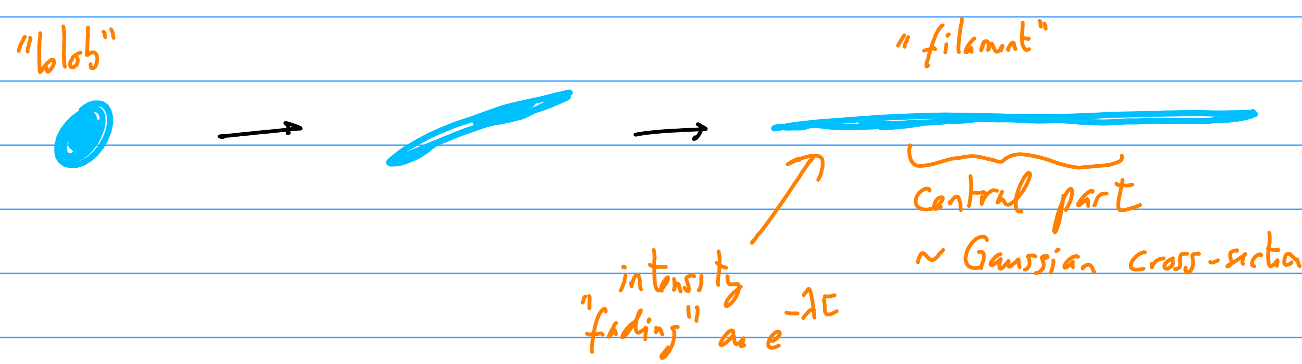
Hence,

$$\theta(\underline{x}, t) \sim e^{-\lambda t} e^{-y^2/2\ell^2}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

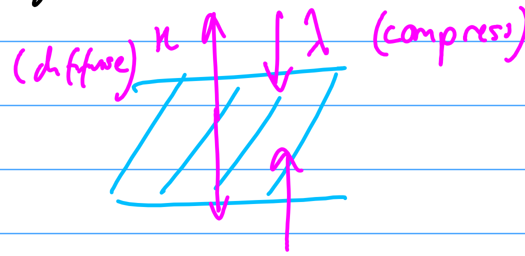
$$\ell = \sqrt{\frac{\kappa}{\lambda}}$$

Batchelor length

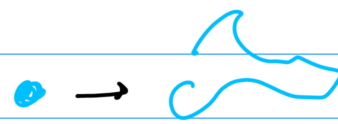
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

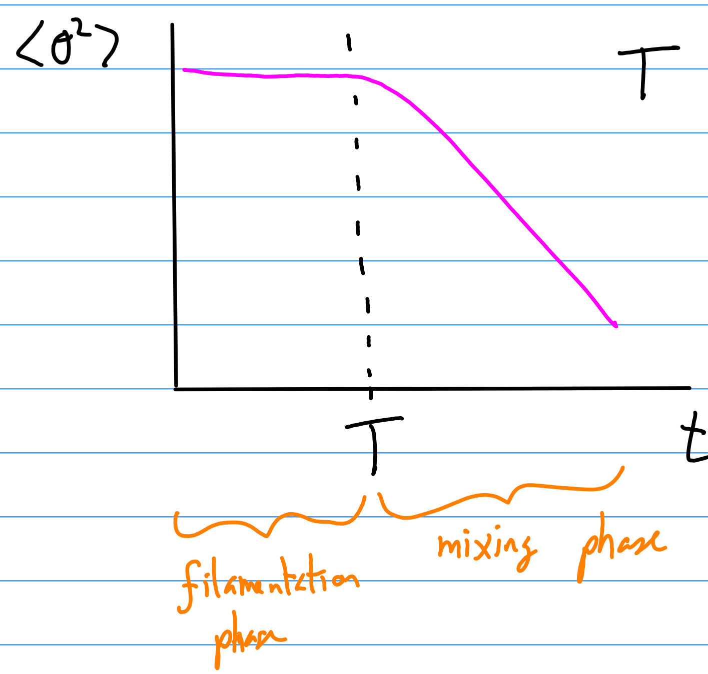
In practical applications, λ is often taken to be the local rate of strain.

ℓ is set by a balance between compression and diffusion



Summary: how mixing proceeds

- A blob is stirred 
- For a while, $\langle \theta^2 \rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order ℓ, diffusion takes over
- After that, $\langle \theta^2 \rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$$T \sim \lambda^{-1} \log \kappa$$

Lecture 2: Linear flows

(AD) $\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta.$

For this lecture, think of θ as a "patch"

Last time we examined $\underline{u} = (\lambda x, -\lambda y)$. Let's try something more general:

$$\underline{u} = \underline{U} + \underline{x}\cdot A, \qquad \nabla\cdot\underline{u} = \text{trace } A = 0.$$

($\underline{U}$ and A: const.)

Let $\langle f\rangle = \int_\Omega f\, dV \qquad (\Omega = \mathbb{R}^2 \text{ or } \mathbb{R}^3)$

Solve (AD) using moments:

$$c_i = \frac{\langle x_i\theta\rangle}{\langle\theta\rangle} \qquad (\partial_t\langle\theta\rangle = 0)$$

(AD) →

$$\partial_t\langle x_i\theta\rangle + \langle x_i \nabla\cdot((\underline{U}+\underline{x}\cdot A)\theta)\rangle = \kappa\langle x_i\nabla^2\theta\rangle$$

$$\partial_t\langle x_i\theta\rangle - \langle (U_j + x_\ell A_{\ell j})\theta\cdot \underbrace{\partial_j x_i}_{\delta_{ji}}\rangle = \kappa\ (0)$$

$$\langle\theta\rangle\partial_t c_i - U_i\langle\theta\rangle - A_{\ell i}\langle\theta\rangle c_\ell = 0$$

$$\boxed{\partial_t \underline{c} = \underline{U} + \underline{c}\cdot A}$$

Motion of center of mass

Next moments:

$$m_{ij} = \frac{\langle x_i x_j \theta\rangle}{\langle\theta\rangle} - c_i c_j$$

Again, multiply (AD) by $x_i x_j$ and $\langle\cdot\rangle$.

