Lecture 3: Stirring by swimming organisms, part 1[a]

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(Dated: 1 June 2016)

The setting of our problem is a large volume V that contains a number of swimmers N, also typically large. The swimmers move independently of each other in randomly directions (see Fig. 1). In the dilute limit that we consider, the velocity field of one swimmer is not significantly affected by the others. A random fluid particle (not too near the edges of the domain), will be displaced by the cumulative action of the swimmers. If we follow the displacements of a large number of well-separated fluid particles, which we treat as independent, we can obtain the full pdf of displacements. Our goal is to derive the exact pdf of displacements from a simple probabilistic model. Our starting point is the model described by Thiffeault and Childress [1] and improved by Lin *et al.* [2], which captures the important features observed in experiments. The calculation we describe is mostly taken from [3].

We examine the distribution of particle displacements for relatively short times, when the swimmers can be assumed to move along straight paths. For this we need the partial-path drift function for a fluid particle, initially at $\boldsymbol{r} = \boldsymbol{r}_0$, affected by a single swimmer:

$$\boldsymbol{\Delta}(\boldsymbol{r}_0, t) = U \int_0^t \boldsymbol{u}(\boldsymbol{r}(s) - \boldsymbol{U}s)\,\mathrm{d}s, \qquad \dot{\boldsymbol{r}} = \boldsymbol{u}(\boldsymbol{r} - \boldsymbol{U}t), \quad \boldsymbol{r}(0) = \boldsymbol{r}_0\,. \tag{1}$$

Here $\boldsymbol{U}t$ is the swimmer's position, with $\boldsymbol{U}$ assumed constant. To obtain $\boldsymbol{\Delta}(\boldsymbol{r}_0, t)$ we must solve the differential equation for each initial condition $\boldsymbol{r}_0$. After using homogeneity and

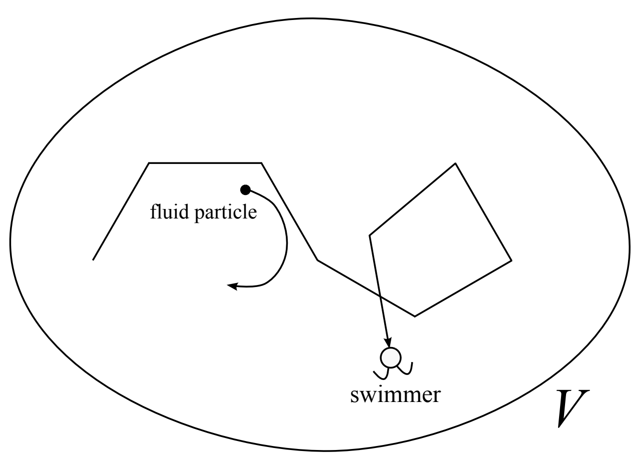

FIG. 1. A fluid particle displaced by a swimmer.

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isotropy, we obtain the probability density of displacements, [4]

$$p_1(\boldsymbol{r},t) = \frac{1}{\alpha_d\, r^{d-1}} \int_V \delta(r - \Delta(\boldsymbol{\eta},t))\, \frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}, \qquad r = \|\boldsymbol{r}\|, \tag{2}$$

where α_d is the area of the unit sphere in d dimensions: $\alpha_2 = 2\pi$, $\alpha_3 = 4\pi$. Here $\boldsymbol{r}$ gives the displacement of the particle from its initial position after a time t, and $p_1(\boldsymbol{r},t)$ is the probability density function of $\boldsymbol{r}$ for one swimmer.

The second moment of $\boldsymbol{r}$ for a single swimmer is

$$\langle r^2\rangle_1 = \int_V r^2 p_1(\boldsymbol{r},t)\,\mathrm{d}V_{\boldsymbol{r}} = \int_V \Delta^2(\boldsymbol{\eta},t)\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}. \tag{3}$$

This goes to zero as $V \to \infty$, since a single swimmer in an infinite volume shouldn't give any fluctuations. If we have N swimmers, the second moment is

$$\langle r^2\rangle_N = N\langle r^2\rangle_1 = n\int_V \Delta^2(\boldsymbol{\eta},t)\,\mathrm{d}V_{\boldsymbol{\eta}} \tag{4}$$

with $n = N/V$ the number density of swimmers. This is nonzero (and might diverge) in the limit $V \to \infty$, reflecting the cumulative effect of multiple swimmers. Note that this expression is exact, within the problem assumptions: it doesn't even require N to be large. It is not at all clear that (4) leads to diffusive behavior, but it does [2, 5, 6]: the "support" of the drift function $\Delta(\boldsymbol{\eta},t)$ typically grows in time: that is, the longer we wait, the larger the number of particles displaced by the swimmer. There is another mechanism, important for microswimmers, by which (4) can grow linearly in time, but it is more involved and will not be discussed here (see [3]).

