GFD Lectures: Swimming & Swirling

2010/06/21

Lecture 1: Stirring & Mixing

Stirring: mechanical action (cause)
Mixing: homogenization of a scalar (effect)

$\theta(\underline{x}, t)$ = concentration, $\underline{u}(\underline{x}, t)$ given

Advection Diffusion eq. (AD)

$$\frac{\partial \theta}{\partial t} + \underline{u}\cdot\nabla\theta = \kappa \nabla^2 \theta, \quad \nabla\cdot\underline{u} = 0 \quad \text{in } \Omega$$

Boundary conditions: $\left.\begin{aligned}\hat{n}\cdot\nabla\theta &= 0\\ \hat{n}\cdot\underline{u} &= 0\end{aligned}\right\}$ on boundary $\partial\Omega$

Let $\langle\cdot\rangle = \int_\Omega \cdot \, dV$

Multiply AD by $m\theta^{m-1}$, integrate:

$$\langle m\theta^{m-1}\partial_t\theta\rangle = \partial_t\langle\theta^m\rangle$$

$$\langle m\theta^{m-1}\underline{u}\cdot\nabla\theta\rangle = \langle\underline{u}\cdot\nabla\theta^m\rangle \overset{\nabla\cdot\underline{u}=0}{=} \langle\nabla\cdot(\underline{u}\theta^m)\rangle$$

$$= \int_{\partial\Omega}\theta^m \underbrace{\underline{u}\cdot\hat{n}}_{0!} \, dS = 0$$

$$\langle m\theta^{m-1}\kappa\nabla^2\theta\rangle = \kappa m\langle\nabla\cdot(\theta^{m-1}\nabla\theta) - \nabla\theta^{m-1}\cdot\nabla\theta\rangle$$

$$= \kappa m\int_{\partial\Omega}\theta^{m-1}\underbrace{\nabla\theta\cdot\hat{n}}_{0!} \, dS - \kappa m(m-1)\langle\theta^{m-2}|\nabla\theta|^2\rangle$$

$$\boxed{\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle}$$

$m=0$ is trivial

$m=1$: $\partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

$m=2$: $\partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle \theta^2 \rangle$ non-increasing!

Let variance $Var = C_2 = \langle \theta^2 \rangle - \underbrace{\langle \theta \rangle^2}_{\text{constant}}$

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

Scenario:

- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = \underbrace{\text{constant}}_{\text{in some sense}}$.

Hence the system is "driven" towards a homogeneous state where

$$\theta(\underline{x}, t) = \langle \theta \rangle = \text{constant}. \qquad (C_2 = 0,\ \langle \theta^2 \rangle = \langle \theta \rangle^2)$$

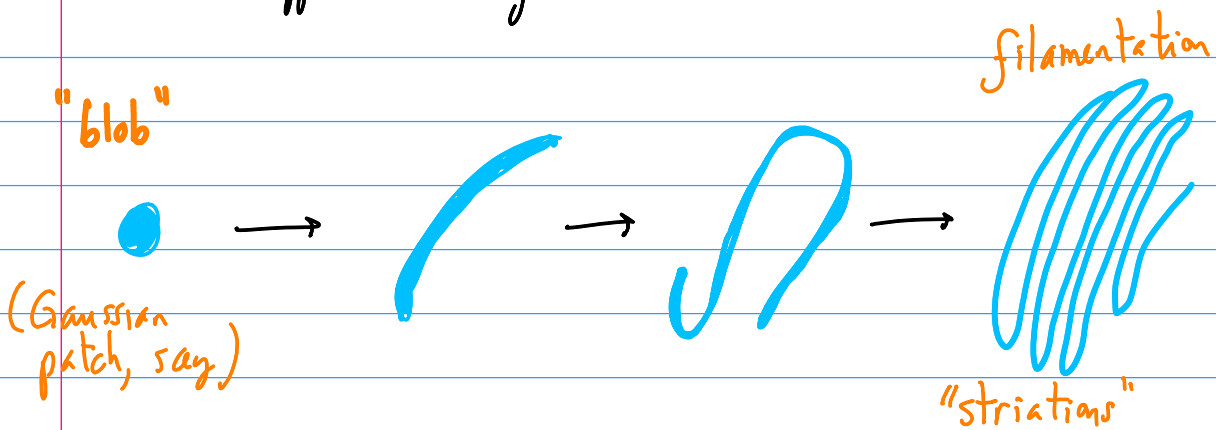
Assume $\langle \theta \rangle = 0$ WLOG

No fluctuations from the mean! When C_2 is small "enough", we say the system is mixed.

Big Q: Where is $\underline{u}(\underline{x}, t)$!? (stirring)

It doesn't appear in the variance equation!

But of course the variance equation is not closed: it depends on $\nabla\theta$. What happens when you stir?



This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle \theta^2 \rangle = -2\kappa \underbrace{\langle |\nabla\theta|^2 \rangle}_{\text{this becomes larger as we stir}}$$

By how much are gradients increased? After all, if $|\nabla\theta|$ becomes too large, then $\langle\theta^2\rangle \to 0$, so there are no gradients anymore!

Answer: for "good" stirring, the system is driven to a state where

$$\kappa \langle |\nabla\theta|^2 \rangle \longrightarrow \boxed{\text{independent of } \kappa}$$

Hence, $\boxed{\nabla\theta \sim \kappa^{-1/2}}$

This is the chaotic/turbulent mixing scenario:

$\dfrac{\partial \langle\theta^2\rangle}{\partial t}$ becomes independent of κ after a "short" transient

This is the Platonic ideal of mixing

(How short? Typically $\sim \log \kappa$)

Furthermore, the smallest scales visible in the concentration field $\theta(\underline{x}, t)$ have size $\sim \sqrt{\kappa}$. (missing a dimensional factor → see later)

Note that $\partial_t \langle \theta^2 \rangle$ independent of κ is crucial: in most applications, κ is tiny!

Heat: $\kappa = 2.2160 \times 10^{-5}\ m^2/s$ at 300K

10 m room: diffusion time $\sim \frac{L^2}{\kappa} = \frac{(10m)^2}{(2\times 10^{-5} m^2/s)} \sim 4.5 \times 10^6$ sec

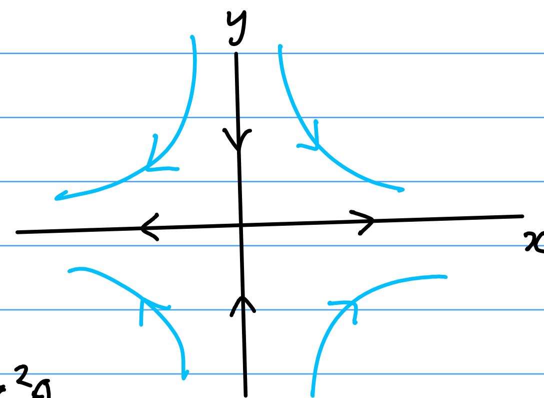
$\sim$ 1300 hours

So we better stir!
Even thermal convection is often enough. ← $\boxed{\sim 53 \text{ days !}}$

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t \theta + \lambda x \partial_x \theta - \lambda y \partial_y \theta = \kappa \nabla^2 \theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

$$-\lambda f - \lambda y f' = \kappa f''$$

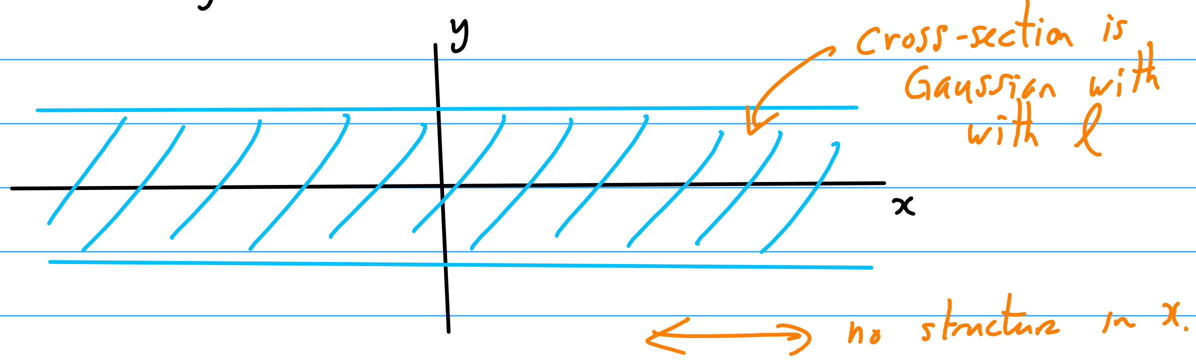
Boundary condition:

$f \to 0$ as $y \to \pm\infty$.

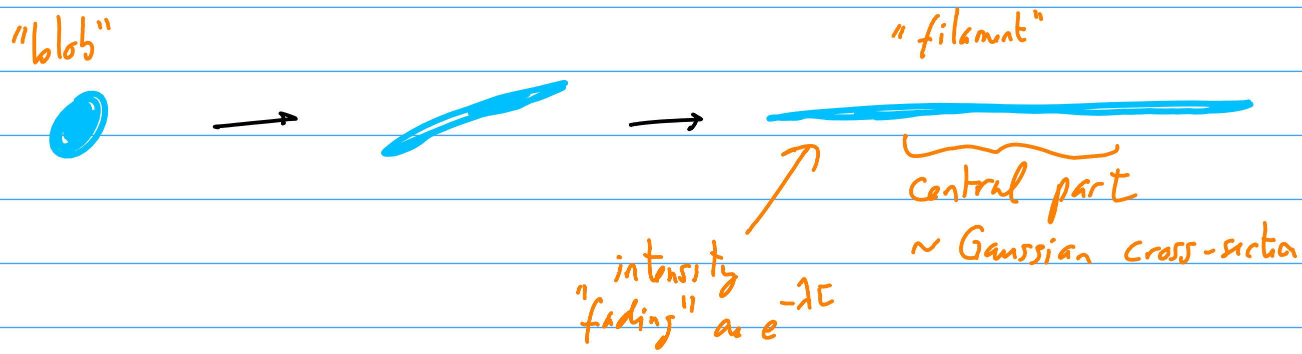
Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \frac{\kappa}{\lambda}$

Hence, $$\boxed{\theta(\underline{x}, t) \sim e^{-\lambda t}\, e^{-y^2/2\ell^2}}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

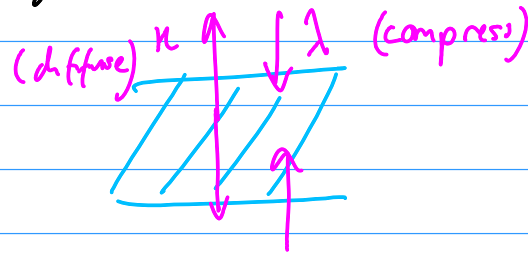
$$\boxed{\ell = \sqrt{\frac{\kappa}{\lambda}}}$$

Batchelor length

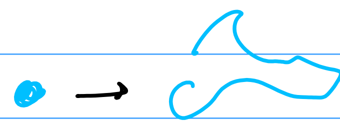
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

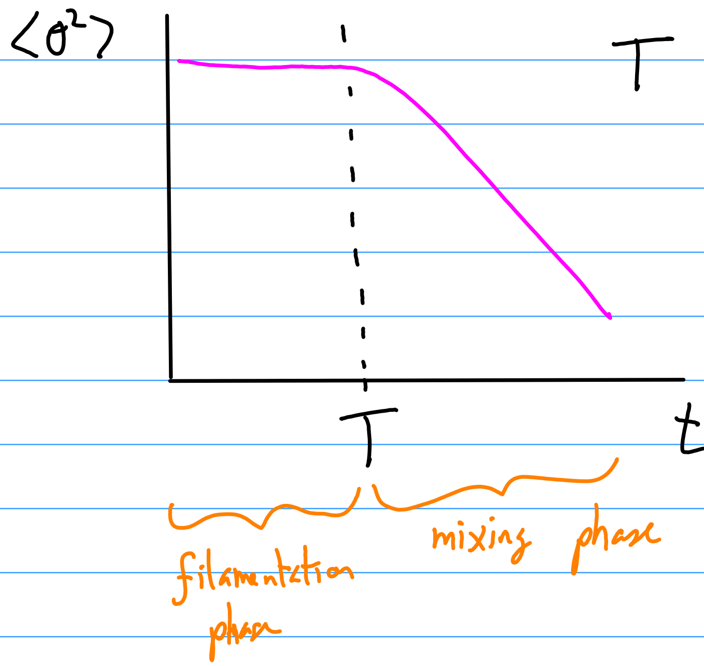
In practical applications, λ is often taken to be the local rate of strain.

ℓ is set by a balance between <u>compression</u> and <u>diffusion</u>



<u>Summary:</u> how mixing proceeds

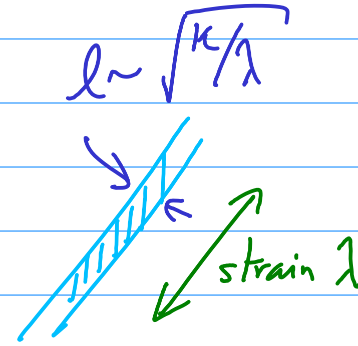
- A blob is stirred 
- For a while, $\langle\theta^2\rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order ℓ, diffusion takes over
- After that, $\langle\theta^2\rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$$\underline{T \sim \lambda^{-1} \log \kappa}$$

Effective Diffusivity



Recall: filaments in chaotic advection

Goal was to compute decay of variance,

$$\langle \theta^2 \rangle \sim e^{-\gamma t} \qquad (\gamma = \lambda \text{ for uniform strain})$$

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} = K_{eff} \nabla^2 \theta \ ?$$

Diffusion arises from noise: $\quad x_n = x_{n-1} + \xi_n$

Assume $\quad \langle \xi_n \rangle = 0, \quad \langle \xi_n^2 \rangle = \sigma^2 \qquad$ i.i.d. (Gaussian)

$$x_n = \underbrace{x_0}_{0} + \sum_{i=1}^{n} \xi_n , \quad \langle x_n \rangle = 0$$

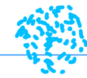
$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle \xi_n^2 \rangle = \overset{\sim\text{"time"}}{n} \sigma^2 \ \underbrace{=}_{\text{by definition}} \ 2Kt$$

$t = nT$

In $\underline{\underline{d}}$ dimensions,

$$\langle x_n^2 + y_n^2 \, (+ z_n^2) \rangle = n \underline{\underline{d}} \sigma^2 = 2dKnT$$

$$\boxed{K_{eff} = \frac{\sigma^2}{2T}}$$

Now if we take a "cloud" of points , and define a density

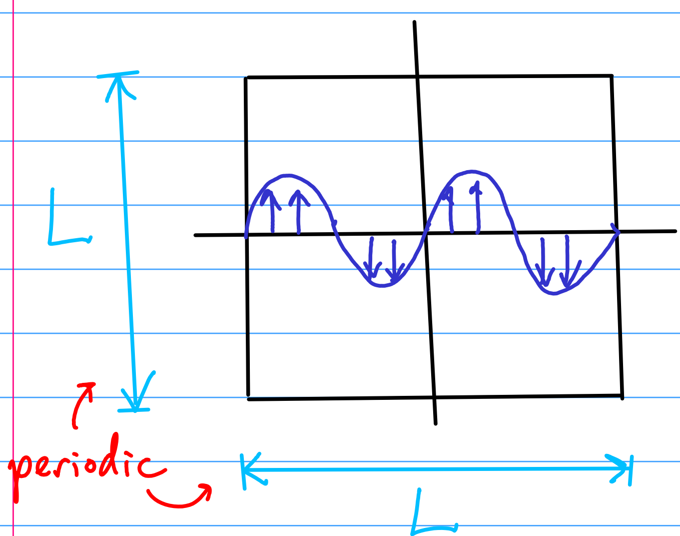
$$\theta(\underline{x}, t) = \text{density of points}$$

Then θ satisfies $\dfrac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta$ if each point evolves independently according to $x' = x + \xi$.

