Lecture 2: Filament model

Last time: $\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta = \kappa\nabla^2\theta$, $\nabla\cdot u = 0$

(AD)

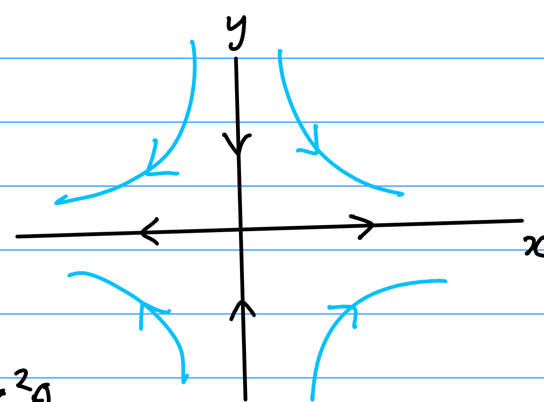
u is stirring (advection)
κ is mixing (diffusion) ← small at first

Let's look at a simple exact solution that illustrates important features.

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t\theta + \lambda x\partial_x\theta - \lambda y\partial_y\theta = \kappa\nabla^2\theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

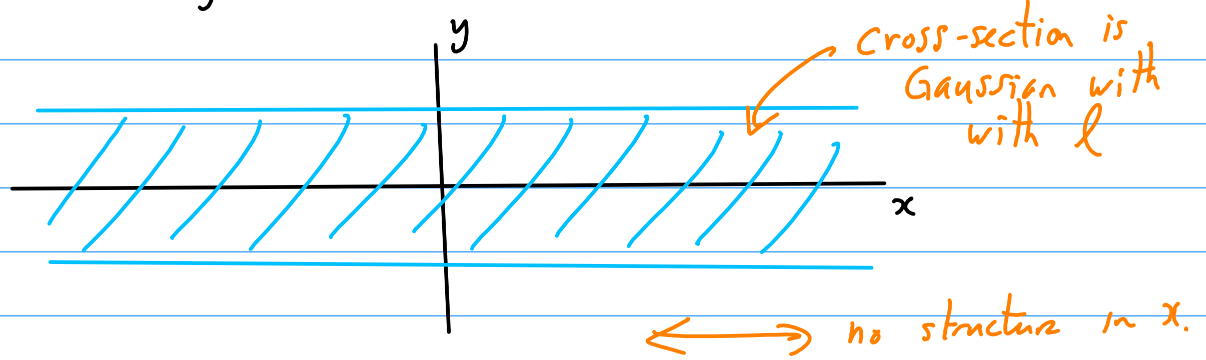
$$-\lambda f - \lambda y f' = \kappa f''$$

Boundary condition: $f \to 0$ as $y \to \pm\infty$.

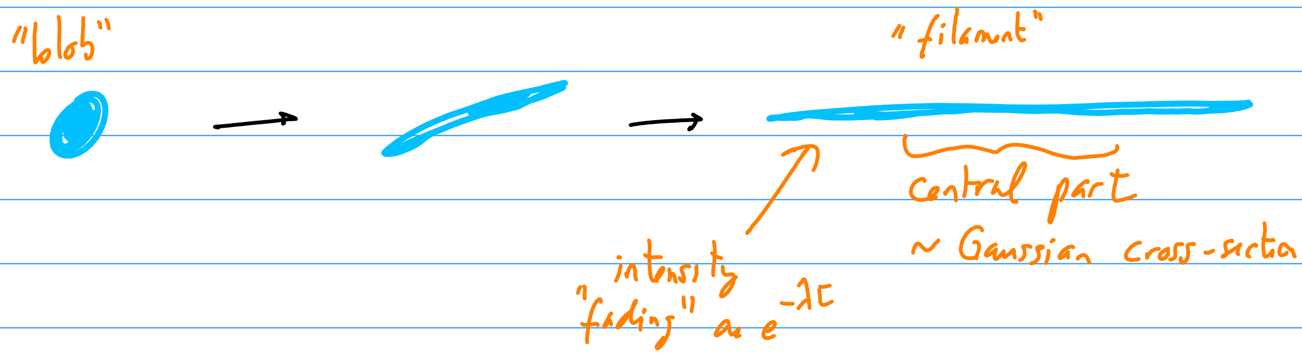
Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \dfrac{\kappa}{\lambda}$

Hence,
$$\boxed{\theta(\underline{x}, t) \sim e^{-\lambda t}\, e^{-y^2/2\ell^2}}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

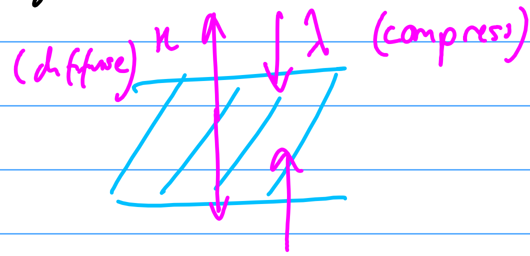
$$\boxed{\ell = \sqrt{\frac{\kappa}{\lambda}}}$$