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = \langle x_i x_j \partial_k((U_k + x_\ell A_{\ell k})\theta)\rangle$$

$$= -\langle (U_k + x_\ell A_{\ell k})(\delta_{ik} x_j + x_i \delta_{jk})\theta\rangle$$

$$= -U_i c_j \langle\theta\rangle - U_j c_i \langle\theta\rangle - A_{\ell i} \underbrace{\langle x_\ell x_j \theta\rangle}_{\langle\theta\rangle(m_{\ell j} + c_\ell c_j)} - A_{\ell j} \underbrace{\langle x_\ell x_i \theta\rangle}_{\langle\theta\rangle(m_{\ell i} + c_\ell c_i)}$$

$$\partial_t(c_i c_j) = c_i \partial_t c_j + c_j \partial_t c_i$$
$$= c_i(U_j + A_{\ell j} c_\ell) + c_j(U_i + A_{\ell i} c_\ell)$$

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = -\left(\partial_t(c_i c_j) + A_{\ell i} m_{\ell j} + A_{\ell j} m_{\ell i}\right)\langle\theta\rangle$$

That's the hard part! Next:

$$\langle x_i x_j \nabla^2\theta\rangle = \langle\theta\,\nabla^2(x_i x_j)\rangle = 2\langle\theta\rangle\,\delta_{ij}$$

So finally:

$$\partial_t m_{ij} = A_{\ell i} m_{\ell j} + A_{\ell j} m_{i\ell} + 2\kappa\,\delta_{ij}$$

Let $(M)_{ij} = m_{ij}$ (symmetric matrix)

$$\boxed{\partial_t M = M\cdot A + A^T\cdot M + 2\kappa I}$$

Moment of inertia equation.
"spread" of patch

Time to solve these equations!

$$\underline{c}(t) = \underline{c}(0)\cdot e^{At} + \underline{U}\cdot\int_0^t e^{A(t-\tau)}\,d\tau$$

$$M(t) = e^{A^T t} \cdot M(0) \cdot e^{At} + 2\kappa \int_0^t e^{A^T(t-\tau)} \cdot e^{A(t-\tau)} d\tau$$

Shear flow: $A = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$, $A^2 = 0$

Can't write as $e^{(A+A^T)(t-\tau)}$ unless $[A, A^T] = 0$ Normal matrix

$$e^{At} = I + At + 0$$

$$\int_0^t e^{A^T(t-\tau)} \cdot e^{A(t-\tau)} d\tau = \int_0^t (I + A^T(t-\tau)) \cdot (I + A(t-\tau))\, d\tau$$

$$= \int_0^t \left(I - (A+A^T)(\tau - t) + A^T \cdot A\, (\tau - t)^2 \right) d\tau$$

$$= tI + (A+A^T) \left.\frac{(\tau-t)^2}{2}\right|_0^t + \left.\frac{1}{3} A^T \cdot A\, (\tau - t)^3\right|_0^t$$

$$= tI + \frac{1}{2}(A+A^T)\, t^2 + \frac{1}{3} A^T \cdot A\, t^3$$

$$A^T A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A + A^T = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

Assume an initial circular blob of size ρ: $M(0) \sim \rho^2 I$

$$e^{A^T t} \cdot M(0) \cdot e^{At} = \rho^2 \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha t & 1 \end{pmatrix} = \rho^2 \begin{pmatrix} 1 + \alpha^2 t^2 & \alpha t \\ \alpha t & 1 \end{pmatrix}$$

The different components of M have different asymptotic growth rates:

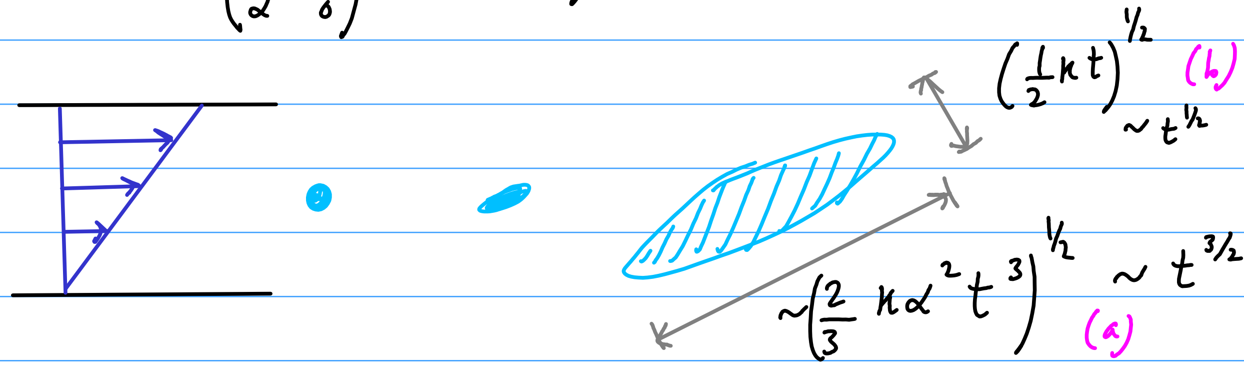
$$M_{11} = \rho^2 (1 + \alpha^2 t^2) + 2\kappa t + \underline{\frac{2}{3} \kappa \alpha^2 t^3}$$

$$M_{22} = \rho^2 + \underline{2\kappa t}$$

$$M_{12} = \rho^2 \alpha t + \underline{\kappa \alpha t^2}$$

So for large time, $M \sim \begin{pmatrix} \frac{2}{3}\kappa\alpha^2 t^3 & \kappa\alpha t^2 \\ \kappa\alpha t^2 & 2\kappa t \end{pmatrix}$ $\quad \det M \sim \frac{1}{3}\kappa^2\alpha^2 t^4$

$\underline{x} \cdot A = (x \;\; y)\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} = (\alpha y, 0)$



Sort of like a filament, except keeps "fattening".

Can use this to predict decay rate: $\text{area}^2 \sim \det M = \frac{1}{3}\kappa^2\alpha^2 t^4$

Concentration at a point $\sim \frac{\langle\theta\rangle}{\text{area}} \sim \boxed{\frac{3\rho^2\theta_0}{\alpha\kappa t^2}}$ — faster!

Compare to purely diffusive case: $M = (\rho^2 + 2\kappa t) I$

concentration $\sim \frac{\rho^2\theta_0}{2\kappa t}$

This speedup is known as Taylor-Aris dispersion or shear dispersion

General 2x2 matrix: $\operatorname{tr} A = 0$, $\det A = -\lambda^2$

Need to compute e^{At}.

Trick: $A^2 - (\operatorname{tr} A)A + (\det A) I = 0 \Rightarrow A^2 = \lambda^2 I$

Cayley-Hamilton theorem $\qquad$ λ can be imaginary

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I \sum_{n \text{ even}} \frac{(\lambda t)^n}{n!} + \sum_{n \text{ odd}} \frac{A \lambda^{n-1} t^n}{n!}$$

$$\boxed{e^{At} = \cosh(\lambda t)\, I + A \lambda^{-1} \sinh(\lambda t)}$$

$$A = \begin{pmatrix} \lambda & 0 \\ \alpha & -\lambda \end{pmatrix} : \quad \boxed{e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ \frac{\alpha}{\lambda} \sinh(\lambda t) & e^{-\lambda t} \end{pmatrix}}$$

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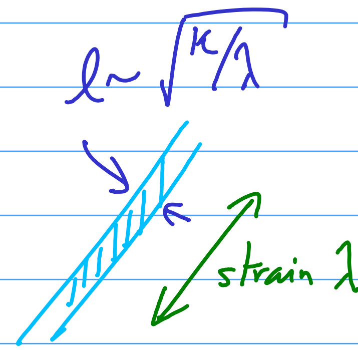
2010/02/17

Lecture 3: Effective Diffusivity



Recall: filaments in chaotic advection

Goal was to compute decay of variance,

$$\langle \theta^2 \rangle \sim e^{-\gamma t} \qquad (\gamma = \lambda \text{ for uniform strain})$$