Of course, this requires "coarse-graining": it is only true if we don't look to closely (scales $\lesssim \sigma$) or too often (time scales $\lesssim T$).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous SINE FLOW.



- Velocity field (shear flow)

$$\underline{u}_H = \left(U \sin\left(\frac{2\pi k y}{L}\right), 0 \right)$$ STEP 1

applied for $0 \leq t < \tau/2$

- $$\underline{u}_V = \left(0, U \sin\left(\frac{2\pi k x}{L}\right) \right)$$ STEP 2

for $\tau/2 \leq t < \tau$.

Can solve $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$ exactly:

STEP 1:
$$x(\tau/2) = x_0 + U\tau/2 \sin\left(\frac{2\pi k y_0}{L}\right)$$
$$y(\tau/2) = y_0$$

$x(\tau/2) = x(\tau)$

STEP 2: $$x(\tau) = x(\tau/2)$$

$$y(\tau) = y(\tau/2) + \frac{U\tau}{2}\sin\left(\frac{2\pi k x(\tau/2)}{L}\right)$$

Write as one map of period τ:

$$\boxed{\begin{aligned} x' &= x + T\sin\left(2\pi k y/L\right) \\ y' &= y + T\sin\left(2\pi k x'/L\right) \end{aligned}} \qquad T \equiv \frac{U\tau}{2}$$

Easy to iterate on a gazillion particles.

note x'! Important for area-preservation (comes from incompressibility)

Example 1: Run Matlab script example(1).

$$L = k = 1, \quad T = 0.1$$

Note how regular the orbits are: for small T the map is effectively a symplectic integrator

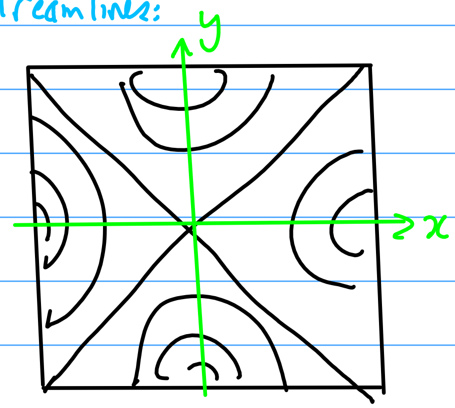
$$\frac{x'-x}{T} = \sin\left(\frac{2\pi k y}{L}\right), \quad \frac{y'-y}{T} = \sin\left(\frac{2\pi k x'}{L}\right)$$

As $T \to 0$, this approximates $\frac{dx}{dt} = \sin\left(\frac{2\pi k y}{L}\right)$, $\frac{dy}{dt} = \sin\left(\frac{2\pi k x}{L}\right)$,

$= \partial\psi/\partial y$ $\qquad = -\partial\psi/\partial x$

or flow with streamfunction: $$\psi = \frac{L}{2\pi k}\left(\cos\left(\frac{2\pi k x}{L}\right) - \cos\left(\frac{2\pi k y}{L}\right)\right)$$

streamlines:



The streamlines aren't traced exactly because T is finite.

Example 2 adds a bit of noise.

$$x' = (\text{sine map}) + \sqrt{2D}\,\xi$$

Gaussian random var. with $\langle \xi^2 \rangle = 1$.

Example 3: $T=1$. Now doesn't approximate a flow at all $\rightarrow$ CHAOTIC.

Example 4: $T=1$, $L=1$, $D=10^{-4}$: "fat" filaments.

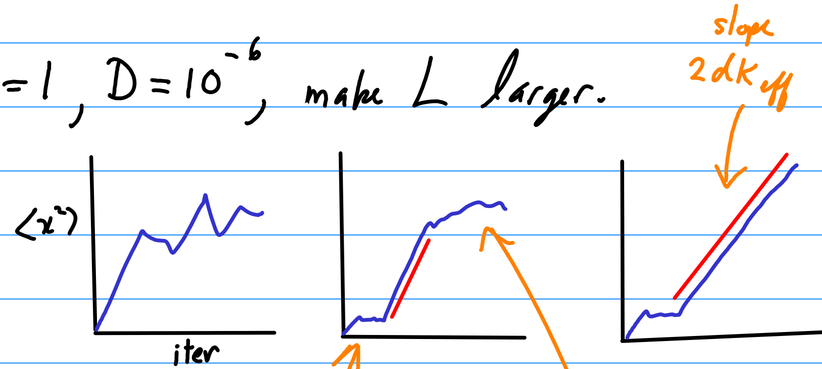
$\rightarrow$ measure width by clicking

$\rightarrow$ repeat for $D=10^{-6}$

$\rightarrow$ observe rough $\sqrt{D}$ scaling for filament width (see Lecture 1)

Example 5: $T=1/2$, $k=1$, $D=10^{-6}$, make L larger.

Plot $\langle x^2 \rangle$ vs iteration



$L=1$ $\quad$ $L=3$ $\quad$ $L=25$

particles are initially very close, so correlated

"chaotic mixing" regime

particles reach sides of box

Hence, the concept of an effective diffusivity makes sense if we look at large scales, such that we cannot see the correlated small scale motions, and long times. (but not too long!)

$\rightarrow$ Useful for turbulence

$$K_{eff} \sim 0.068 \gg D = 10^{-6}$$

Note that the "cross" shape evident in the pattern is not captured.

Lecture 2: Stirring by swimming organisms

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23 June 2010

Biomixing

A controversial proposition:

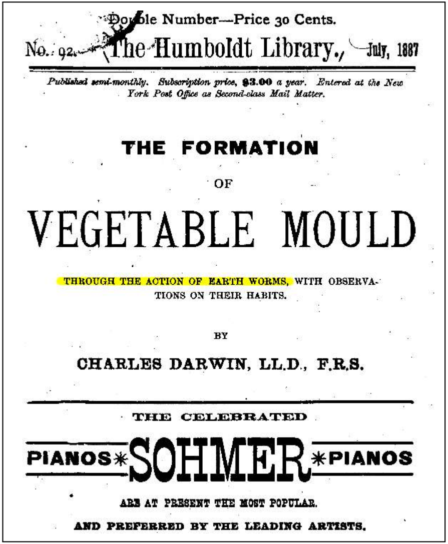
- There are many regions of the ocean that are relatively quiescent, especially in the depths (1 hairdryer/ km^3);
- Yet mixing occurs: nutrients eventually get dredged up to the surface somehow;
- What if organisms swimming through the ocean made a significant contribution to this?
- There could be a local impact, especially with respect to feeding and schooling;
- Also relevant in suspensions of microorganisms (Viscous Stokes regime).

Bioturbation

The earliest case studied of animals 'stirring' their environment is the subject of Darwin's last book.

This was suggested by his uncle and future father-in-law Josiah Wedgwood II, son of the famous potter.

"I was thus led to conclude that all the vegetable mould over the whole country has passed many times through, and will again pass many times through, the intestinal canals of worms."



Double Number—Price 30 Cents.

No. 92. The Humboldt Library, July, 1887

Published semi-monthly. Subscription price, $3.00 a year. Entered at the New York Post Office as Second-class Mail Matter.

THE FORMATION

OF

VEGETABLE MOULD

THROUGH THE ACTION OF EARTH WORMS, WITH OBSERVATIONS ON THEIR HABITS.

BY

CHARLES DARWIN, LL.D., F.R.S.

THE CELEBRATED

PIANOS * SOHMER * PIANOS

ARE AT PRESENT THE MOST POPULAR.

AND PREFERRED BY THE LEADING ARTISTS.

Munk's Idea

Though it had been mentioned earlier, the first to seriously consider the role of ocean biomixing was Walter Munk (1966):

Abyssal recipes

WALTER H. MUNK*

(*Received* 31 *January* 1966)

Abstract—Vertical distributions in the interior Pacific (excluding the top and bottom kilometer) are not inconsistent with a simple model involving a constant upward vertical velocity $w \approx 1{\cdot}2$ cm day^{-1} and eddy diffusivity $\kappa \approx 1{\cdot}3$ cm^2 sec^{-1}. Thus temperature and salinity can be fitted by exponential-like solutions to $[\kappa \cdot d^2/dz^2 - w \cdot d/dz]\, T, S = 0$, with $\kappa/w \approx 1$ km the appropriate "scale height." For Carbon 14 a decay term must be included, $[\quad]\,{}^{14}C = \mu\,{}^{14}C$; a fitting of the solution to the observed ^{14}C distribution yields $\kappa/w^2 \approx 200$ years for the appropriate "scale time," and permits w and

"...I have attempted, without much success, to interpret [the eddy diffusivity] from a variety of viewpoints: from mixing along the ocean boundaries, from thermodynamic and biological processes, and from internal tides."

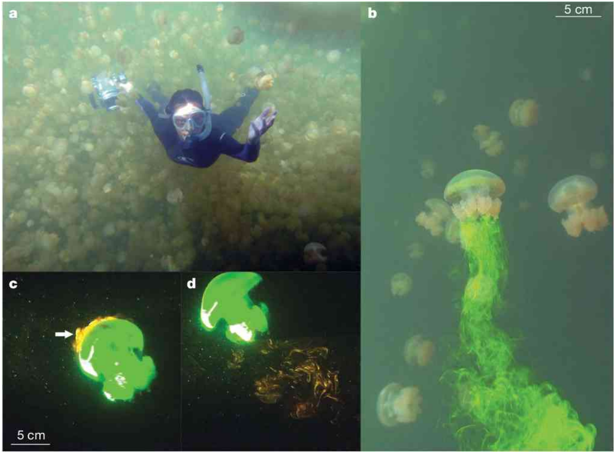
Basic claims

The idea lay dormant for almost 40 years; then

- Huntley & Zhou (2004) analyzed the swimming of 100 (!) species, ranging from bacteria to blue whales. Turbulent energy production is $\sim 10^{-5}\ \mathrm{W\ kg^{-1}}$ for 11 representative species.
- Total is comparable to energy dissipation by major storms.
- Another estimate comes from the solar energy captured: 63 TeraW, something like 1% of which ends up as mechanical energy (Dewar *et al.*, 2006).
- Kunze *et al.* (2006) find that turbulence levels during the day in an inlet were 2 to 3 orders of magnitude greater than at night, due to swimming krill.

In situ experiments

Katija & Dabiri (2009) looked at jellyfish:



[movie 1] (Palau's Jellyfish Lake.)

Displacement by a moving body

86 Mr. J. Clerk-Maxwell *on* [Mar. 10,

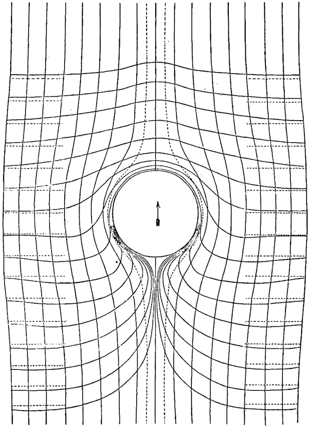


FIG. 1.
Fluid flowing past a fixed cylinder.

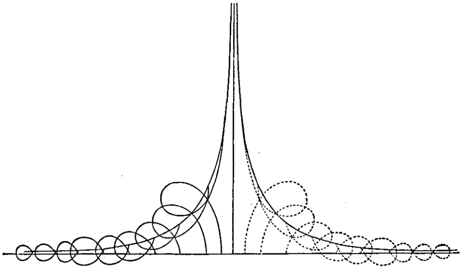
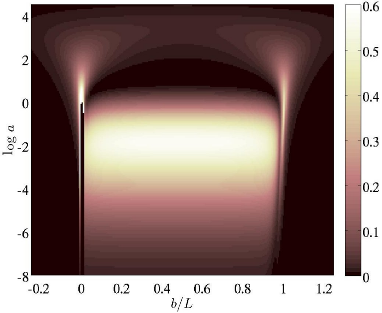


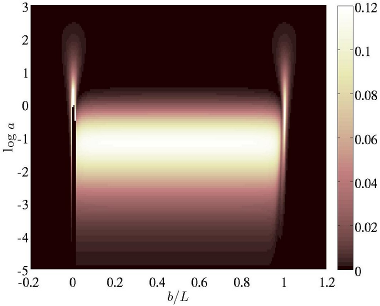
FIG. 2.
Paths of particles of the fluid when a cylinder moves through it.