Batchelor length

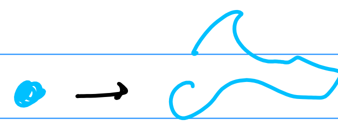
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

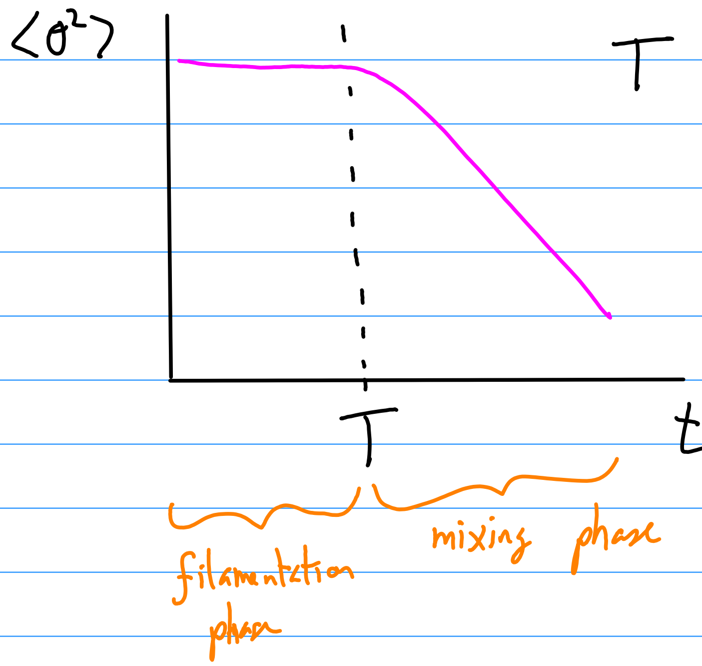
In practical applications, λ is often taken to be the local rate of strain.

ℓ is set by a balance between compression and diffusion



Summary: how mixing proceeds

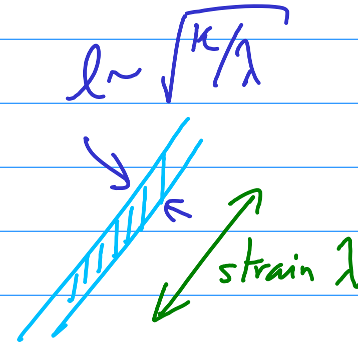
- A blob is stirred 
- For a while, $\langle \theta^2 \rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order ℓ, diffusion takes over
- After that, $\langle \theta^2 \rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$$T \sim \lambda^{-1} \log \kappa$$

Effective Diffusivity



Recall: filaments in chaotic advection

Goal was to compute decay of variance,

$$\langle \theta^2 \rangle \sim e^{-\gamma t} \qquad (\gamma = \lambda \text{ for uniform strain})$$

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} = K_{eff} \nabla^2 \theta \;?$$

Diffusion arises from noise: $\quad x_n = x_{n-1} + \xi_n$

Assume $\langle \xi_n \rangle = 0, \quad \langle \xi_n^2 \rangle = \sigma^2$ $\quad$ i.i.d. (Gaussian)

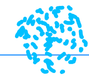
$$x_n = \underbrace{x_0}_{0} + \sum_{i=1}^{n} \xi_n, \qquad \langle x_n \rangle = 0$$

$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle \xi_n^2 \rangle = \overset{\sim \text{"time"}}{n}\sigma^2 \underbrace{=}_{\text{by definition}} 2Kt \qquad t = nT$$

In $\underline{\underline{d}}$ dimensions,

$$\langle x_n^2 + y_n^2 \,(+ z_n^2) \rangle = n\underline{\underline{d}}\sigma^2 = 2dKnT$$

$$\boxed{K_{eff} = \frac{\sigma^2}{2T}}$$

Now if we take a "cloud" of points , and define a density

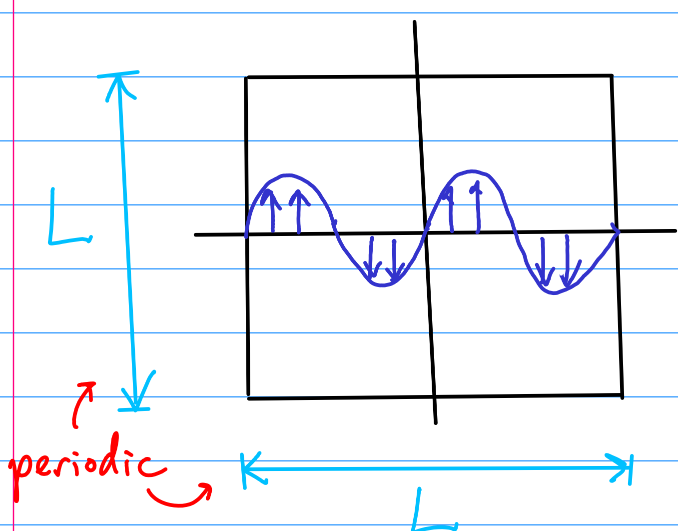
$$\theta(\underline{x}, t) = \text{density of points}$$

Then θ satisfies $\frac{\partial \theta}{\partial t} = K \nabla^2 \theta$ if each point evolves independently according to $x' = x + \xi$.