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} = K_{\text{eff}} \nabla^2 \theta \; ?$$

Diffusion arises from noise: $\quad x_n = x_{n-1} + \xi_n$

Assume $\langle \xi_n \rangle = 0, \quad \langle \xi_n^2 \rangle = \sigma^2$ (i.i.d. (Gaussian))

$$x_n = \underbrace{x_0}_{0} + \sum_{i=1}^{n} \xi_n, \qquad \langle x_n \rangle = 0$$

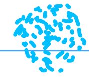
$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle \xi_n^2 \rangle = \underset{\sim \text{"time"}}{n} \sigma^2 = \underbrace{2 \kappa t}_{\text{by definition}}$$

$t = nT$

In $\underline{\underline{d}}$ dimensions,

$$\langle x_n^2 + y_n^2 \; (+ z_n^2) \rangle = n \underline{\underline{d}} \sigma^2 = 2 d \kappa n T$$

$$\boxed{K_{\text{eff}} = \frac{\sigma^2}{2T}}$$

Now if we take a "cloud" of points  , and define a density

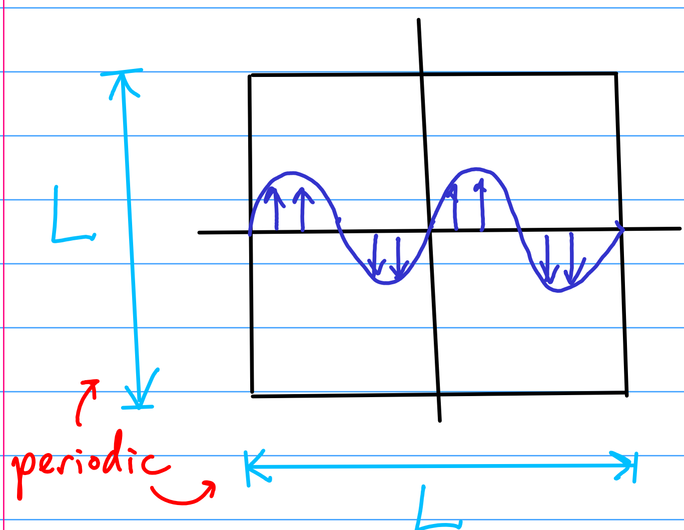
$$\theta(\underline{x}, t) = \text{density of points}$$

Then θ satisfies $\frac{\partial \theta}{\partial t} = K \nabla^2 \theta$ if each point evolves independently according to $x' = x + \xi$.

Of course, this requires "coarse-graining": it is only true if we don't look to closely (scales $\lesssim \sigma$) or too often (time scales $\lesssim T$).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous SINE FLOW.



- Velocity field (shear flow)

$$\underline{u}_H = \left(U \sin\left(\frac{2\pi k y}{L}\right), 0 \right)$$ STEP 1

applied for $0 \lesssim t < \tau/2$

- $$\underline{u}_V = \left(0, U \sin\left(\frac{2\pi k x}{L}\right) \right)$$ STEP 2

for $\tau/2 \lesssim t < \tau$.

Can solve $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$ exactly:

STEP 1: $$x(\tau/2) = x_0 + U\tau/2 \sin\left(\frac{2\pi k y_0}{L}\right)$$

$$y(\tau/2) = y_0$$

$x(\tau/2) = x(\tau)$

STEP 2: $x(\tau) = x(\tau/2)$

$$y(\tau) = y(\tau/2) + \frac{U\tau}{2}\sin\left(\frac{2\pi k x(\tau/2)}{L}\right)$$

Write as one map of period τ:

$$x' = x + T\sin(2\pi k y/L)$$
$$y' = y + T\sin(2\pi k x'/L)$$

$$T \equiv \frac{U\tau}{2}$$

Easy to iterate on a gazillion particles.

note x'! Important for area-preservation (comes from incompressibility)

Example 1: Run Matlab script example(1).

$$L = k = 1, \quad T = 0.1$$

Note how regular the orbits are: for small T the map is effectively a symplectic integrator

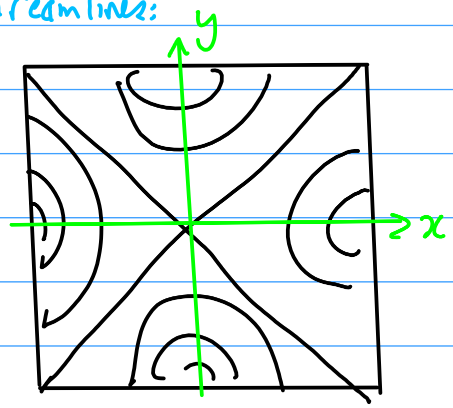
$$\frac{x'-x}{T} = \sin\left(\frac{2\pi k y}{L}\right), \quad \frac{y'-y}{T} = \sin\left(\frac{2\pi k x'}{L}\right)$$

As $T \to 0$, this approximates $\frac{dx}{dt} = \sin\left(\frac{2\pi k y}{L}\right)$, $\frac{dy}{dt} = \sin\left(\frac{2\pi k x}{L}\right)$,

$= \partial\psi/\partial y \qquad = -\partial\psi/\partial x$

or flow with streamfunction: $\psi = \frac{L}{2\pi k}\left(\cos\left(\frac{2\pi k x}{L}\right) - \cos\left(\frac{2\pi k y}{L}\right)\right)$

streamlines:



The streamlines aren't traced exactly because T is finite.

<u>Example 2</u> adds a bit of noise.

$$x' = (\text{sine map}) + \sqrt{2D}\,\xi$$

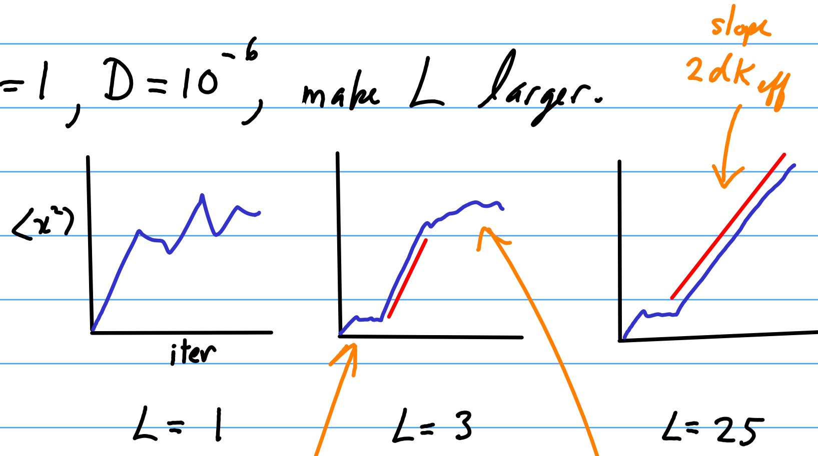
↑ Gaussian random var. with $\langle \xi^2 \rangle = 1$.

<u>Example 3:</u> $T = 1$. Now doesn't approximate a flow at all → CHAOTIC.