Maxwell (1869); Darwin (1953); Eames *et al.* (1994); Eames & Bush (1999)

Cylinders and spheres: Displacements



$\Delta_L^2(a,b)\,a$ (cylinder)



$\Delta_L^2(a,b)\,a^2$ (sphere)

Displacement for cylinders

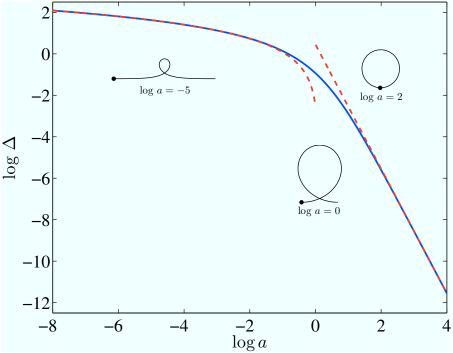
Small a: $\Delta \sim -\log a$

Large a: $\Delta \sim a^{-3}$

(Darwin, 1953)

$\int_0^1 \Delta^2(a)da \simeq 2.31$

$\int_1^\infty \Delta^2(a)da \simeq .06$

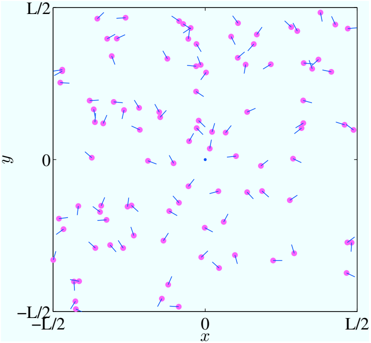


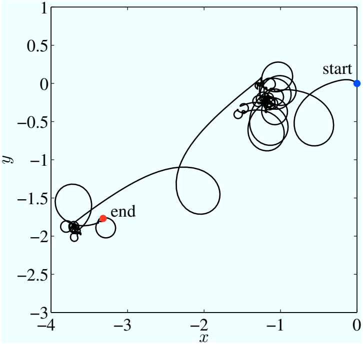
$\Longrightarrow$ 97% dominated by "head-on" collisions (similar for spheres)

Numerical simulation

- Validate theory using simple simple simulations;
- Large periodic box;
- N swimmers (cylinders of radius 1), initially at random positions, swimming in random direction with constant speed $U = 1$;
- Target particle initially at origin advected by the swimmers;
- Since dilute, superimpose velocities;
- Integrate for some time, compute $|\mathbf{x}(t)|^2$, repeat for a large number N_{real} of realizations, and average.

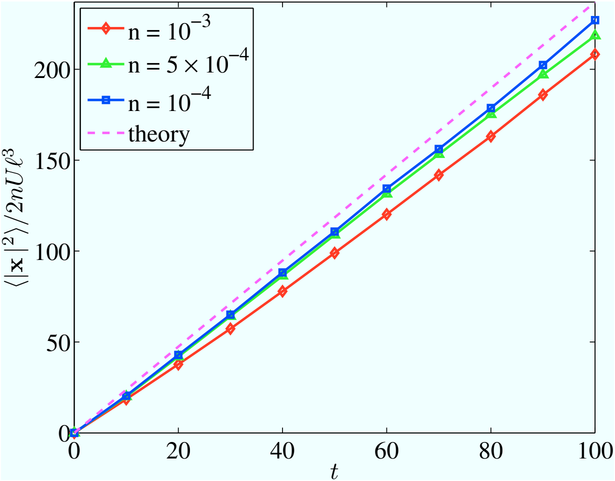
A ‘gas’ of swimmers



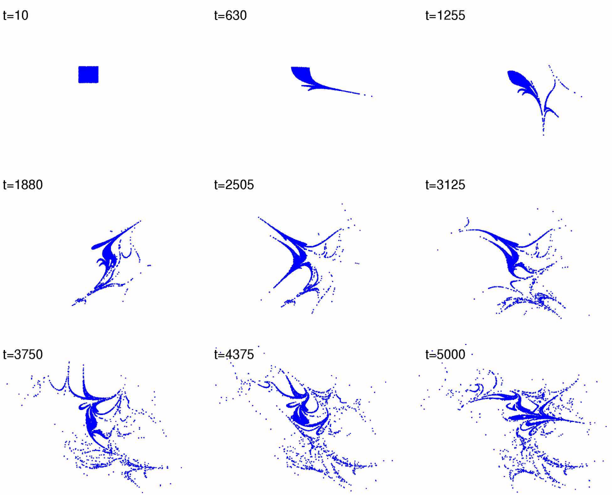


[movie 2] $N = 100$ cylinders, box size $= 1000$

How well does the dilute theory work?

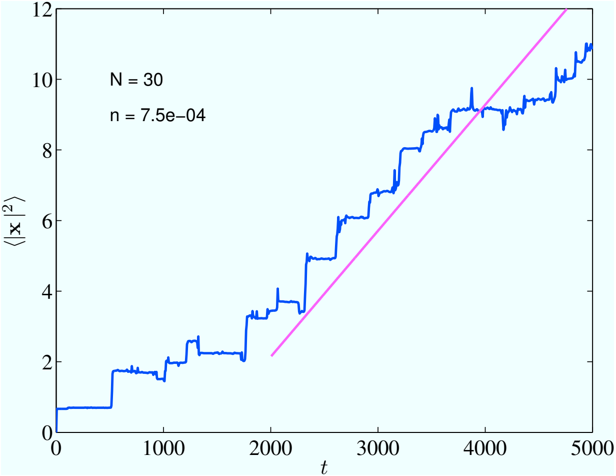


Cloud of particles



[movie 3] (30 cylinders)

Cloud dispersion proceeds by steps

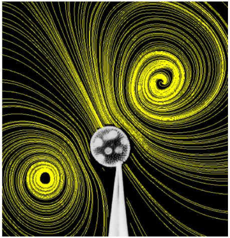


Squirmers

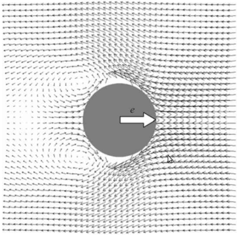
Considerable literature on transport due to microorganisms: Wu & Libchaber (2000); Hernandez-Ortiz *et al.* (2006); Saintillian & Shelley (2007); Underhill *et al.* (2008); Ishikawa (2009); Leptos *et al.* (2009)

Lighthill (1952), Blake (1971), and more recently Ishikawa *et al.* (2006) have considered squirmers:

- Sphere in Stokes flow;
- Steady velocity specified at surface, to mimic cilia;
- Steady swimming condition imposed (no net force on fluid).



(Drescher *et al.*, 2009)



(Ishikawa *et al.*, 2006)

Typical squirmer

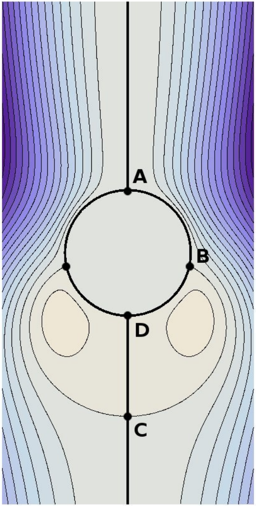
3D axisymmetric streamfunction for a typical squirmer, in cylindrical coordinates (ρ, z):

$$\psi = -\tfrac{1}{2}\rho^2 + \frac{1}{2r^3}\,\rho^2 + \frac{3\beta}{4r^3}\,\rho^2 z\left(\frac{1}{r^2} - 1\right)$$

where $r = \sqrt{\rho^2 + z^2}$, $U = 1$, radius of squirmer $= 1$.

Note that $\beta = 0$ is the sphere in potential flow.

We will use $\beta = 5$ for most of the remainder.

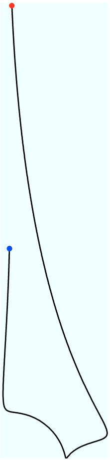


Particle motion for squirmer

A particle near the squirmer's swimming axis initially (blue) moves towards the squirmer.

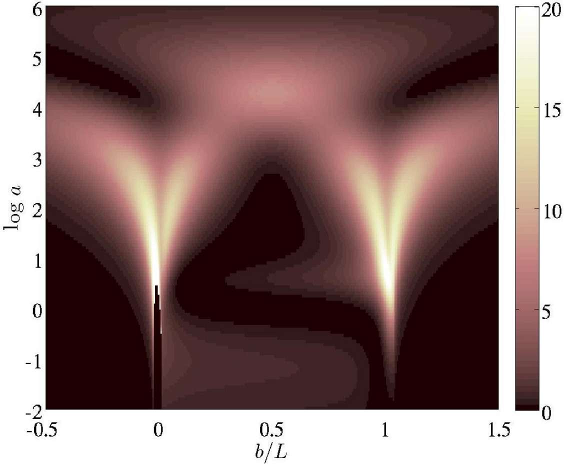
After the squirmer has passed the particle follows in the squirmer's wake.

(The squirmer moves from bottom to top.)

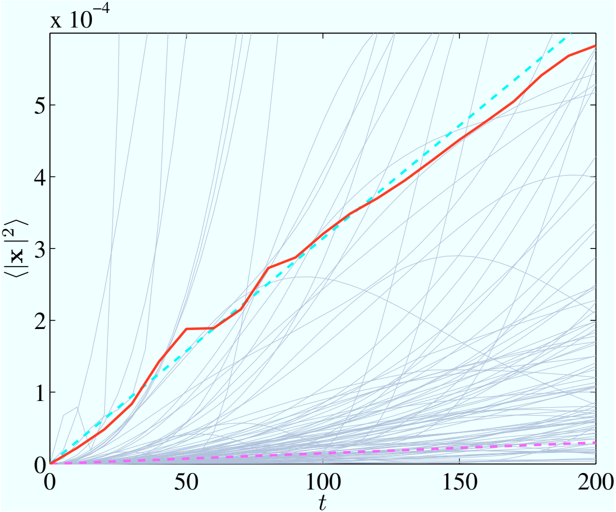


[movie 4]

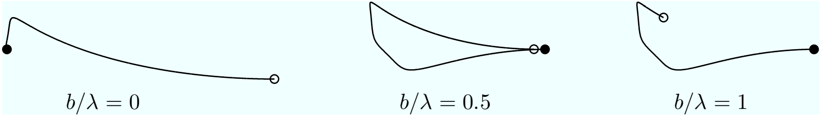
Squirmer displacements



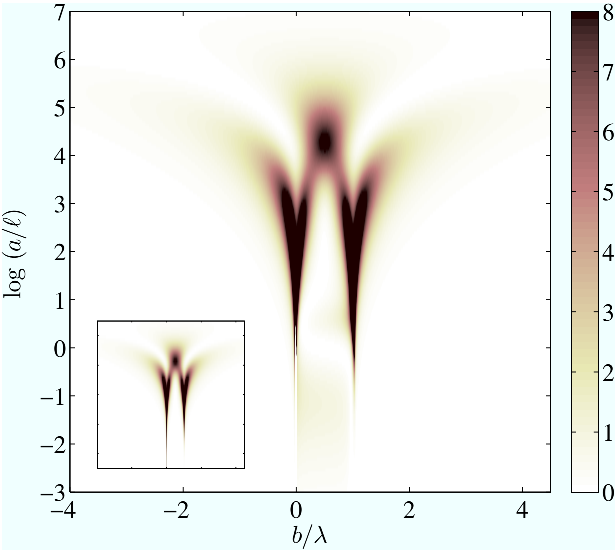
Squirmers: Transport



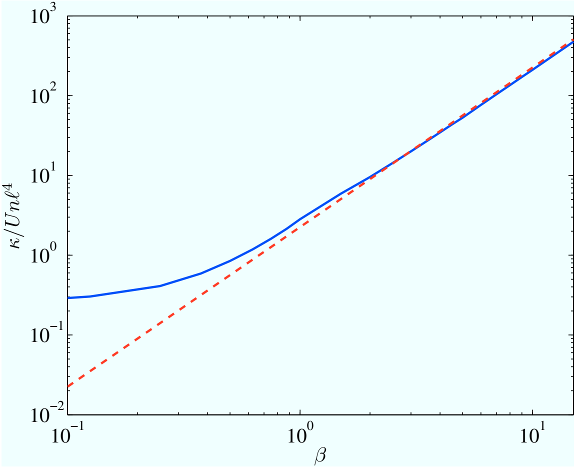
Squirmers: Trajectories



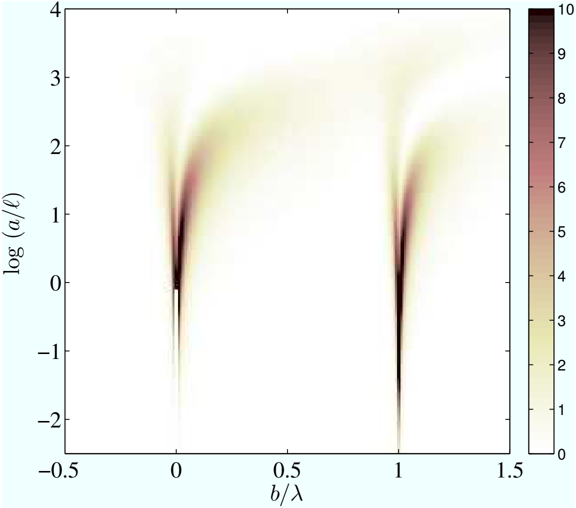
Far field: Displacements



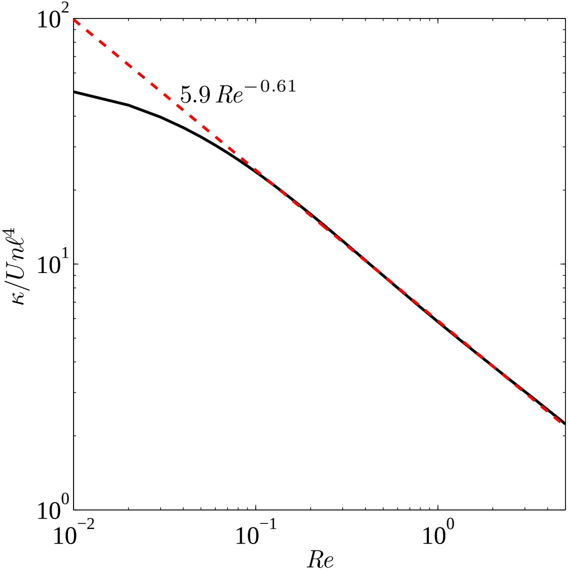
Far field: transport



Finite Reynolds number: Displacements



Finite Reynolds number: Transport



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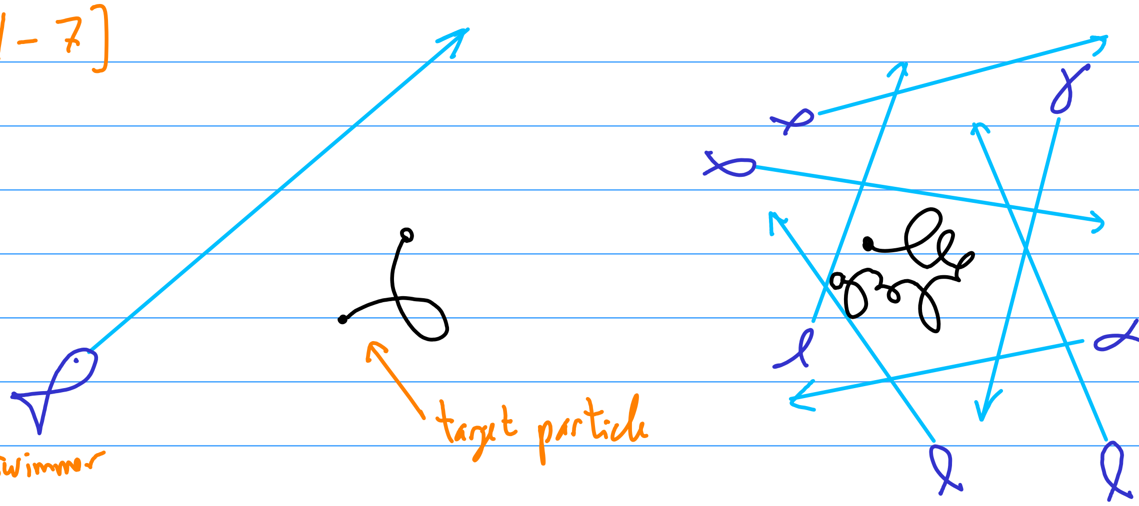
GFD Lectures: Swimming & Swirling