Of course, this requires "coarse-graining": it is only true if we don't look to closely (scales $\lesssim \sigma$) or too often (time scales $\lesssim T$).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous SINE FLOW.



- Velocity field (shear flow)

$$\underline{u}_H = \left(U \sin\left(\frac{2\pi k y}{L}\right), 0\right)$$ STEP 1

applied for $0 \leq t < \tau/2$

- $$\underline{u}_V = \left(0, U \sin\left(\frac{2\pi k x}{L}\right)\right)$$ STEP 2

for $\tau/2 \leq t < \tau$.

Can solve $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$ exactly:

STEP 1: $$x(\tau/2) = x_0 + U\tau/2 \sin\left(\frac{2\pi k y_0}{L}\right)$$

$$y(\tau/2) = y_0$$

STEP 2: $x(\tau) = x(\tau/2)$

$x(\tau/2) = x(\tau)$

$$y(\tau) = y(\tau/2) + \frac{U\tau}{2}\sin\left(\frac{2\pi k x(\tau/2)}{L}\right)$$

Write as one map of period τ:

$$\boxed{\begin{aligned} x' &= x + T\sin(2\pi k y/L) \\ y' &= y + T\sin(2\pi k x'/L) \end{aligned}}$$

$$T \equiv \frac{U\tau}{2}$$

Easy to iterate on a gazillion particles.

note x'! Important for area-preservation (comes from incompressibility)

Example 1: Run Matlab script example(1).

$$L = k = 1, \quad T = 0.1$$

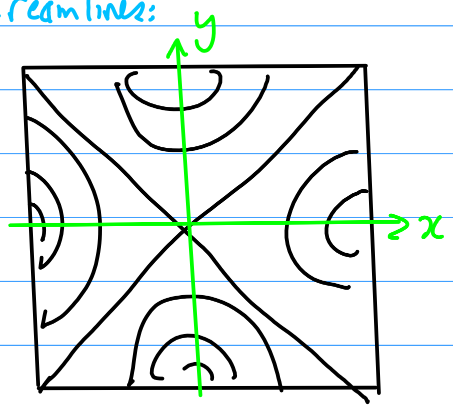
Note how regular the orbits are: for small T the map is effectively a symplectic integrator

$$\frac{x'-x}{T} = \sin\left(\frac{2\pi k y}{L}\right), \quad \frac{y'-y}{T} = \sin\left(\frac{2\pi k x'}{L}\right)$$

As $T \to 0$, this approximates $\frac{dx}{dt} = \sin\left(\frac{2\pi k y}{L}\right) = \partial\psi/\partial y$, $\frac{dy}{dt} = \sin\left(\frac{2\pi k x}{L}\right) = -\partial\psi/\partial x$,

or flow with streamfunction: $\psi = \frac{L}{2\pi k}\left(\cos\left(\frac{2\pi k x}{L}\right) - \cos\left(\frac{2\pi k y}{L}\right)\right)$

streamlines:



The streamlines aren't traced exactly because T is finite.

Example 2 adds a bit of noise.

$$x' = (\text{sine map}) + \sqrt{2D}\,\xi$$

Gaussian random var. with $\langle \xi^2 \rangle = 1$.

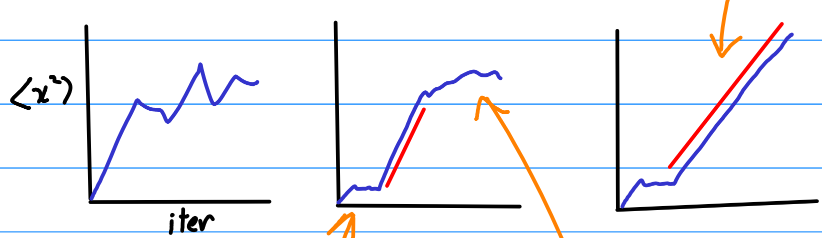
Example 3: $T = 1$. Now doesn't approximate a flow at all $\rightarrow$ CHAOTIC.

Example 4: $T = 1$, $L = 1$, $D = 10^{-4}$: "fat" filaments.
- $\rightarrow$ measure width by clicking
- $\rightarrow$ repeat for $D = 10^{-6}$
- $\rightarrow$ observe rough $\sqrt{D}$ scaling for filament width (see Lecture 1)

Example 5: $T = 1/2$, $k = 1$, $D = 10^{-6}$, make L larger.

Plot $\langle x^2 \rangle$ vs iteration

slope $2dK_{eff}$



$L = 1$ $\quad$ $L = 3$ $\quad$ $L = 25$

particles are initially very close, so correlated

"chaotic mixing" regime

particles reach sides of box

Hence, the concept of an effective diffusivity makes sense if we look at large scales, such that we cannot see the correlated small scale motions, and long times. (but not too long!)

$\rightarrow$ Useful for turbulence

$$K_{eff} \sim 0.068 \gg D = 10^{-6}$$

Note that the "cross" shape  evident in the pattern is not captured.