<u>Example 4:</u> $T = 1$, $L = 1$, $D = 10^{-4}$: "fat" filaments.
→ measure width by clicking
→ repeat for $D = 10^{-6}$
→ observe rough $\sqrt{D}$ scaling for filament width (see Lecture 1)

<u>Example 5:</u> $T = 1/2$, $k = 1$, $D = 10^{-6}$, make L larger.

Plot $\langle x^2 \rangle$ vs iteration



Hence, the concept of an effective diffusivity makes sense if we look at large scales, such that we cannot see the correlated small scale motions, and long times. (but not too long!)
→ <u>Useful for turbulence</u>

$$K_{eff} \sim 0.068 \gg D = 10^{-6}$$

<u>Note that the "cross" shape  evident in the pattern is not captured.</u>

IMA Tutorial: Transport & Mixing

2010/03/03

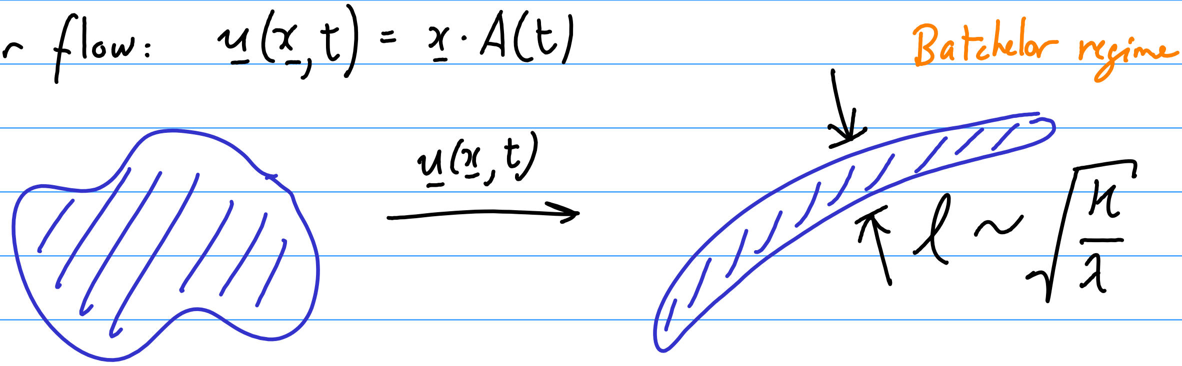
Lecture 4: Stochastic Models (part 1)

Antonsen et al. '96
Balkovsky & Fouxon '99

(AD) $\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta.$

Ω = domain = $\mathbb{R}^d$

Linear flow: $\underline{u}(\underline{x},t) = \underline{x}\cdot A(t)$



$\langle f\rangle = \int_\Omega f\,dV$, $\qquad \langle \underline{x}\theta\rangle = 0$ (center of mass)

(moment of inertia tensor)

$(M_{ij}) = m_{ij} = \dfrac{\langle x_i x_j \theta\rangle}{\langle\theta\rangle}$, $\qquad \dot{M} = M\cdot A + A^T\cdot M + 2\kappa I$

Let $M = RDR^T$, $\quad R$ orthogonal, D diagonal

$$\dot{M} = \dot{R}DR^T + RD\dot{R}^T + R\dot{D}R^T = RDR^TA + A^TRDR^T + 2\kappa I$$

$$R^T\dot{R}D + D\dot{R}^TR + \dot{D} = \underbrace{DR^TAR}_{\tilde{A}} + \underbrace{R^TA^TRD}_{\tilde{A}^T} + 2\kappa I$$

Now: $\dfrac{d}{dt}(R^TR) = \dot{R}^TR + R^T\dot{R} = \dfrac{d}{dt}(I) = 0$, so $(R^T\dot{R})^T = \dot{R}^TR = -R^T\dot{R}$

$\Rightarrow R^T\dot{R}$ is antisymmetric

$[R^T\dot{R}D]_{ii} = (R^T\dot{R})_{ik}D_{ki} = (R^T\dot{R})_{ii}D_{ii} = 0$

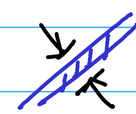
(no sum)

$$\dot D_{ii} = D_{il}\tilde A_{li} + \tilde A_{li} D_{li} + 2\kappa$$

$$\dot D_{ii} = 2\tilde A_{ii} D_{ii} + 2\kappa$$

Write $D_{ii} = e^{2\rho_i}$, with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d$.

$\dot D_{ii} = 2e^{2\rho_i}\dot\rho_i$, $\boxed{\dot\rho_i = \tilde A_{ii} + \kappa e^{-2\rho_i}}$

Great equation:
- $\tilde A = R^T A R$ → rotated velocity gradient matrix.
- $e^{-2\rho_i}$ → negligible unless $\rho_i < 0$ 

compression

Moral: the directions of contraction or compression play an important role.

Now we need an equation for R: off-diagonal terms.

$$[R^T\dot R D]_{ij} = (R^T\dot R)_{il} D_{lj} = (R^T\dot R)_{ij} D_{jj}, \quad i \neq j$$

$$[D\dot R^T R]_{ij} = D_{il}(\dot R^T R)_{lj} = D_{ii}(\dot R^T R)_{ij} = -(R^T\dot R)_{ij} D_{ii}$$

(no sum over i,j)

$$(D_{jj} - D_{ii})(R^T\dot R)_{ij} = D_{ii}\tilde A_{ij} + \tilde A_{ji} D_{jj}$$

$$(R^T\dot R)_{ij} = \Omega_{ij} \iff \boxed{\dot R = R\Omega}$$

$$\boxed{\Omega_{ij} = \frac{e^{2\rho_i}\tilde A_{ij} + e^{2\rho_j}\tilde A_{ji}}{e^{2\rho_j} - e^{2\rho_i}}}$$

Not completely obvious what this means...

$(= 0 \text{ for } i = j)$

Almost always true for long time, esp. in 2D, 3D with $\rho_1 + \rho_2 (+\rho_3) = 0$. Usually a symmetry can break this, or fails locally.

Assume we have separation between the eigenvalues: $e^{2\rho_i} \gg e^{2\rho_j}$, $i<j$

$$\Omega_{ij} \simeq \frac{e^{2\rho_i}\tilde{A}_{ij} + e^{2\rho_j}\tilde{A}_{ji}}{e^{2\rho_j} - e^{2\rho_i}} = -\tilde{A}_{ij}, \quad i<j$$

$$\Omega_{ij} \simeq \begin{cases} -\tilde{A}_{ij}, & i<j \\ \tilde{A}_{ji}, & i>j \end{cases}$$

(large t)

Independent of eigenvalues!

Can solve: $\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}$ since $\tilde{A}$ indep. of ρ_i

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\left[1 + 2\kappa e^{-2\rho_{0i}}\int_0^t \exp\left(-2\mathcal{A}_i(t')\right)dt'\right]$$

diffusion (↑ κ)

where

$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')\,dt'$$

When diffusion negligible: $\rho_i(t) = \rho_{i0} + \int_0^t \tilde{A}_{ii}(t')\,dt'$

In fact, solving the equations for ρ_i, R, $\kappa = 0$, is not a bad way of computing Lyapunov exponents:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t}\rho_i(t) \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$$

(Some numerical issues regarding orthogonality of R.)