2010/06/23

Lecture 2: Stirring by swimming organisms

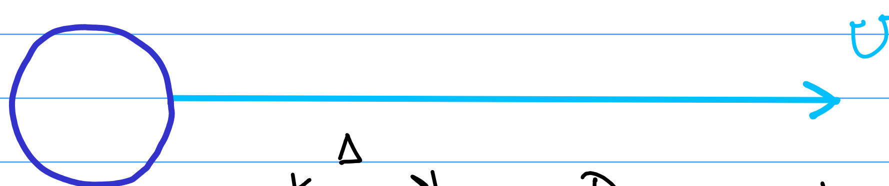
[Slide 1-7]

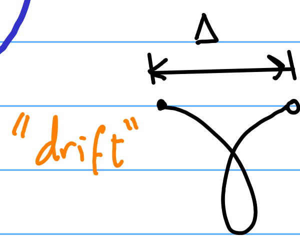
Vision:



Target particle moves under the influence of many swimmers

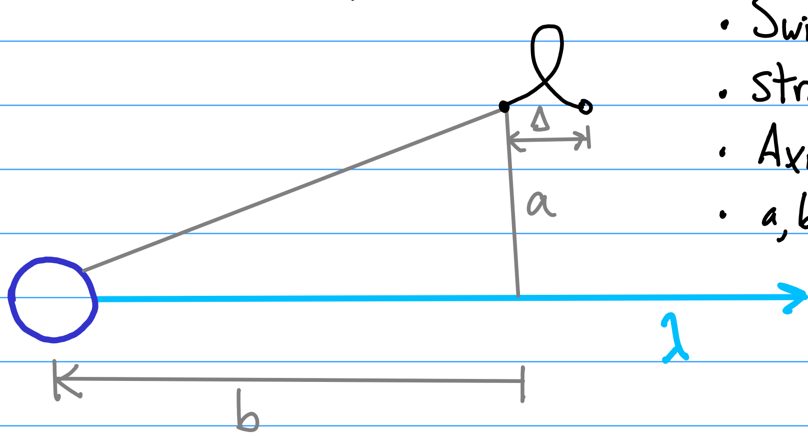
Single swimmer: take a cylinder





Darwin Jr.'s "elastica"
(in potential flow)

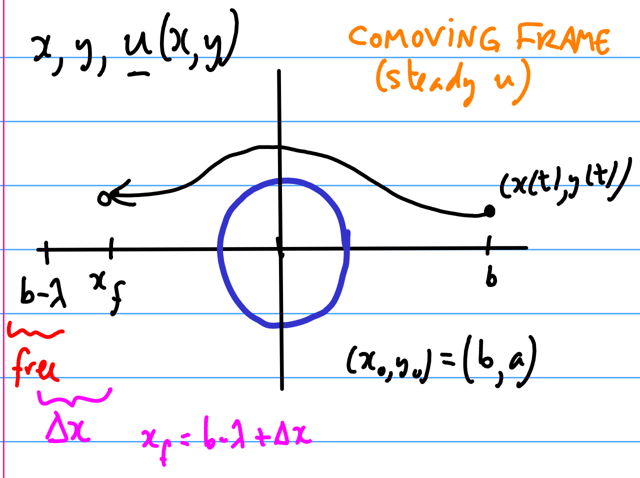
How do we compute Δ?

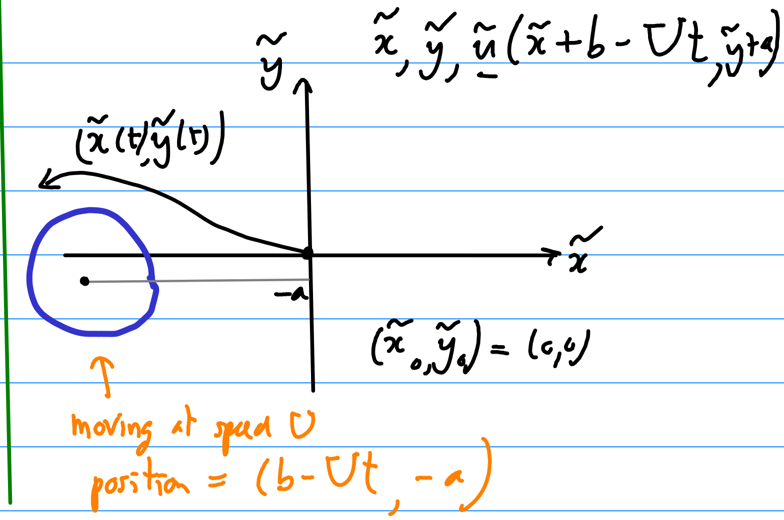


- Swimming velocity is U (const)
- Straight line for distance λ.
- Axially-symmetric, steady swimmer
- a, b are "impact parameters" $(a>0)$

Compute $\Delta_\lambda(a,b)$

Do 2D case (axisymmetric 3D similar):





$$\frac{d\tilde{x}}{dt} = \tilde{u}(\tilde{x}(t)+b-Ut, \tilde{y}(t)+a), \qquad x = \tilde{x}+b-Ut$$

$$\frac{dx}{dt} + U = \tilde{u}(x,y) \iff \boxed{\frac{dx}{dt} = -U + \tilde{u}(x,y) = u(x,y)}$$

autonomous (better!)

$$-\lambda + \Delta x = \int_0^{T=\lambda/U} u(x(t), y(t))\, dt$$

need both

For y: $\frac{dy}{dt} = \tilde{v}(x,y)$

Alternate form: $T = \frac{\lambda}{U} = \int_b^{x_f} \frac{dx}{u(x,y)}$, $x_f = b - \lambda + \Delta x$

how far particle moves when "free-streaming"

$$\frac{\lambda}{U} = \int_b^{b-\lambda+\Delta x} \frac{dx}{u} = -\int_b^{b-\lambda+\Delta x} \frac{dx}{|u|} = \int_{b-\lambda+\Delta x}^{b} dx/|u|$$

$$= \int_{b-\lambda}^{b} \frac{dx}{|u|} + \int_{b-\lambda+\Delta x}^{b-\lambda} \frac{dx}{|u|}$$

If particle doesn't move much and $|b-\lambda|$ "large", then $|u| \simeq U$

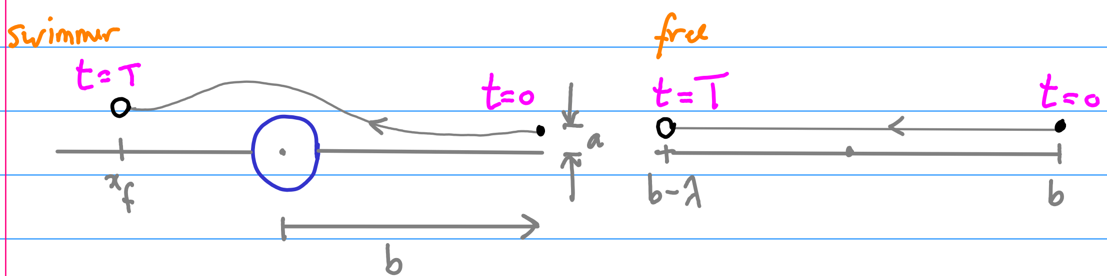
$$\frac{\lambda}{U} \simeq \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\Delta x}{U} \iff \Delta x = \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\lambda}{U}$$

$$\boxed{\Delta x \simeq \int_{b-\lambda}^{b} \left(\frac{1}{|u|} - \frac{1}{U} \right) dx}$$

Better form, since now can take $b \to \infty$, $b-\lambda \to -\infty$ if we want.

"Rayleigh form"

Intuitively, this formula measures the "lag" behind a free-streaming particle:



$T = \dfrac{\lambda}{U}$

2D incompressible: $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

$$\psi(x_f, a+\Delta y) = \psi(b, a)$$ Same streamline

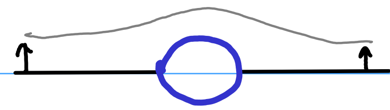
$$\psi(b-\lambda+\Delta x, a+\Delta y) = \psi(b, a)$$ solve for Δy, given Δx

If $|b-\lambda| \gg \Delta x$, $\boxed{\psi(b-\lambda, a+\Delta y) \simeq \psi(b,a)}$ solve for Δy

If also $\Delta y \ll a$, $\psi(b-\lambda, a) + \Delta y \underbrace{\partial_y \psi(b-\lambda, a)}_{u(b-\lambda, a)} \simeq \psi(b,a)$

$$\Delta y \simeq \frac{\psi(b,a) - \psi(b-\lambda, a)}{u(b-\lambda, a)}$$

Now for infinite λ, we have:



$$\Delta y = \frac{\psi(\infty, a) - \psi(-\infty, a)}{U} = 0!$$

$\boxed{\Delta y = 0}$ for $\lambda \to \infty$, $b - \lambda \to -\infty$

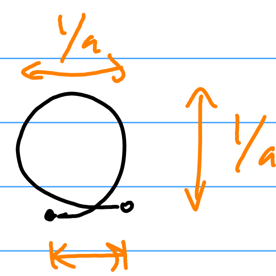
Cylinder in potential flow:

$$\psi(x,y) = -Uy\left(1 - \frac{\ell^2}{x^2+y^2}\right) \qquad \text{Set } U = \ell = 1$$

in fixed frame

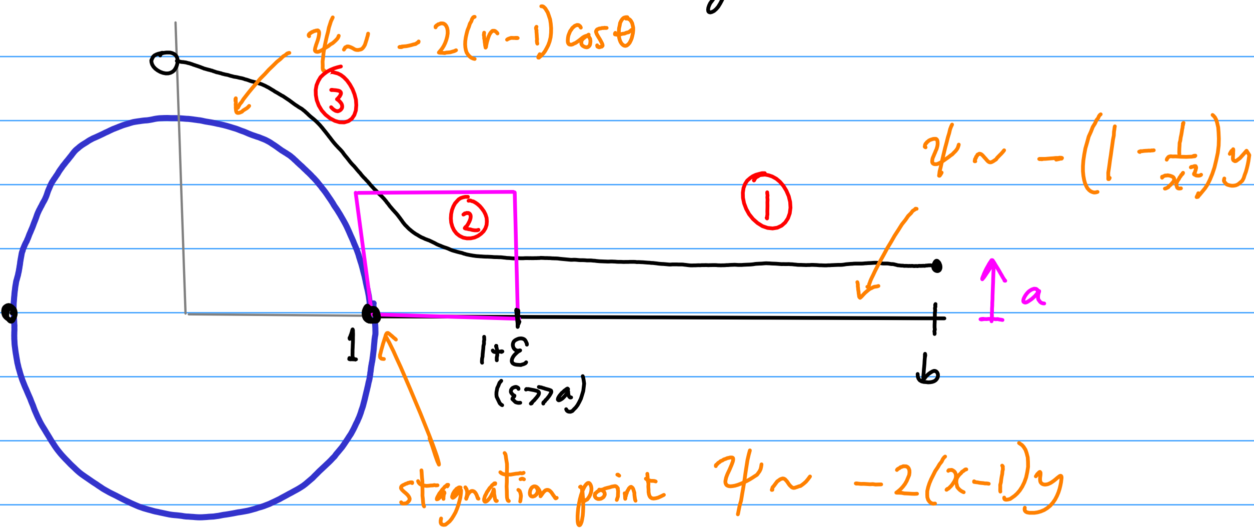
Far away, $\tilde{\psi} \sim \frac{y}{r^2}$, so $\tilde{u} \sim \frac{1}{r^2}$

However, trajectories are almost closed,



Net result is $\Delta(a) \sim \frac{1}{a^3}$ Much smaller than overall excursion!

The limit $a \ll 1$ is more interesting:



free flow U (to the left)

Need to calculate $\int \left(\frac{1}{u}+1\right)dx$ over each region ①, ②, ③.