From (2) with $d = 2$ we can compute $p_1(x,t)$, the marginal distribution for one coordinate:

$$p_1(x,t) = \int_{-\infty}^{\infty} p_1(\boldsymbol{r},t)\,\mathrm{d}y = \int_V \int_{-\infty}^{\infty} \frac{1}{2\pi r}\,\delta(r - \Delta(\boldsymbol{\eta},t))\,\mathrm{d}y\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}. \tag{5}$$

Since $r^2 = x^2 + y^2$, the δ-function will capture two values of y, and with the Jacobian included we obtain

$$p_1(x,t) = \frac{1}{\pi}\int_V \frac{1}{\sqrt{\Delta^2(\boldsymbol{\eta},t) - x^2}}\,[\Delta(\boldsymbol{\eta},t) > |x|]\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}\,, \tag{6}$$

where $[A]$ is an indicator function: it is 1 if A is true, 0 otherwise.

The marginal distribution in the three-dimensional case proceeds the same way from (2) with $d = 3$:

$$p_1(x,t) = \tfrac{1}{2}\int_V \frac{1}{\Delta(\boldsymbol{\eta},t)}\,[\Delta(\boldsymbol{\eta},t) > |x|]\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}. \tag{7}$$

For summing the displacements due to multiple swimmers, we need the characteristic function of $p_1(x,t)$, defined by the Fourier transform

$$\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1 = \int_{-\infty}^{\infty} p_1(x,t)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \tag{8}$$

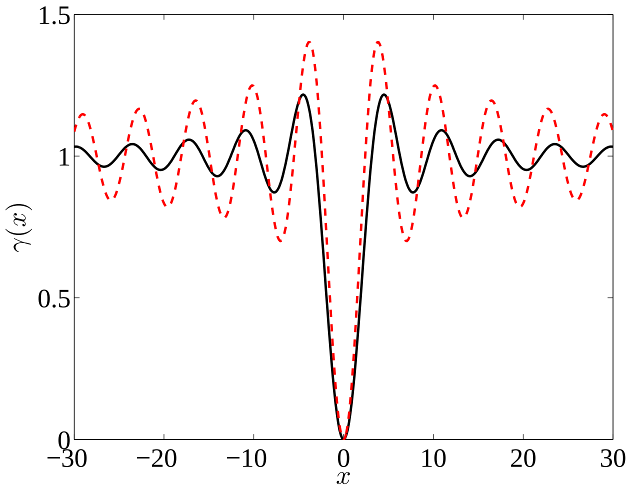

FIG. 2. The function $\gamma_d(x)$ defined by (10) for $d = 3$ (solid) and $d = 2$ (dashed).

For the three-dimensional pdf (7), the characteristic function is

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1 &= \tfrac{1}{2}\int_V \frac{1}{\Delta(\boldsymbol{\eta},t)} \int_{-\infty}^{\infty} [\Delta(\boldsymbol{\eta},t) > |x|]\, \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V} \\ &= \tfrac{1}{2}\int_V \frac{1}{\Delta(\boldsymbol{\eta},t)} \int_{-\Delta}^{\Delta} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V} \\ &= \int_V \mathrm{sinc}\,(k\Delta(\boldsymbol{\eta},t))\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}\end{aligned}$$

where $\mathrm{sinc}\,x := x^{-1}\sin x$ for $x \neq 0$, and $\mathrm{sinc}\,0 := 1$. For the two-dimensional pdf (6), we have

$$\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1 = \int_V J_0(k\Delta(\boldsymbol{\eta},t))\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V} \tag{9}$$

where $J_0(x)$ is a Bessel function of the first kind.

Here there is a subtlety that will creep up when dealing when taking the limit of infinite volume. As $\|\boldsymbol{\eta}\| \to \infty$, the displacement $\Delta(\boldsymbol{\eta},t)$ naturally goes to zero. However, because $J_0(0) = 1$ and $\mathrm{sinc}(0) = 1$, this means the integrand goes to one at infinity, which means the integral will go to V. This would create problems later when passing to the 'thermodynamic limit' $N, V \to \infty$. To remedy this, we add and subtract one in the integrand and use $\int_V \mathrm{d}V_{\boldsymbol{\eta}}/V = 1$. We then define (see Fig. 2)

$$\gamma_d(x) := \begin{cases} 1 - J_0(x), & d = 2; \\ 1 - \mathrm{sinc}\,x, & d = 3, \end{cases} \tag{10}$$

and write the two cases for the characteristic function together as

$$\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1 = 1 - \Gamma_d(k,t)/V. \tag{11}$$

where[1]

$$\Gamma_d(k,t) := \int_V \gamma_d(k\Delta(\boldsymbol{\eta},t))\,\mathrm{d}V_{\boldsymbol{\eta}}\,. \tag{12}$$

[1] Note that our normalization of Γ_d is different than in [3].

We have $\gamma_d(0) = \gamma_d'(0) = 0$, $\gamma_d''(0) = 1/d$, so $\gamma_d(\xi) \sim (1/2d)\,\xi^2 + \mathrm{O}(\xi^4)$ as $\xi \to 0$. For large argument, $\gamma_d(\xi) \to 1$ as $\xi \to \infty$.