Convergence: famous Oseledec Multiplicative ergodic theorem

Now comes the stochastic part: could have formulated things in terms of an SDE. But we take a shortcut:

$$\rho_i(t) = \rho_{0i} + \sum_t \tilde{A}_{ii}$$

sum of uncorrelated random numbers (more later)

What is PDF of $\rho_i(t)$?

Recall: if x_i are i.i.d. and $X = \sum_{i=1}^{N} x_i$

$\overline{x_i} = \xi$

$\overline{x_i^2} - \overline{x_i}^2 = \sigma^2$

What is PDF of X? CENTRAL LIMIT THEOREM

$$P(X,N) \sim \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left(-\frac{(X - N\xi)^2}{2N\sigma^2}\right)$$

($N\xi$: mean of X)

Valid for: (i) $N \gg 1$; (ii) $X - N\xi < \sqrt{N}\sigma$

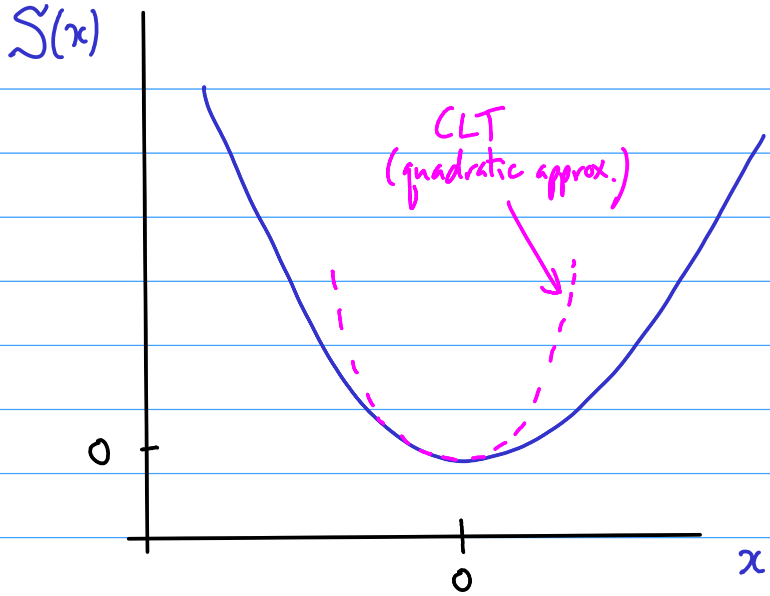
This second restriction is less commonly stated: it tells us that the CLT is not valid in the tails. The CLT tends to vastly underestimate the probability of rare events, or black swans as is trendy to call them these days. These tails matter for mixing.

More generally,

$$P(X,N) \sim \exp\left(-N S\left(\frac{X - N\xi}{N}\right)\right)$$

Large deviation form

$S(x)$ is a convex function with $S(0) = S'(0) = 0$.



$$S(x) = \overset{0}{S(0)} + \overset{0}{S'(0)}\, x + \frac{1}{2} S''(0)\, x^2 + \dots$$

$$S\left(\frac{X - N\xi}{N}\right) = \frac{1}{2} S''(0) \frac{(X - N\xi)^2}{N^2} + \dots$$

$$\exp\left(-N S\left(\frac{X}{N} - \xi\right)\right) \sim \exp\left(-S''(0) \frac{(X - N\xi)^2}{2N}\right)$$

Compare to CLT: $\boxed{S''(0) = \frac{1}{\sigma^2}}$

Can also express in terms of mean: $x = \frac{X}{N}$

$$P(x, N) \sim \exp\left(-N S(x - \xi)\right)$$

Example: Binomial distribution for x_i (-1 or 1, mean 0)

$$p(x_i) = \frac{1}{2}\delta(x_i + 1) + \frac{1}{2}\delta(x_i + 1)$$

$$e^{-s(k)} = \int p(\xi)\, e^{-ik\xi}\, d\xi \qquad \text{characteristic function}$$

$$= \frac{1}{2}\left(e^{ik} + e^{-ik}\right) = \cos k$$

For the mean $x = \frac{1}{N}\sum x_i$:

$$P(x,N) = \int p(x_1)\cdots p(x_N)\,\delta\!\left(\frac{x_1+\cdots+x_N}{N} - x\right) dx_1\cdots dx_N$$

$$e^{-S(k)} = \int P(x,N)\,e^{-ikx}\,dx$$

$$= \int p(x_1)\cdots p(x_N)\,e^{-ik(x_1+\cdots+x_N)/N}\,dx_1\cdots dx_N$$

$$= \prod_{i=1}^{N}\int p(x_i)\,e^{-ikx_i/N}\,dx_i = \left(\int p(\xi)\,e^{-ik\xi/N}\,d\xi\right)^N$$

$$= \left(e^{-s(k/N)}\right)^N = \cos^N(k/N)$$

Inverse Fourier

$$P(x,N) = \frac{1}{2\pi}\int e^{-S(k)}\,e^{ikx}\,dk = \frac{1}{2\pi}\int \cos^N\!\left(\frac{k}{N}\right) e^{ikx}\,dk$$

$$= \frac{N}{2\pi}\int \cos^N K\; e^{iNKx}\,dK\,,\quad K = k/N.$$

$$= \frac{N}{2\pi}\int e^{N(\log\cos K + iKx)}\,dK$$

For N large, look for saddle (stationary) point:

$$\frac{d}{dK}\underbrace{(\log\cos K + iKx)}_{H(K,x)} = -\tan K + ix = 0 \text{ when } K = K_{sp}.$$

$$\tan K_{sp} = -ix$$

$$H(k,x) = H(k_{sp},x) + \overset{0}{\widetilde{H'(k_{sp},x)(k-k_{sp})}} + \frac{1}{2}H''(k_{sp},x)(k-k_{sp})^2 + \cdots$$

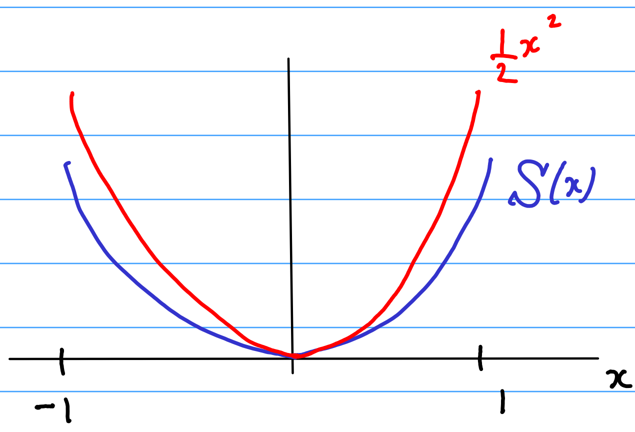
With this approximation the inverse transform is a Gaussian integral.
Get finally (skip some steps ... see Acosta lecture notes)

$$P(x,N) = \sqrt{\frac{NS''(0)}{2\pi}}\, e^{-NS(x)}, \quad \text{with}$$

$$\boxed{S(x) = -\frac{1}{2}(x+1)\log\left(\frac{1-x}{x+1}\right) + \log(1-x)} \qquad -1 \lesssim x \lesssim 1$$

Note $S(0) = 0$, $S'(x) = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right)$, so $S'(0) = 0$

$$S''(x) = \frac{1}{1-x^2}, \quad \text{so} \quad S''(0) = 1$$



$S(x)$ is called the
rate function
Cramer function
entropy function

Next lecture: what this has to do with mixing!