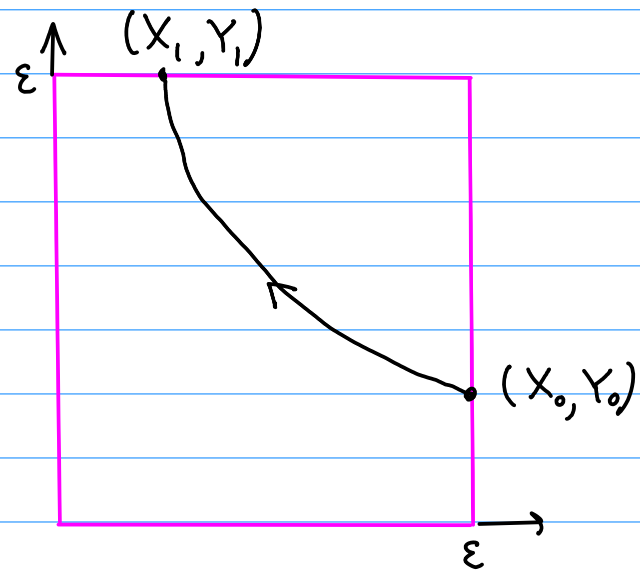
Region 1: $\psi_0 = \psi(b,a) = -(1-b^{-2})\,a$

$$u = -(1-x^{-2}), \quad T_1 = \int_b^{1+\varepsilon} \left(\frac{1}{u}+1\right) dx = \int_b^{1+\varepsilon} \frac{dx}{1-x^2}$$

transit time

After using $\varepsilon \ll 1$, $b \gg 1$: $T_1 \simeq \frac{1}{2}\log(2/\varepsilon) + \varepsilon/4 - b^{-1} + O(\varepsilon^2, b^{-2})$

Region 2:



$X = x-1, \quad Y = y$

$\psi = -2XY$

neglect

At X_0, Y_0,

$$\psi = -2\underbrace{X_0}_{\varepsilon}Y_0 = -(1-b^{-2})a$$

$$\Rightarrow Y_0 = \frac{a}{2\varepsilon}$$

$(X_0, Y_0) = (\varepsilon, a/2\varepsilon)$

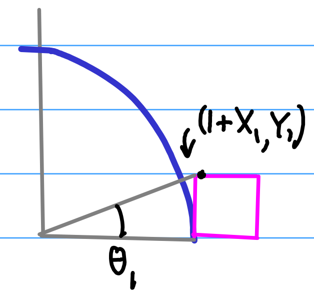
But also $Y_1 = \varepsilon$, so $X_1 = a/2\varepsilon$.

small

$$T_2 = \int_{X_0}^{X_1} \left(\frac{1}{u}+1\right) dx = \int_{\varepsilon}^{a/2\varepsilon} \left(\frac{1}{(-2X)}+1\right) dx = -\frac{1}{2}\log\left(\frac{a}{2\varepsilon^2}\right) + \frac{a}{2\varepsilon} - \varepsilon$$

$$u = -1 + \frac{\cos 2\theta}{r^2}$$

Region 3:



$$T_3 = \int_{\theta_1}^{\pi/2} \left(\frac{1}{u} + 1\right) \underbrace{\frac{dx}{d\theta}}_{-r\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\theta_1}^{\pi/2} \frac{\cos 2\theta}{\sin\theta}\, d\theta \qquad , \ \theta_1 \text{ small.}$$

$$T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\theta_1 + O(\theta_1^2)$$

$$\tan\theta_1 = \frac{Y_1}{1+X_1} = \frac{\varepsilon}{1 + a/\varepsilon} \simeq \varepsilon(1 - a/\varepsilon) = \varepsilon - a$$

$$\therefore\ T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\varepsilon + \frac{1}{2}\frac{a}{\varepsilon} + O\left((a/\varepsilon)^2\right)$$

Add everything together:

divergent $\log\varepsilon$ terms cancel

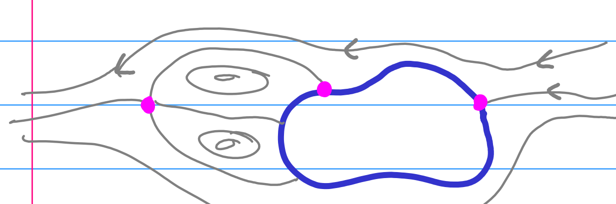
$$T = T_1 + T_2 + T_3 = \left(\frac{1}{2}\log(2/\cancel{\varepsilon}) - 6^{-1}\right) + \left(-\frac{1}{2}\log\left(\frac{a}{2\cancel{\varepsilon}}\right)\right) + \left(-1 + \frac{1}{2}\log 2 - \cancel{\frac{1}{2}\log\varepsilon}\right)$$

$$T = \underbrace{-\frac{1}{2}\log a}_{} - 1 + \frac{3}{2}\log 2 - 6^{-1} \quad \text{to leading order.}$$

Dominant term for small a.
Comes only from region 2, near stagnation point.

$T \to \infty$ for $a \to 0$. particle gets stuck!

The total drift is given by $2T$, since the body is fore-aft symmetric.



In general, the coefficient of $\log a$ is given by summing over the linearization coeffs for each (hyperbolic) stagnation pt encountered. (not true for no-slip!)

Note that to pick up the $-\log a$ contribution, the target particle must come in the vicinity of the stagnation point.

$$\Delta_\lambda(a,b) = \begin{cases} -\log a &, \quad 0 \leq b \leq \lambda \\ \text{(neglect)} &, \quad \text{otherwise} \end{cases}$$

Effective diffusivity:

What we have: $\Delta_\lambda(a,b)$ Need: effective diffusivity

Constants: U, ℓ, λ, n Random: a, b

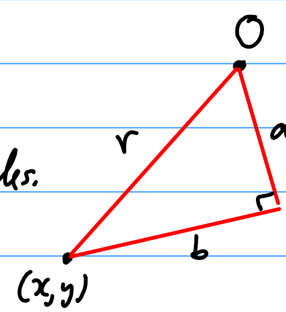
number density

If we pick a random point in space, what is PDF of a, b?

$$\frac{1}{V}\, dx\, dy \;\rightarrow\; p(a,b)\, da\, db$$

volume

Assume target particle at origin:



- Hard way: compute (a,b) from (x,y), transform variables.
- Easier: note (a,b) just like (x,y), but rotated, and $a>0$.

irrelevant, by isotropy

Hence: $$\frac{1}{V}\, dx\, dy = \frac{1}{V}\, 2\, da\, db \qquad \text{2D}$$

3D

In 3D, $\frac{1}{V}\,dx\,dy\,dz = \frac{1}{V}\,2\pi a\,da\,db$ ← like cylindrical coordinates, integrated over θ.

Now, assume target particle is "kicked" by swimmer:

$$\underline{x}_N = \underbrace{\underline{x}_0}_{0} + \sum_{k=1}^{N} \Delta_\lambda(a_k, b_k)\,\hat{r}_k \qquad a_k,\ b_k,\ \hat{r}_k \text{ random independent \& identical}$$

On average, particle goes nowhere: $\langle \underline{x}_N \rangle = 0$

$$\langle |\underline{x}_N|^2 \rangle = \sum_{k=1}^{N} \langle \Delta_\lambda^2(a_k, b_k)\,\hat{r}_k \cdot \hat{r}_k \rangle + \text{vanishing cross terms } \langle \hat{r}_k \rangle = 0$$

$$= N \langle \Delta_\lambda^2(a, b) \rangle$$

$$= \frac{N}{V} \int \Delta_\lambda^2(a, b)\,2\,da\,db \qquad \text{2D}$$

What is N? # of "collisions" $N = t/T$ ← elapsed time, ← "mean free time"

$T = \lambda / U$ λ = mean free path

Hence, $\langle |x(t)|^2 \rangle = \frac{Ut}{\lambda} \underbrace{\frac{1}{V}}_{\text{What about this?}} \int \Delta_\lambda^2(a, b)\,2\,da\,db$ Only one swimmer, so $\frac{1}{V} = n$, the number density

$x_N \to x(t)$

$$\langle |x(t)|^2 \rangle = 2\frac{Unt}{\lambda} \int \Delta_\lambda^2(a, b)\,da\,db = 2dkt = 4kt$$

↑ dimension of space ↑ effective diff

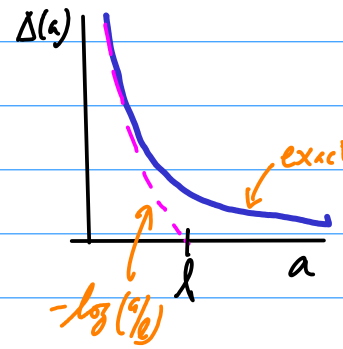
This depends on integral of squared displacement.
Actual mass displaced could be 0!

$$\kappa = \boxed{\begin{cases} \dfrac{U n}{2\lambda} \displaystyle\int \Delta_\lambda^2(a,b)\, da\, db & \text{2D} \\[2ex] \dfrac{\pi U n}{3\lambda} \displaystyle\int \Delta_\lambda^2(a,b)\, a\, da\, db & \text{3D} \end{cases}}$$

effective diffusivity

Recall our approximate form for cylinder: $\Delta_\lambda(a,b) = \begin{cases} -\log a, & 0 \leq b \leq \lambda \\ 0, & \text{otherwise} \end{cases}$ ← ℓ

Cylinder: $\kappa \simeq \dfrac{2 U n}{\cancel{\lambda}} \cancel{\lambda} \displaystyle\int_0^\ell \log^2\left(\frac{a}{\ell}\right) da$



$$\int \log^2 x\, dx = x \log^2 x - 2x \log x + 2x$$

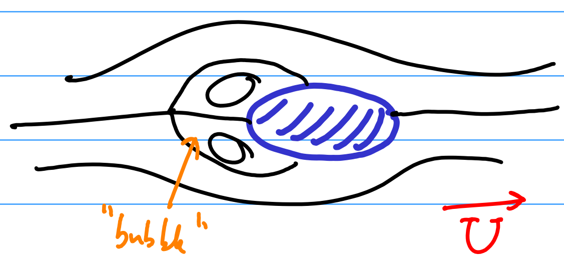
$$\int_0^1 \log^2 x\, dx = 2 \qquad \text{(numerical answer: 2.37)}$$

$$\boxed{\kappa \simeq U n \ell^3} \qquad (\text{numerical: } \kappa = 1.19\, U n \ell^3)$$

Note that this is completely independent of λ!

⇒ see computer simulations

Another example: consider a swimmer with a bubble "wake":



If a particle is trapped in the bubble, moves by λ.

$$\Delta_\lambda(a,b) = \begin{cases} \lambda, & \text{particle inside bubble} \\ \text{neglect}, & \text{otherwise} \end{cases}$$

Total volume of bubble

$$6\kappa = \frac{2Un}{\cancel{\lambda}} \int_{\substack{\text{inside}\\\text{bubble}}} \lambda^{\cancel{2}}\, da\, db = Un\lambda V_{bubble}$$

The 2 goes away since $2\,da\,db$ is volume element

$$\boxed{\kappa = \frac{1}{6} Un\lambda V_{bubble}}$$

V_{bubble} = area in 2D
= volume in 3D

Now this depends on path length λ. This can be much larger than for untrapped fluid. Real swimmer probably in between

(Viscous swimmer with boundary layer: $\kappa \sim \log\lambda$)

swimmer	λ-dependence	far/near field dominance
potential (slip)	none	near
viscous (slip) squirmer	none	far
viscous (no-slip)	$\log\lambda$	near
trapped	λ	near

More topics:
- Green-Kubo
- Wakes
- Far field
- Levy flights
- Stratification

GFD Lectures: Swimming & Swirling 2010/06/24

Lecture 3: Local Stretching Theories

Antonsen et al. '96
Balkovsky & Fouxon '99

(AD) $\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa \nabla^2 \theta.$

For this lecture, think of θ as a "patch"

Last time we examined $\underline{u} = (\lambda x, -\lambda y)$. Let's try something more general:

$$\underline{u} = \underbrace{\underline{U}}_{\text{const.}} + \underline{x}\cdot \underbrace{A}_{\text{const.}}, \qquad \nabla\cdot\underline{u} = \operatorname{trace} A = 0.$$

Let $\langle f \rangle = \int_\Omega f \, dV \qquad (\Omega = \mathbb{R}^2 \text{ or } \mathbb{R}^3)$

Solve (AD) using moments: $\boxed{c_i = \dfrac{\langle x_i \theta\rangle}{\langle\theta\rangle}}$ $\qquad (\partial_t \langle\theta\rangle = 0)$

(AD) $\rightarrow$ $\partial_t \langle x_i \theta\rangle + \langle x_i \nabla\cdot((\underline{U} + \underline{x}\cdot A)\theta)\rangle = \kappa \langle x_i \nabla^2\theta\rangle$

$$\partial_t \langle x_i \theta\rangle - \langle (U_j + x_\ell A_{\ell j})\theta \cdot \underbrace{\partial_j x_i}_{\delta_{ji}} \rangle = \kappa \, (0)$$

$$\langle\theta\rangle \partial_t c_i - U_i \langle\theta\rangle - A_{\ell i} \langle\theta\rangle c_\ell = 0$$

$$\boxed{\partial_t \underline{c} = \underline{U} + \underline{c}\cdot A}$$

Motion of center of mass

Next moments: $\boxed{m_{ij} = \dfrac{\langle x_i x_j \theta\rangle}{\langle\theta\rangle} - c_i c_j}$

Again, multiply (AD) by $x_i x_j$ and $\langle\cdot\rangle$.

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = \langle x_i x_j \partial_k((U_k + x_\ell A_{\ell k})\theta)\rangle$$

$$= -\langle (U_k + x_\ell A_{\ell k})(\delta_{ik} x_j + x_i \delta_{jk})\theta\rangle$$

$$= -U_i c_j \langle\theta\rangle - U_j c_i \langle\theta\rangle - A_{\ell i} \underbrace{\langle x_\ell x_j \theta\rangle}_{\langle\theta\rangle(m_{\ell j} + c_\ell c_j)} - A_{\ell j} \underbrace{\langle x_\ell x_i \theta\rangle}_{\langle\theta\rangle(m_{\ell i} + c_\ell c_i)}$$

$$\partial_t (c_i c_j) = c_i \partial_t c_j + c_j \partial_t c_i$$
$$= c_i(U_j + A_{\ell j} c_\ell) + c_j (U_i + A_{\ell i} c_\ell)$$

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = -\left(\partial_t(c_i c_j) + A_{\ell i} m_{\ell j} + A_{\ell j} m_{\ell i}\right)\langle\theta\rangle$$

That's the hard part! Next:

$$\langle x_i x_j \nabla^2\theta\rangle = \langle\theta\,\nabla^2(x_i x_j)\rangle = 2\langle\theta\rangle\,\delta_{ij}$$

So finally:

$$\partial_t m_{ij} = A_{\ell i} m_{\ell j} + A_{\ell j} m_{i\ell} + 2\kappa\,\delta_{ij}$$

Let $(M)_{ij} = m_{ij}$ (symmetric matrix)

$$\boxed{\partial_t M = M\cdot A + A^T\cdot M + 2\kappa I}$$

Moment of inertia equation.
"spread" of patch

Time to solve these equations!

$$\underline{c}(t) = \underline{c}(0)\cdot e^{At} + \underline{U}\cdot\int_0^t e^{A(t-\tau)}\,d\tau$$

$$M(t) = e^{A^T t} \cdot M(0) \cdot e^{At} + 2\kappa \int_0^t \underbrace{e^{A^T(t-\tau)} \cdot e^{A(t-\tau)}}_{\text{Can't write as } e^{(A+A^T)(t-\tau)} \text{ unless } [A,A^T]=0} d\tau$$

Normal matrix

Let $M = RDR^T$, R orthogonal, D diagonal

$$\dot{M} = \dot{R}DR^T + RD\dot{R}^T + R\dot{D}R^T = RDR^TA + A^TRDR^T + 2\kappa I$$

$$R^T\dot{R}D + D\dot{R}^TR + \dot{D} = D\underbrace{R^TAR}_{\tilde{A}} + \underbrace{R^TA^TR}_{\tilde{A}^T}D + 2\kappa I$$

Now: $\frac{d}{dt}(R^TR) = \dot{R}^TR + R^T\dot{R} = \frac{d}{dt}(I) = 0$, so $(R^T\dot{R})^T = \dot{R}^TR = -R^T\dot{R}$

$$\Rightarrow R^T\dot{R} \text{ is antisymmetric}$$

$$[R^T\dot{R}D]_{ii} = (R^T\dot{R})_{ik}D_{ki} = (R^T\dot{R})_{ii}D_{ii} = 0$$

(no sum)

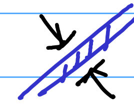
$$\dot{D}_{ii} = D_{il}\tilde{A}_{li} + \tilde{A}_{li}D_{li} + 2\kappa$$

$$\dot{D}_{ii} = 2\tilde{A}_{ii}D_{ii} + 2\kappa$$

Write $D_{ii} = e^{2\rho_i}$, with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d$.

$$\dot{D}_{ii} = 2e^{2\rho_i}\dot{\rho}_i, \qquad \boxed{\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}}$$

Great equation: • $\tilde{A} = R^T A R$ → rotated velocity gradient matrix.