We will need the following simple result:

Proposition 1. *Let $y(\varepsilon) \sim \mathrm{o}(\varepsilon^{-M/(M+1)})$ as $\varepsilon \to 0$ for an integer $M \geq 1$; then*

$$(1 - \varepsilon y(\varepsilon))^{1/\varepsilon} = \exp\left(-\sum_{m=1}^{M} \frac{\varepsilon^{m-1} y^m(\varepsilon)}{m}\right)\left(1 + \mathrm{o}(\varepsilon^0)\right), \quad \varepsilon \to 0. \tag{13}$$

Proof. Observe that $\varepsilon y(\varepsilon) \sim \mathrm{o}(\varepsilon^{1/(M+1)}) \to 0$ as $\varepsilon \to 0$. Writing $(1 - \varepsilon y)^{1/\varepsilon} = \mathrm{e}^{\varepsilon^{-1}\log(1-\varepsilon y)}$, we expand the exponent as a convergent Taylor series:

$$\begin{aligned}
(1 - \varepsilon y)^{1/\varepsilon} &= \exp\left(-\varepsilon^{-1}\sum_{m=1}^{\infty} \frac{(\varepsilon y)^m}{m}\right) \quad (\text{converges since } \varepsilon y \sim \mathrm{o}(\varepsilon^{1/(M+1)}))\\
&= \exp\left(-\varepsilon^{-1}\left(\sum_{m=1}^{M} \frac{(\varepsilon y)^m}{m} + \mathrm{O}((\varepsilon y)^{M+1})\right)\right)\\
&= \exp\left(-\varepsilon^{-1}\sum_{m=1}^{M} \frac{(\varepsilon y)^m}{m}\right)\exp\left(\mathrm{O}(\varepsilon^M y^{M+1})\right)\\
&= \exp\left(-\varepsilon^{-1}\sum_{m=1}^{M} \frac{(\varepsilon y)^m}{m}\right)\left(1 + \mathrm{o}(\varepsilon^0)\right).
\end{aligned}$$ □

Since we are summing their independent displacements, the characteristic function for N swimmers is $\langle \mathrm{e}^{\mathrm{i}kx}\rangle_N = \langle \mathrm{e}^{\mathrm{i}kx}\rangle_1^N$. From (11),

$$\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1^N = (1 - \Gamma_d(k,t)/V)^{nV}, \tag{14}$$

where we used $N = nV$, with n the number density of swimmers. Let's examine the assumption of Proposition 1 for $M = 1$ applied to (14), with $\varepsilon = 1/V$ and $y = \Gamma_d(k,t)$. For $M = 1$, the assumption of Proposition 1 requires

$$\Gamma_d(k,t) \sim \mathrm{o}(V^{1/2}), \qquad V \to \infty. \tag{15}$$

A stronger divergence with V means using a larger M in Proposition 1, but we shall not need to consider this here. Note that it is not possible for $\Gamma_d(k,t)$ to diverge faster than $\mathrm{O}(V)$, since $\gamma_d(x)$ is bounded. In order for $\Gamma_d(k,t)$ to diverge that fast, the displacement must be bounded away from zero as $V \to \infty$, an unlikely situation which can be ruled out.

Assuming that (15) is satisfied, we use Proposition 1 with $M = 1$ to make the large-volume approximation

$$\langle \mathrm{e}^{\mathrm{i}kx}\rangle_1^N = (1 - \Gamma_d(k,t)/V)^{nV} \sim \exp\left(-n\,\Gamma_d(k,t)\right), \quad V \to \infty. \tag{16}$$

If the integral $\Gamma_d(k,t)$ is convergent as $V \to \infty$ we have achieved a volume-independent form for the characteristic function, and hence for the distribution of x for a fixed swimmer density.

A comment is in order about evaluating (12) numerically: if we take $|k|$ to ∞, then $\gamma_d(k\Delta) \to 1$, and thus $\Gamma_d \to V$, which then leads to e^{-N} in (16). This is negligible as long as the number of swimmers N is moderately large. In practice, this means that $|k|$ only needs to be large enough that the argument of the decaying exponential in (16) is of order one, that is

$$n\,\Gamma_d(k_{\max},t) \sim \mathrm{O}(1). \tag{17}$$

Wavenumbers $|k| > k_{\max}$ do not contribute to (16). (We are assuming monotonicity of $\Gamma_d(k,t)$ for $k > 0$, which will hold for our case.) Note that (17) implies that we need larger wavenumbers for smaller densities n: a typical fluid particle then encounters very few swimmers, and the distribution should be far from Gaussian.

We finally recover the pdf of x as the inverse Fourier transform

$$p_n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-n\,\Gamma_d(k,t)\right) \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{18}$$

Note that we write $p_n(x,t)$ now instead of $p_N(x,t)$, where n is the number density, since N has been taken to infinity. In the next lecture we'll use this formula to evaluate $p_n(x,t)$ for several model swimmers.

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