IMA Tutorial: Transport & Mixing 2010/03/11

Lecture 5: Stochastic Models (part 2)

More refs:
Falkovich et al. 2001
Zeldovich et al. 1984

(moment of inertia tensor)

$$\dot{M} = M \cdot A + A^T \cdot M + 2\kappa I, \qquad M = R D R^T$$

R orthogonal, D diagonal

Eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d$.

$$\boxed{\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}}$$

$\tilde{A}_{ii} = R^T A R$, evolves independent of ρ_i for $\rho_1 \gg \rho_2 \gg \cdots \gg \rho_d$.

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\left[1 + 2\kappa e^{-2\rho_{0i}} \int_0^t \exp\left(-2\mathcal{A}_i(t')\right) dt'\right]$$

where

$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')\,dt'$$

(see Falkovich et al. for description as SDE.)

For $\kappa = 0$, we argued that if $\tilde{A}_{ii}$ is a random var., then ρ_i are distributed according to large deviation form (for large t).

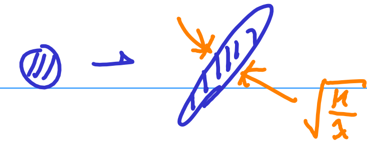
$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{\theta(\rho_1)}_{\substack{\text{ordering} \\ \rho_1 \geq \rho_2}} \underbrace{\delta(\rho_1 + \rho_2)}_{\text{incompressibility}}$$

in 2D ($d=2$). (return 3D later)

$$\lambda_1 = \lim_{t\to\infty} \rho_1 / t = \text{Lyapunov exp.} > 0 \quad (\text{for chaotic flows})$$

$\theta(x) =$ step function

What happens with diffusion? Recall "filament":



The contracting direction "stabilizes" near the Batchelor width $\sqrt{\frac{\kappa}{\lambda_1}}$.

↳ or "freezes"

Shraiman & Siggia '94

Chertkov et al. '97

Balkovsky & Fouxon '99

$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) P_{stab}(\rho_2)$$

($P_{stab}(\rho_2)$: stationary distribution.)

If we assume, say, an initial Gaussian "patch" of passive scalar, then the concentration at a point scales as

$$\theta(\underline{x}, t) \sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2}$$

$$= \exp\left(-\sum_i \rho_i\right)$$

(indep. of $\underline{x}$)

Expected value:

$$\langle \theta^\alpha \rangle (t) \sim \int e^{-\alpha \sum \rho_i} \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) P_{stab}(\rho_2)\, d\rho_1\, d\rho_2$$

Non-exponential function of t (neglect) ↘

$$\sim \int e^{-\alpha \rho_1} \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) d\rho_1$$

← Do the ρ_2 integral

Use $h_i = \rho_i / t$ as variable:

expected value, not integral ↘

$$\langle \theta^\alpha \rangle (t) \sim \int e^{-\alpha h_1 t}\, e^{-t\, S(h_1 - \lambda_1)}\, dh_1$$

$h_1 \to h$

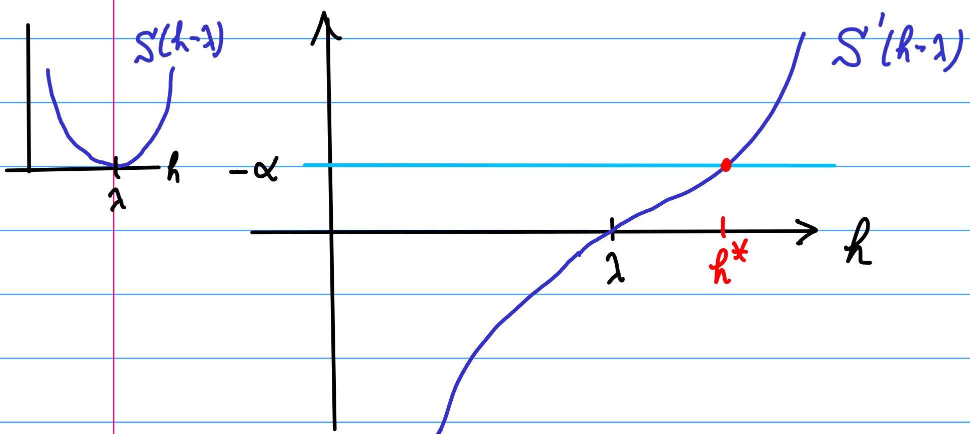
$\lambda_1 \to \lambda$

$$\boxed{\langle \theta^\alpha \rangle (t) \sim \int e^{-t(\alpha h + S(h - \lambda))}\, dh}$$

Let $H(h) = \alpha h + S(h-\lambda)$.

For large time, the integral is dominated by saddle point h^*:

$$H'(h^*) = 0 = \alpha + S'(h^* - \lambda)$$



Because of convexity of S, h^* is unique.

We then have $H(h) = H(h^*) + \frac{1}{2} H''(h^*)(h-h^*)^2 + \ldots$

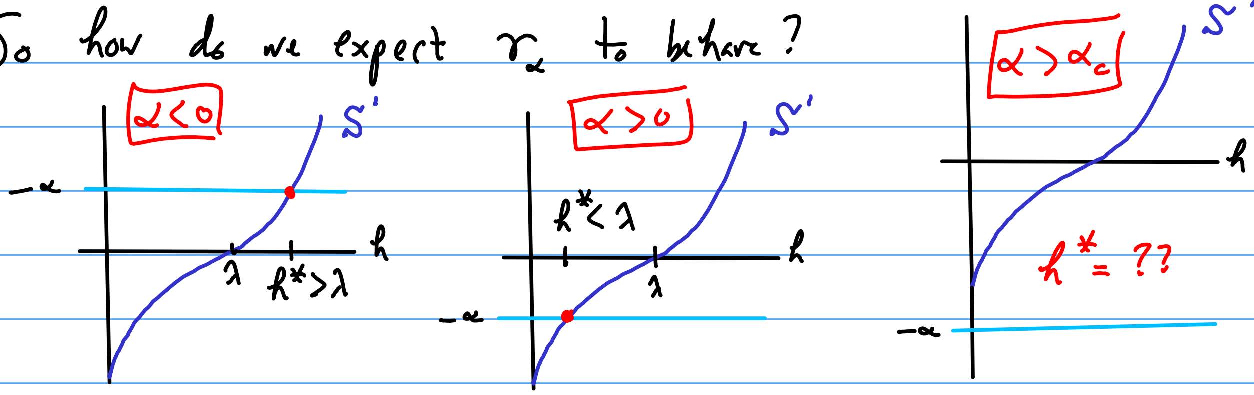
which we use to evaluate the integral. Find:

$$\langle \theta^\alpha \rangle (t) \sim e^{-\gamma_\alpha t}, \quad \text{where } \gamma_\alpha = H(h^*)$$

Note that we do not have $\langle \theta^\alpha \rangle \sim e^{-\alpha \gamma t}$, which would be the case if θ decayed the same pointwise everywhere.

$$\text{Kurtosis} \sim \frac{\langle \theta^\alpha \rangle}{\langle \theta \rangle^\alpha} \sim e^{-\gamma_\alpha t}$$

So how do we expect γ_α to behave?