• $e^{-2\rho_i}$ → negligible unless $\rho_i < 0$ 

compression

Moral: the directions of contraction or compression play an important role.

Now we need an equation for R: off-diagonal terms.

$$[R^T \dot{R} D]_{ij} = (R^T \dot{R})_{il} D_{lj} = (R^T \dot{R})_{ij} D_{jj} \quad , \quad i \neq j$$

$$[D \dot{R}^T R]_{ij} = D_{il} (\dot{R}^T R)_{lj} = D_{ii} (\dot{R}^T R)_{ij} = -(R^T \dot{R})_{ij} D_{ii}$$

(no sum over i, j)

$$(D_{jj} - D_{ii})(R^T \dot{R})_{ij} = D_{ii} \tilde{A}_{ij} + \tilde{A}_{ji} D_{jj}$$

$$(R^T \dot{R})_{ij} = \Omega_{ij} \iff \boxed{\dot{R} = R \Omega}$$

$$\boxed{\Omega_{ij} = \frac{e^{2\rho_i} \tilde{A}_{ij} + e^{2\rho_j} \tilde{A}_{ji}}{e^{2\rho_j} - e^{2\rho_i}}}$$

Not completely obvious what this means...

$(= 0$ for $i = j)$

Almost always true for long time, esp. in 2D, 3D with $\rho_1 + \rho_2 (+\rho_3) = 0$.
Usually a symmetry can break this, or fails locally.

Assume we have separation between the eigenvalues: $e^{2\rho_i} \gg e^{2\rho_j}$, $i < j$

$$\Omega_{ij} \simeq \frac{e^{2\rho_i} \tilde{A}_{ij} + \cancel{e^{2\rho_j} \tilde{A}_{ji}}}{\cancel{e^{2\rho_j}} - e^{2\rho_i}} = -\tilde{A}_{ij} \quad , \quad i < j$$

$$\boxed{\Omega_{ij} \simeq \begin{cases} -\tilde{A}_{ij}, & i<j \\ \tilde{A}_{ji}, & i>j \end{cases}}$$
(large t)

Independent of eigenvalues!

Can solve: $\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}$ since $\tilde{A}$ indep. of ρ_i

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\left[1 + 2\kappa e^{-2\rho_{0i}} \int_0^t \exp\left(-2\mathcal{A}_i(t')\right)dt'\right]$$

diffusion

where
$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')dt'$$

When diffusion negligible: $\rho_i(t) = \rho_{i0} + \int_0^t \tilde{A}_{ii}(t')dt'$

In fact, solving the equations for ρ_i, R, $\underline{\underline{\kappa=0}}$, is not a bad way of computing <u>Lyapunov exponents:</u>

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t}\rho_i(t)} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$$

(Some numerical issues regarding orthogonality of R.)

Convergence: famous Oseledec Multiplicative ergodic theorem

Now comes the stochastic part: could have formulated things in terms of an SDE. But we take a shortcut:

$$\rho_i(t) = \rho_{0i} + \sum_t \tilde{A}_{ii}$$

$\Longleftarrow$ sum of uncorrelated random numbers (more later)

What is PDF of $p_i(t)$?

<u>Recall</u>: if x_i are i.i.d. and $X = \sum_{i=1}^{N} x_i$ $\quad \overline{x_i} = \xi$, $\quad \overline{x_i^2} - \overline{x_i}^2 = \sigma^2$

What is PDF of X ? CENTRAL LIMIT THEOREM

mean of X

$$P(X,N) \sim \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left(-\frac{(X - N\xi)^2}{2N\sigma^2}\right)$$

Valid for: (i) $N \gg 1$; (ii) $\underline{\underline{X - N\xi < \sqrt{N}\,\sigma}}$

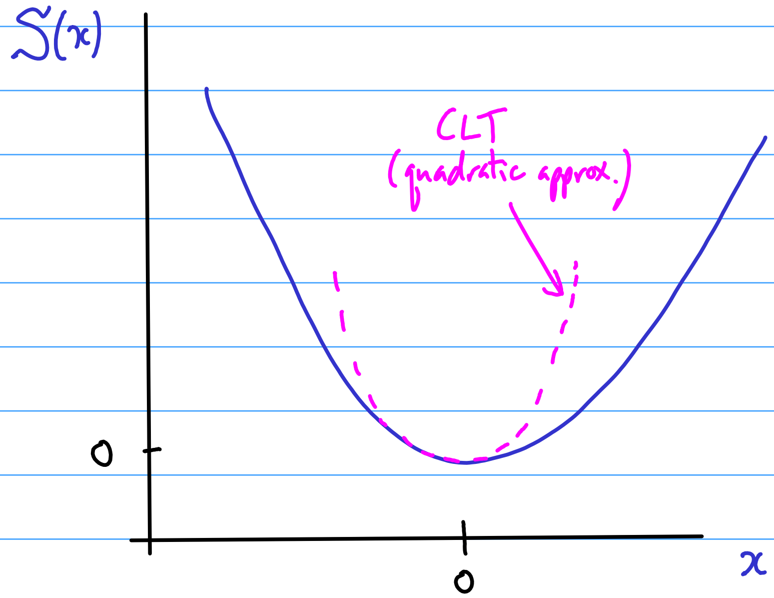
This second restriction is less commonly stated: it tells us that the CLT is not valid in the <u>tails</u>. The CLT tends to vastly underestimate the probability of rare events, or <u>black swans</u> as is trendy to call them these days. These tails matter for mixing.

More generally,

$$\boxed{P(X,N) \sim \exp\left(-N\,S\left(\frac{X - N\xi}{N}\right)\right)}$$

Large deviation form

$S(x)$ is a convex function with $S(0) = S'(0) = 0$.



$$S(x) = \overset{0}{S(0)} + \overset{0}{S'(0)}\,x + \frac{1}{2} S''(0)\, x^2 + \dots$$

$$S\left(\frac{X - N\xi}{N}\right) = \frac{1}{2} S''(0) \frac{(X - N\xi)^2}{N^2} + \dots$$

$$\exp\left(-N S\left(\frac{X}{N} - \xi\right)\right) \sim \exp\left(-S''(0)\frac{(X - N\xi)^2}{2N}\right)$$

Compare to CLT: $\boxed{S''(0) = \frac{1}{\sigma^2}}$

Can also express in terms of mean: $x = \frac{X}{N}$

$$P(x, N) \sim \exp\left(-N S(x - \xi)\right)$$

<u>Example:</u> Binomial distribution for x_i $(-1$ or 1, mean $0)$

$$p(x_i) = \frac{1}{2}\delta(x_i + 1) + \frac{1}{2}\delta(x_i + 1)$$

$$e^{-s(k)} = \int p(\xi)\, e^{-ik\xi}\, d\xi \qquad \text{characteristic function}$$

$$= \frac{1}{2}\left(e^{ik} + e^{-ik}\right) = \cos k$$

For the mean $x = \frac{1}{N}\sum x_i$:

$$P(x, N) = \int p(x_1) \cdots p(x_N)\, \delta\left(\frac{x_1 + \cdots + x_N}{N} - x\right) dx_1 \cdots dx_N$$

$$e^{-S(k)} = \int P(x, N)\, e^{-ikx}\, dx$$

$$= \int p(x_1) \cdots p(x_N)\, e^{-ik(x_1 + \cdots + x_N)/N}\, dx_1 \cdots dx_N$$

$$= \prod_{i=1}^{N} \int p(x_i)\, e^{-ikx_i/N}\, dx_i = \left(\int p(\xi)\, e^{-ik\xi/N}\, d\xi\right)^N$$

$$= \left(e^{-s(k/N)}\right)^N = \cos^N(k/N)$$

Inverse Fourier

$$P(x, N) = \frac{1}{2\pi} \int e^{-S(k)}\, e^{ikx}\, dk = \frac{1}{2\pi} \int \cos^N\left(\frac{k}{N}\right) e^{ikx}\, dk$$

$$= \frac{N}{2\pi} \int \cos^N K\; e^{iNKx}\, dK, \quad K = k/N.$$

$$= \frac{N}{2\pi} \int e^{N(\log \cos K + iKx)}\, dK$$

For N large, look for saddle (stationary) point:

$$\frac{d}{dK} \underbrace{(\log \cos K + iKx)}_{H(K,x)} = -\tan K + ix = 0 \text{ when } K = K_{sp}.$$

$$\tan K_{sp} = -ix$$

$$H(k,x) = H(k_{sp},x) + H'(k_{sp},x)(k-k_{sp}) + \frac{1}{2}H''(k_{sp},x)(k-k_{sp})^2 + \cdots$$

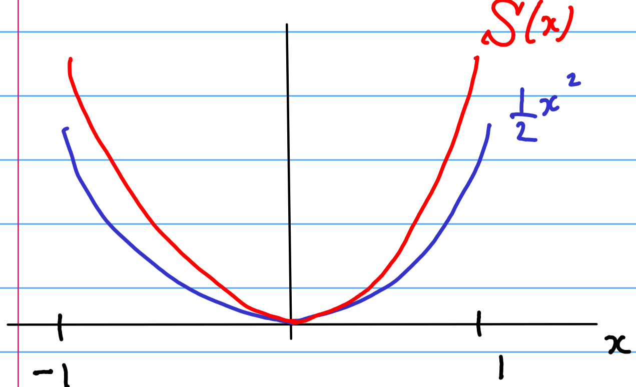
With this approximation the inverse transform is a Gaussian integral.
Get finally (skip some steps ... see Aosta lecture notes)

$$P(x,N) = \sqrt{\frac{N S''(0)}{2\pi}}\; e^{-N S(x)}, \quad \text{with}$$

$$\boxed{S(x) = -\frac{1}{2}(x+1)\log\left(\frac{1-x}{x+1}\right) + \log(1-x)} \qquad -1 \le x \le 1$$

Note $S(0) = 0$, $S'(x) = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right)$, so $S'(0) = 0$

$$S''(x) = \frac{1}{1-x^2}, \quad \text{so} \quad S''(0) = 1$$



$S(x)$ is called the
rate function
Cramer function
entropy function

For this case the Gaussian form overestimates the probability in the tails (not typical)

More refs:
Falkovich et al. 2001
Zeldovich et al. 1984

What this has to do with mixing?

For $\kappa = 0$, we argued that if A_{ii} is a random var., then ρ_i are distributed according to large deviation form (for large t).

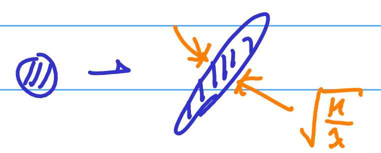
$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{\theta(\rho_1)}_{\substack{\text{ordering} \\ \rho_1 \geq \rho_2}} \underbrace{\delta(\rho_1 + \rho_2)}_{\text{incompressibility}}$$

in 2D ($d = 2$). (return 3D later)

$\theta(x) =$ step function

$$\lambda_1 = \lim_{t \to \infty} \rho_1 / t = \text{Lyapunov exp.}$$
> 0 (for chaotic flows)

What happens with diffusion? Recall "filament":



The contracting direction "stabilizes" near the Batchelor width $\sqrt{\frac{\kappa}{\lambda_1}}$.
↳ or "freezes"

Shraiman & Siggia 199?
Chertkov et al. 1997
Balkovsky & Fouxon 199?

$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{P_{stab}(\rho_2)}_{\text{stationary distribution.}}$$

If we assume, say, an initial Gaussian "patch" of passive scalar, then the concentration at a point scales as

$$\theta(\underline{x}, t) \sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2}$$

indep. of $\underline{x}$

$$= \exp\left(-\sum \rho_i\right)$$

Expected value:

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha \sum \rho_i} \exp\left(-t S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) P_{stab}(\rho_2)\, d\rho_1\, d\rho_2$$

Non-exponential function of t (neglect)

$$\sim \int e^{-\alpha \rho_1} \exp\left(-t S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) d\rho_1$$

← Do the ρ_2 integral

expected value, not integral

Use $h_i = \rho_i / t$ as variable:

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha h_1 t}\, e^{-t S(h_1 - \lambda_1)}\, dh_1$$

$$\boxed{\langle \theta^\alpha \rangle(t) \sim \int e^{-t(\alpha h + S(h - \lambda))}\, dh}$$

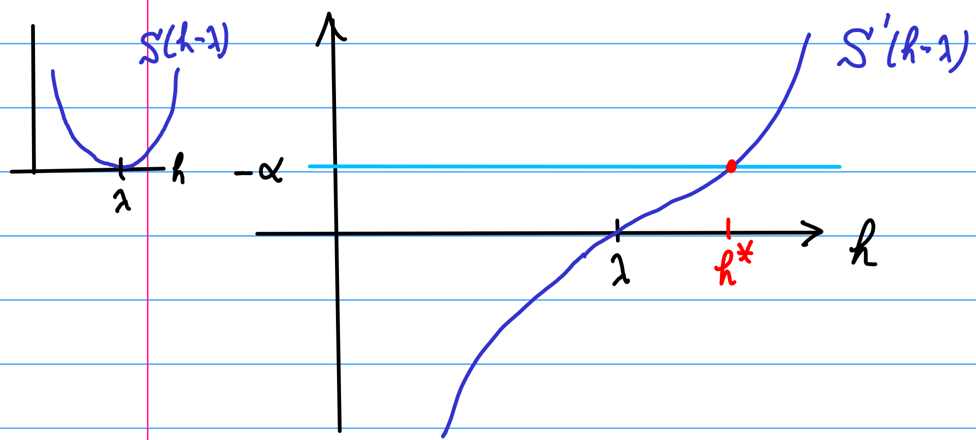
$h_1 \to h$

$\lambda_1 \to \lambda$

Let $H(h) = \alpha h + S(h-\lambda)$.

For large time, the integral is dominated by saddle point h^*:

$$H'(h^*) = 0 = \alpha + S'(h^* - \lambda)$$



Because of convexity of S, h^* is unique.

We then have $H(h) = H(h^*) + \frac{1}{2} H''(h^*)(h-h^*)^2 + \dots$

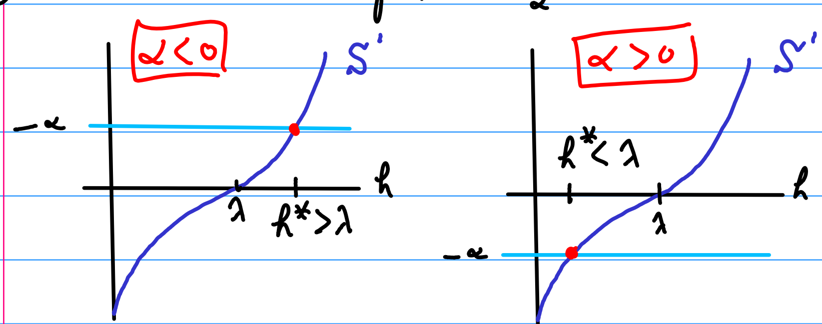
which we use to evaluate the integral. Find:

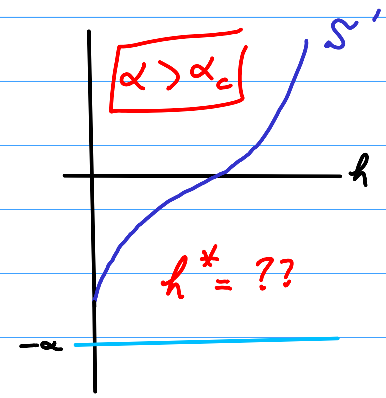
$$\langle \theta^\alpha \rangle (t) \sim e^{-\gamma_\alpha t}, \quad \text{where } \gamma_\alpha = H(h^*)$$

Note that we <u>do not</u> have $\langle \theta^\alpha \rangle \sim e^{-\alpha \gamma t}$, which would be the case if θ decayed the same pointwise everywhere.

$$\text{Kurtosis} \sim \frac{\langle \theta^\alpha \rangle}{\langle \theta \rangle^\alpha} \sim e^{-\gamma_\alpha t}$$

So how do we expect γ_α to behave?



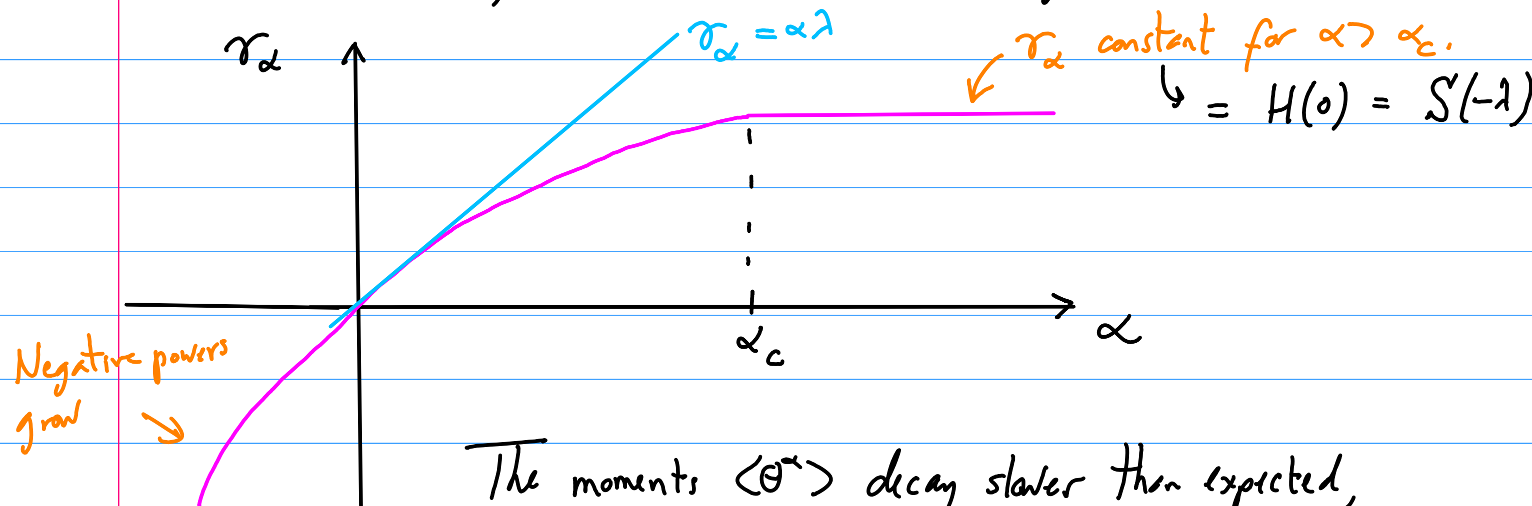


We have $\gamma_0 = 0$, since $S'(h-\lambda) = 0$ at $h=\lambda$, and $S(0)=0$.

↳ $\langle \theta^0 \rangle = 0$ ok!

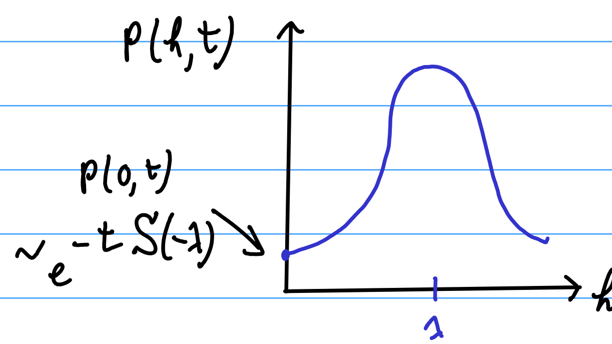
Hence, γ_α changes sign at $\alpha = 0$.

What happens for $\alpha > \alpha_c$? No saddle point, since would require $h^* < 0$ (not allowed). Hence, take $h^* = 0$ (slowest decay)



The moments $\langle \theta^\alpha \rangle$ decay slower than expected, all the more so for larger α: <u>INTERMITTENCY</u>

Why the leveling-off? For large α, $\langle \theta^\alpha \rangle$ is dominated by realizations with large θ, that is, having experienced little stretching. For $\alpha > \alpha_c$, these are all that matter, so γ_α is the rate of decay of realizations with <u>no stretching</u>,



All this was for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.

Lecture 4: Mixing in the presence of sources and sinks

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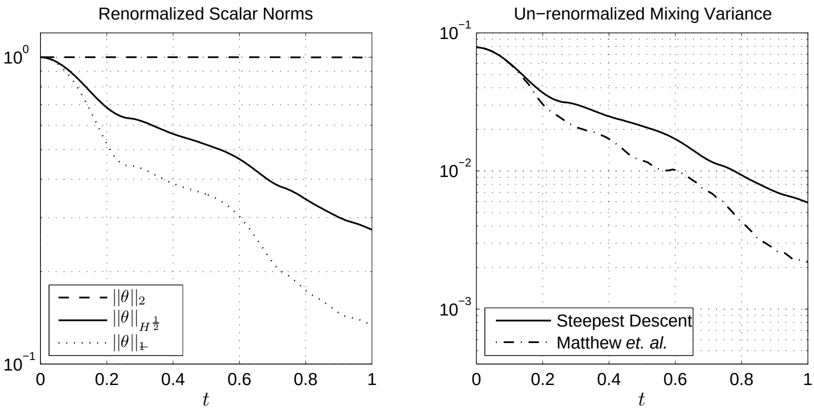
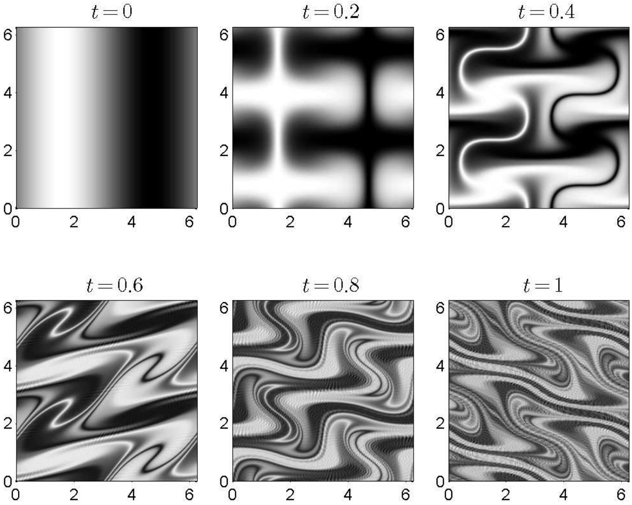
28 June 2010

Optimal control vs steepest descent



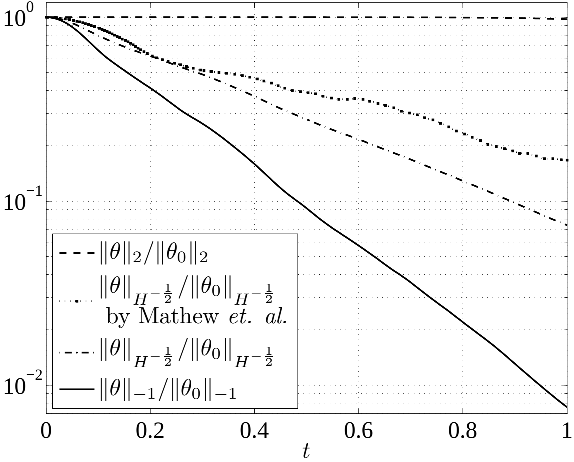
(from Lin, Thiffeault, Doering.)

Steepest descent of $\dot{H}^{-1}$



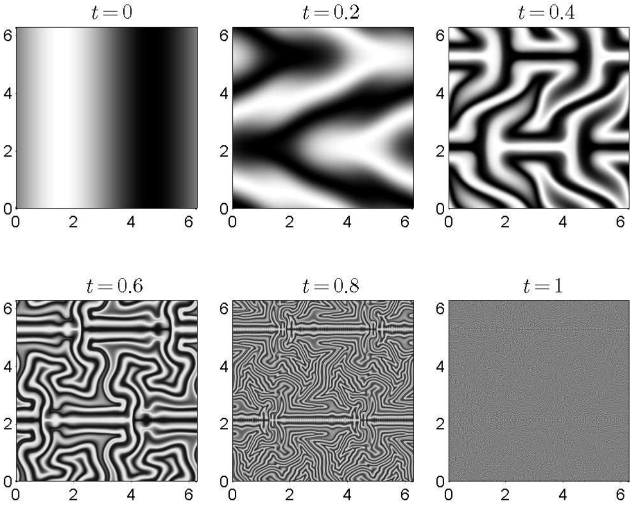
(from Lin, Thiffeault, Doering.)

Optimal control vs steepest descent: any flow



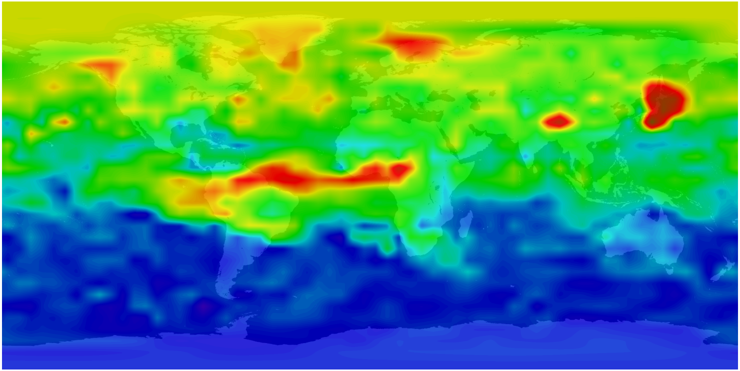
(from Lin, Thiffeault, Doering.)

Steepest descent of $\dot{H}^{-1}$: any flow



(from Lin, Thiffeault, Doering.)

Sources and sinks: CO in the atmosphere



Red corresponds to high levels of CO (450 parts per billion) and blue to low levels (50 ppb). Note the immense clouds due to grassland and forest fires in Africa and South America. (Photo NASA/NCAR/CSA.)

Matlab code: Minimize norm with fmincon

```
function [psi,Effq] = velopt(psi0,src,kappa,q,L,scalefac)

% Problem parameters for Matlab's optimizer fmincon.
psi0 = psi0(:); problem.x0 = psi0(2:end);
problem.objective = @(x) normHq2(x,src,kappa,q,L,scalefac);
problem.nonlcon = @(x) nonlcon(x,src,kappa,q,L,scalefac);
problem.solver = 'fmincon';
problem.options = optimset('Display','iter','TolFun',1e-10,...
   'GradObj','on','GradConstr','on',...
   'algorithm','interior-point');

[psi,Hq2] = fmincon(problem);

% Mixing efficiency: call normHq2 with no flow to get pure-conduction solution.
Effq = sqrt(normHq2(zeros(size(psi)),src,kappa,q,L,scalefac) / Hq2);

psi = reshape([0;psi],size(src)); % Convert psi back into a square grid
```

Matlab code: Right-hand side function

```
function [varargout] = normHq2(psi,src,kappa,q,L,scalefac)

N = size(src,1); src = src(:);

% 2D Differentiation matrices and negative-Laplacian
[Dx,Dy,Dxx,Dyy] = Diffmat2(N,L); mlap = -(Dxx+Dyy);
if q ~= 0 && q ~= -1, error('This code only supports q = 0 or -1.'); end

psi = [0;psi]; ux = Dy*psi; uy = -Dx*psi;
ugradop = diag(sparse(ux))*Dx + diag(sparse(uy))*Dy;

if q == 0
  Aop2 = (-ugradop + kappa*mlap);
elseif q == -1
  Aop2 = mlap*(-ugradop + kappa*mlap);
end
Aop1 = (ugradop + kappa*mlap)*Aop2;
% Solve for chi, dropping corner point to fix normalisation.
chi = [0; Aop1(2:end,2:end) \ src(2:end)];
theta = Aop2*chi;

% The squared H^q norm of theta.
varargout{1} = L^2*sum(theta.^2)/N^2 * scalefac;

if nargout > 1
  % Gradient of squared-norm Hq2.
  gradHq2 = 2*((Dx*theta).*(Dy*chi) - (Dy*theta).*(Dx*chi));
  varargout{2} = gradHq2(2:end) / N^2 * scalefac;
end
```

Matlab code: Constraints

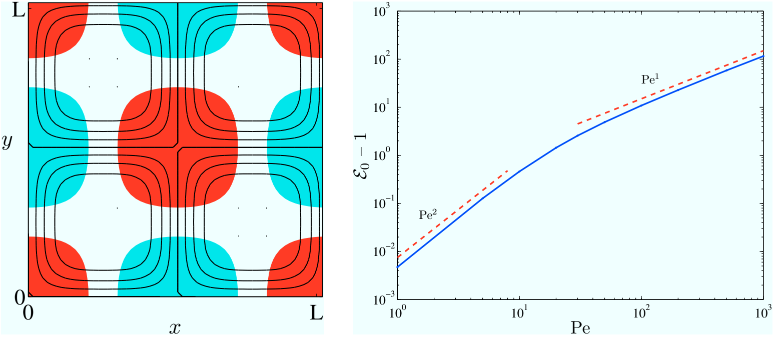
```
function [c,ceq,gc,gceq] = nonlcon(psi,src,kappa,q,L,scalefac)

psi = [0;psi]; N = size(src,1);
c = []; gc = [];

[Dx,Dy,Dxx,Dyy] = Diffmat2(N,L); % 2D Differentiation matrices
U2 = L^2*(sum((Dx*psi).^2 + (Dy*psi).^2)/N^2);
ceq(1) = (U2-1) * scalefac;

if nargout > 2
  % Gradient of constraints
  mlappsi = -(Dxx+Dyy)*psi;
  gceq(:,1) = 2*mlappsi(2:end) / N^2 * scalefac;
end
```

Optimal stirring flow



Left: Optimal stirring velocity field (streamlines) for the source $\sin x \sin y$, for $\mathrm{Pe} = 10$. Right: Dependence on Péclet number of the optimal mixing efficiency $\mathcal{E}_0$. For small Pe the optimal streamfunction $\rightarrow (\sqrt{2}\pi)^{-1} \cos x \cos y$.