We have $\gamma_0 = 0$, since $S'(h-\lambda)=0$ at $h=\lambda$, and $S(0)=0$.

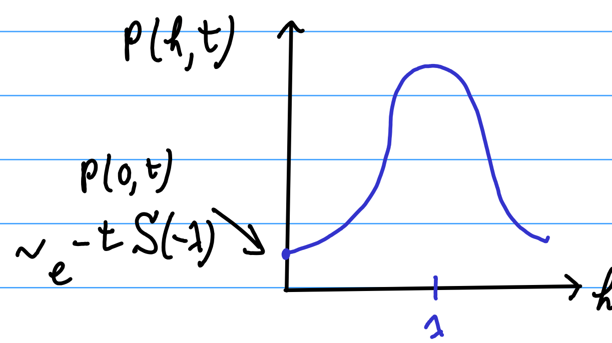
$\hookrightarrow \langle \theta^0 \rangle = 0$ ok!

Hence, γ_α changes sign at $\alpha = 0$.

What happens for $\alpha > \alpha_c$? No saddle point, since would require $h^* < 0$ (not allowed). Hence, take $h^* = 0$ (slowest decay)

The moments $\langle \theta^\alpha \rangle$ decay slower than expected, all the more so for larger α: <u>INTERMITTENCY</u>

Why the leveling-off? For large α, $\langle \theta^\alpha \rangle$ is dominated by realizations with large θ, that is, having experienced little stretching. For $\alpha > \alpha_c$, these are all that matter, so γ_α is the rate of decay of realizations with no stretching,



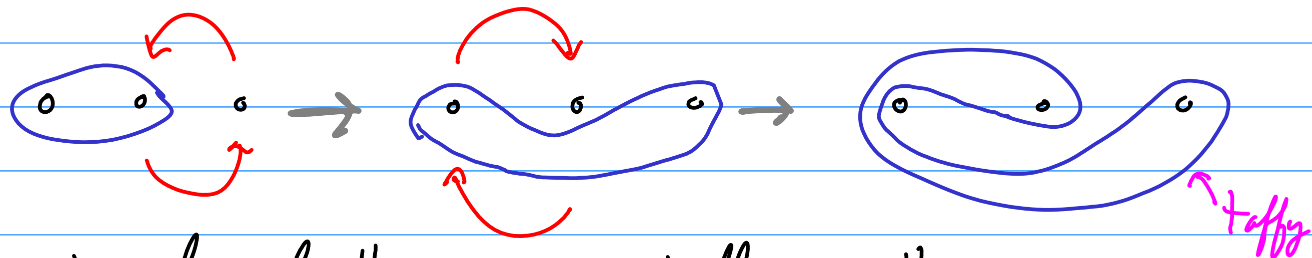
All this was for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.

IMA Tutorial: Transport & Mixing

2010/04/01

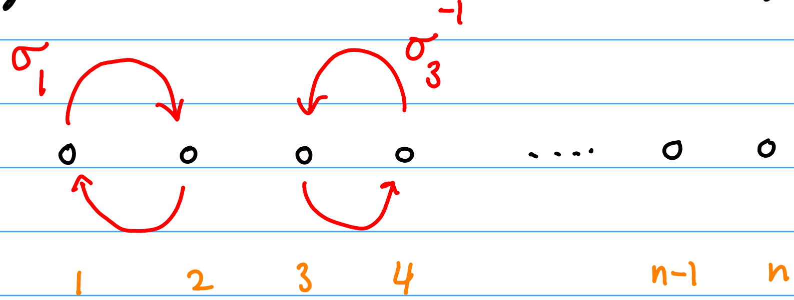
Lecture 6: Topological Mixing

Stirring by moving rods [movie] { fluids (viscous), elastic bodies (bread, taffy) }



Repeat: line length grows exponentially in this case.

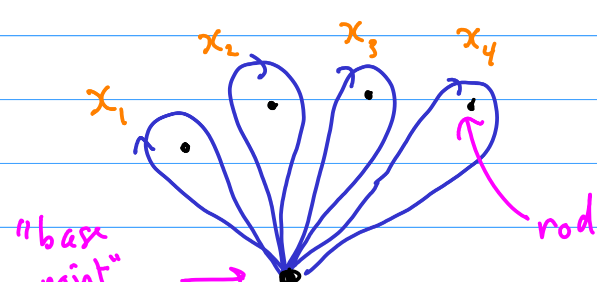
In general, can represent rod motions using generators of braid group:



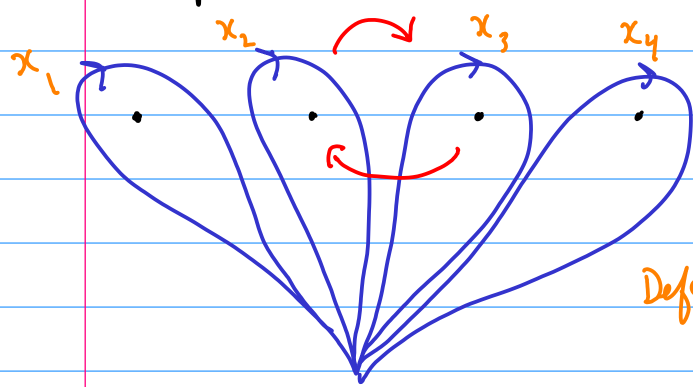
n-1 generators, $\{\sigma_1, \dots, \sigma_{n-1}\}$ + inverses.