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GFD Lectures: Swimming & Swirling 2010/06/28

Lecture 4: Mixing in the Presence of Sources and Sinks

Part 1: Norms

$$\partial_t \theta + \underline{u} \cdot \nabla\theta = \kappa \nabla^2 \theta, \quad \nabla \cdot \underline{u} = 0 \qquad \text{(AD)}$$

in Ω a $\begin{cases} \cdot \text{ bounded domain with zero-flux conditions} \\ \text{OR} \\ \cdot \text{ periodic domain} \end{cases}$

Assume $\int_\Omega \theta \, d\Omega = 0$. Let $\|\theta\|_2^2 = \int_\Omega \theta^2 d\Omega$ L^2-norm Also VARIANCE

Recall: $\frac{d}{dt} \|\theta\|_2^2 = -2\kappa \|\nabla\theta\|_2^2$ Equation of variance decay.

Variance (L^2-norm) would seem a good measure of mixing. But it requires knowledge of small-scales in θ, which we may not care about. Wouldn't it be better to blindly solve:

$$\text{(A)} \quad \frac{\partial\theta}{\partial t} + \underline{u} \cdot \nabla\theta = 0 \quad ?$$

since often we don't care how something is homogenized

But then: $\frac{d}{dt} \|\theta\|_2^2 = 0$, so can't use variance.

The advection equation (A) takes us closer to the ergodic theory sense of mixing.

In ergodic theory, we think of an operator $S^t : \Omega \to \Omega$, which is obtained from the solution of (A). S^t "moves forward" a patch of dye $\theta_0(\underline{x})$ to $\theta(\underline{x}, t)$.

For a patch A, Vol (A) (or Area (A)) is the <u>Lebesgue measure</u> of A

$\theta(\underline{x}, t)$ only 0 or 1, say

Because of incompressibility, S^tA has the same volume as A. S^t is <u>measure-preserving</u>.

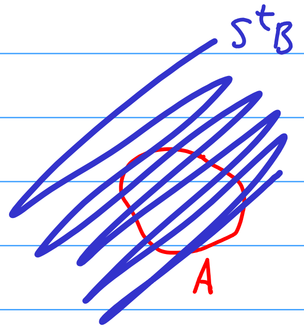
Now for the definition of <u>MIXING</u> (in the sense of ergodic theory):

$$\boxed{\lim_{t \to \infty} \mathrm{Vol}(A \cap S^t(B)) = \mathrm{Vol}(A)\,\mathrm{Vol}(B)}$$

for all patches A, B in Ω.

usual def'n has $-t$, but S invertible here so it's the same.

What does this mean? Imagine B has been mixed:



Recall S^tB has the same volume as B. So if it "fills the space" Ω as best as it can, its intersection with A can only be as large as Vol (B), but also Vol (A). This is true for any A, B, so every blob must spread everywhere.

This is actually called STRONG MIXING. It implies ergodicity, but not the other way around.

Notice that this follows our intuition for what "good mixing" is, but no diffusion is needed.

In fact, the arbitrary "reference patch" A is a bit like a function that we project on. This suggests another def'n, which is more "analytic":

Weak convergence:

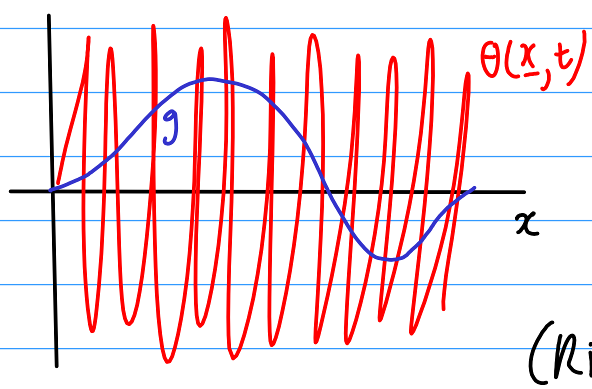
$$\lim_{t\to\infty} \langle \theta(\underline{x},t), g \rangle = 0$$

(θ converges to zero weakly)

for all functions $g(\underline{x})$ in $L^2(\Omega)$

Here: $\langle f, g \rangle = \int_\Omega f(\underline{x})\, g(\underline{x})\, d\Omega$, and a function f is in $L^2(\Omega)$ if $\int_\Omega |f|^2 d\Omega < \infty$ (for example, δ-functions are not)

Weak convergence is equivalent to mixing. Why?



θ is no vanishing, but it is getting wigglier, so

$$\int \theta g \, dx \to 0$$

(Riemann-Lebesgue lemma)

But neither the def'n of mixing and weak convergence are that useful in practice: hard to compute something over all functions $g(\underline{x})$!

But there is a simpler way: Mathew, Mezic, & Petzold introduced the mix-norm, which is basically a negative Sobolev norm:

The "dot" is because we assume $\int\theta=0$

$$\|\theta\|_{\dot{H}^q} := \|\nabla^q\theta\|_2$$

Sobolev norm for $\dot{H}^q(\Omega)$ ($q=0$ is L^2 norm)

We can interpret this norm for $q<0$ as well! This is easiest on a periodic domain:

$$\boxed{\|\theta\|^2_{\dot{H}^q} = \sum_{\underline{k}}{}' |\underline{k}|^{2q}\,|\hat{\theta}_{\underline{k}}|^2}$$

Note $\hat{\theta}_{\underline{0}}=0$ (mean)

For $q<0$, $\|\theta\|_{\dot{H}^q}$ smooths θ before taking the L^2 norm.

Theorem (Mathew-Mezic-Petzold, Doering-Lin-T)

$$\lim_{t\to\infty}\|\theta\|_{\dot{H}^q}=0,\ q<0 \iff \theta \text{ converges weakly to } 0.$$

(proof is short, but a bit technical.)

Upshot: we can track any of these norms to determine if a system is mixing. q controls how much smoothing is imposed.

This makes optimization easier, for instance.

Time-evolution of $\dot{H}^q$-norms: (w/o diffusion)

$\nabla^{-1}\sim -i\underline{k}/|\underline{k}|^2$

$$\frac{d}{dt}\|\theta\|^2_{\dot{H}^{-1}} = \langle\nabla^{-1}\theta\cdot\nabla u\cdot\nabla^{-1}\theta\rangle$$

$$\frac{d}{dt}\|\theta\|^2_{\dot{H}^{1}} = -\langle\nabla\theta\cdot\nabla u\cdot\nabla\theta\rangle$$

NOT conserved even in the absence of diffusion.

(other norms are uglier)

Mathew, Mezic, Grivopoulos, Vaidya, Petzold: use optimal control to optimize decay of $\|\theta\|^2_{\dot{H}^{-1/2}}$ (nonlocal in time)

Lin, Doering, T: maximize instantaneous decay rate of $\|\theta\|^2_{\dot{H}^{-1}}$

(local in time, easier, almost as good)

SLIDES pages 2-5

PART 2: SOURCES AND SINKS

$$\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta + \underbrace{s(\underline{x},t)}_{\substack{\text{sources / sinks} \\ >0 \quad <0}} \qquad \text{(ADs)}$$

$$(\nabla\cdot\underline{u} = 0)$$

(SLIDE p. 6)

Assume: $\int_\Omega s(\underline{x},t)\,d\Omega = 0$ (otherwise subtract the mean)

More convenient to think of $\underset{\text{sources}}{\text{hot}}$ / $\underset{\text{sinks}}{\text{cold}}$

For simplicity, restrict to time-independent $s(\underline{x})$.
Then system achieves a steady-state: (unlike decaying problem)

$$\underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta + s \qquad \text{let } \mathcal{L} \equiv \underline{u}\cdot\nabla - \kappa\nabla^2$$

$$\hookrightarrow \quad \mathcal{L}\theta = s \quad \text{or} \quad s = \underbrace{\mathcal{L}^{-1}}_{\text{integral operator}}\theta$$

Note that $\kappa \neq 0$ is needed to reach steady-state.

So, assuming the system has reached a steady-state, how do we measure the "quality of mixing"?

Can look at norms $\|\theta\|_{\dot{H}^q}$ ($q=0$ is standard deviation)

But what do we compare to?

One possibility: $\dfrac{\|\theta\|_{\dot{H}^q}}{\|s\|_{\dot{H}^q}}$ Pretty good, but has units of inverse time.

Prefer <u>mixing enhancement factors</u>:

$$\boxed{\mathcal{E}_q = \frac{\|\tilde{\theta}\|_{\dot{H}^q}}{\|\theta\|_{\dot{H}^q}}}$$

$$\tilde{\mathcal{L}} = -\kappa\nabla^2$$
$$\tilde{\mathcal{L}}\tilde{\theta} = s$$

$\tilde{\theta}$ is the solution in the absence of stirring. (purely diffusive)

Since $\|\theta\|_{\dot{H}^q}$ is usually <u>decreased</u> by stirring, $\mathcal{E}_q$ measures the enhancement over the pure-diffusion state.

Several properties given in Doering & T, Shaw, Doering, & T.

For instance, can we have $\mathcal{E}_q < 1$, i.e., can stirring ever be worse than not stirring?

Consider $\mathcal{E}_1 = \dfrac{\|\nabla\tilde{\theta}\|_2}{\|\nabla\theta\|_2}$.

$$\tilde{\theta} = \tilde{\mathcal{L}}^{-1}s = (-\kappa\nabla^2)^{-1}s = -\kappa^{-1}\nabla^{-2}s \Rightarrow \nabla\tilde{\theta} = -\kappa^{-1}\nabla^{-1}s.$$

Also: $\mathcal{L}\theta = s \Rightarrow \langle\theta\,\mathcal{L}\theta\rangle = \langle s\theta\rangle \qquad \langle\cdot\rangle = \int_\Omega \cdot\, d\Omega$

$$\langle\cancel{\theta\,u\cdot\nabla\theta}\rangle - \kappa\langle\theta\nabla^2\theta\rangle = \langle s\theta\rangle$$

$= \langle\nabla\cdot(u\theta^2/2)\rangle = 0$

$$\kappa\langle|\nabla\theta|^2\rangle = \langle\theta s\rangle = \langle\theta\,\underbrace{\nabla\cdot\nabla^{-1}}_{=1}s\rangle$$

$$\kappa\|\theta\|^2_{\dot{H}^1} = -\langle\nabla\theta\cdot\nabla^{-1}s\rangle = \kappa\langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle$$

$$\|\theta\|^2_{\dot{H}^1} = \langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle \le \underbrace{\|\nabla\theta\|_2}_{\|\theta\|_{\dot{H}^1}}\underbrace{\|\nabla\tilde{\theta}\|_2}_{\|\tilde{\theta}\|_{\dot{H}^1}}$$

Cauchy-Schwartz inequality

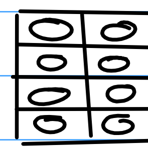
$$\therefore \|\theta\|_{\dot{H}^1} \leq \|\tilde{\theta}\|_{\dot{H}^1} \iff \boxed{\mathcal{E}_1 \geq 1}$$

This is somewhat counter-intuitive: gradients are usually increased by stirring! However, here we're talking about gradients in a steady-state, affected by diffusion.

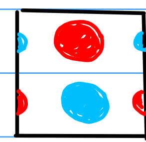
What about the other ones, $\mathcal{E}_q$, $q \neq 1$? Do we have $\mathcal{E}_q \geq 1$?

We tried and failed to prove this, because it isn't true. Following a challenge by Charlie Doering, Jeff Weiss came up with something like:

$$\underline{u} = (2 \sin x \cos 2y, \; -\cos x \sin 2y)$$



$$s = (\cos x - \tfrac{1}{2}) \sin y$$



(Péclet = 4)

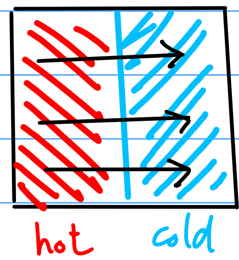
This manages to "concentrate" the source sink distribution more that under pure diffusion, and

$$\mathcal{E}_0 \simeq .978, \qquad \mathcal{E}_{-1} \simeq .945$$

Slightly less than 1! Not a dramatic effect, but it's there!
(more later)

OPTIMIZATION: What kinds of flow give the largest $\mathcal{E}_q$, given source/sink distribution $s(\underline{x})$? (FIXED ENERGY)

Surprising example: $s(\underline{x}) = \sin x$ (periodic B.C.)



optimal: $\underline{u} = U\hat{x}$ Constant flow!

(see Shaw-T-Doering, Plasting-Young)

This example demonstrates that with body sources the best stirring has more to do with transport than with creation of small scales.

SOLVE NUMERICALLY for more complicated sources. (SLIDE p.7, Matlab)