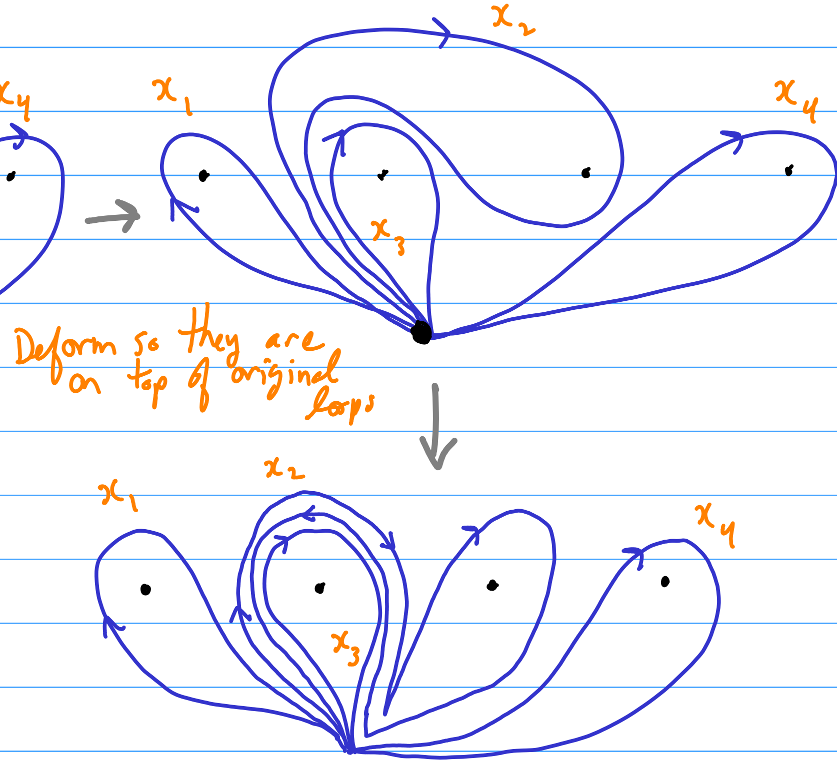
The previous example can be written $\sigma_2^{-1}\sigma_1$. (read left-to-right — conventions differ)

So at what rate will the taffy grow? One way is to look at action of braid group on loops, which are generators of the fundamental group, π_1.



Swap two rods:





Hence, σ_2 induces:

$$x_1 \mapsto x_1$$
$$x_2 \mapsto x_2 x_3 x_2^{-1}$$
$$x_3 \mapsto x_2$$
$$x_4 \mapsto x_4$$

← This is an automorphism of the free group π_1

In general, σ_i induces

$$x_i \mapsto x_i x_{i+1} x_i^{-1}$$
$$x_{i+1} \mapsto x_i$$
$$x_j \mapsto x_j \ , \quad j \neq i, i+1$$

$\{x_1, \cdots, x_n\}$ are the generators for the free group π_1(disc with n holes)

Alternate set of generators: $y_k = x_1 \cdots x_k$

For the above, easy to see $y_1 \mapsto y_1$

$$y_2 = x_1 x_2 \mapsto x_1 (x_2 x_3 x_2^{-1})$$
$$= (x_1 x_2 x_3)(x_1 x_2)^{-1} x_1$$
$$= y_3 y_2^{-1} y_1$$
$$y_3 = x_1 x_2 x_3 \mapsto x_1 (x_2 x_3 \cancel{x_2^{-1}}) \cancel{x_2} = y_3$$
$$y_4 = x_1 x_2 x_3 x_4 \mapsto y_4$$

Hence, σ_i acts as

$$\boxed{\begin{array}{ll} y_i \mapsto y_{i+1} y_i^{-1} y_{i-1} & \\ y_j \mapsto y_j & , \ j \neq i \end{array}}$$

slightly simpler

Now, the length of lines (similar to topological entropy) hooked on the rods will grow at the same rate as symbols:

<u>Example:</u> $\sigma_2^{-1} y_2 = y_1 y_2^{-1} y_3$,

$$\sigma_1 (\sigma_2^{-1} y_2) = \sigma_1(y_1)\, \sigma_1(y_2^{-1})\, \sigma_1(y_3)$$
$$= y_2 y_1^{-1} y_1^{-1} y_3$$

One symbol (y_2) went to 4 symbols ($y_2 y_1^{-1} y_1^{-1} y_3$) after $\sigma_2^{-1}\sigma_1$.

But we need the <u>asymptotic growth</u> (independent of choice of generators)

How?

"Abelianize" → treat like linear algebra.

$$y_i \mapsto y_{i+1} - y_i + y_{i-1}$$

$$\sigma_1 \to \tilde{K}_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \tilde{K}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

From this we can get the growth in the # of symbols, but with lots of cancellations. In fact,

$$\sigma_2^{-1}\sigma_1 \to \tilde{K}_1 \tilde{K}_2^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ has eigenvalues on the unit circle → no growth!

But this was only a lower bound. For an upper bound, put absolute values everywhere!

$$\sigma_1 \to K_1 = \begin{pmatrix} +1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & +1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

But also $\sigma_1^{-1} \to K_1$, $\sigma_2^{-1} \to K_2$! (not a representation.)

$$\sigma_2^{-1}\sigma_1 \to K_1 K_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has eigenvalues $\phi_1^2, \phi_1^{-2}, 1$, where

$\phi_1 = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio

This is an upper bound. It happens to be sharp, which can easily be shown by other means (Burau representation: action on double cover)

OK, so what? Let's define an "efficiency": the topological entropy per generator (TEPG).

$$\text{TEPG} = \frac{\log(\text{growth induced by periodic motion of } n \text{ rods})}{\min \# \text{ of generators in sequence}}$$

Example above: $\text{TEPG}(\sigma_2^{-1}\sigma_1) = \frac{\log(\phi_1^2)}{2} = \log \phi_1$

Let's prove that this is optimal

Consider the set $\{K_i\}_{i=1,\dots,n}$ of the abelianized-absolute value action of σ_i.

Any sequence of σ's corresponds to a product of K's.

The growth of loops in π_1 is given by

$$\rho(M_1 \cdots M_k) \quad , \quad M_j \in \{K_i\}$$

↑ spectral radius (largest eigenvalue in modulus)

If we normalize by how many generators, get $\rho^{1/k}(M_1 \cdots M_k)$.

Define: $$\rho_k(\{K_i\}) = \sup\left\{\rho(M_1 \cdots M_k) : M_j \in \{K_i\}\right\}$$

ρ_k gives the growth of the "best" product.

Define: $$\rho(\{K_i\}) = \limsup_{k\to\infty} \rho_k(\{K_i\})$$ JOINT SPECTRAL RADIUS (Rota & Strang, 1962)

Hard to compute!

Easier: $$\hat{\rho}_k(\{K_i\}) = \sup\left\{\|M_1 \cdots M_k\|_1 : M_j \in \{K_i\}\right\}$$

↳ 1-norm of matrices: sup over |column-sums|

For any matrix,

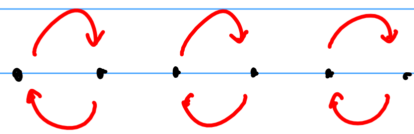
$M K_i$ changes the column sums by summing: $cs_{i\pm1} \mapsto cs_i + cs_{i\pm1}$

Hence, at best get a Fibonacci sequence: $\hat{\rho}_k(\{K_i\}) = F_{k+2}$.

Since $\lim_{k\to\infty} F_{k+2}^{1/k} = \phi_1$, we have an upper bound.

This upper bound can be realized.

There is a related problem where we count simultaneous motions as one:



1 operation

The entropy per operation in this case converges to the silver ratio!

$(1+\sqrt{2})$